1. (a) Let R be a commutative ring with multiplicative identity $1 \neq 0$. Define units and zero divisors of R.

   (b) Find all units in $\overline{R} = \mathbb{Z}/(18)$.

   (c) Compute $\overline{15}^{-1}$ in $\mathbb{Z}_{7564}$.

2. Let $f(x) = \overline{5}x^4 + \overline{3}x^3 + \overline{1}$ and $g(x) = \overline{3}x^2 + \overline{2}x + \overline{1}$ in $\mathbb{Z}_7[x]$. Find the greatest common divisor $d(x) = (f(x), g(x))$, as well as polynomials $h(x)$ and $k(x)$ such that

$$d(x) = k(x)f(x) + h(x)g(x).$$

3. Factorize the following polynomials into products of irreducibles:

   (a) $f(x) = 2x^5 + 5x^4 + 4x^3 + 7x^2 + 7x + 2 \in \mathbb{Q}[x]$;

   (b) $f(x) = x^3 + x^2 + x + \overline{1} \in \mathbb{Z}_2[x]$;

   (c) $f(x) = x^{11} + 1 \in \mathbb{Z}_{11}[x]$.

4. (a) Define an ideal of a commutative ring with multiplicative identity.

   (b) Prove: Let R be a commutative ring with multiplicative identity $1 \neq 0$, A is an ideal of R and "$\sim$" is a relation on R defined by

$$x, y \in R,\ x \sim y \Leftrightarrow x - y \in A\ .$$

   Then: (i) "$\sim$" is an equivalence relation.

   (ii) $x \sim y,\ u \sim v \Rightarrow x + u \sim y + v,\ xu \sim yv$.

5. (a) Let R be an integral domain. Define a prime element in R; define an irreducible element in R.

   (b) Determine which of the polynomials below are irreducible over $\mathbb{Q}$:

      i. $f(x) = 5x^5 + 9x^4 + 15x^3 + 3x^2 + 6x + 3$

      ii. $f(x) = x^4 + 15x^3 + 7$. Justify your answer.

   (c) Prove the following result: If F is a field and $f(x) \in F[x]$, then $f(x)$ is irreducible if and only if $f(x)$ is a prime.

6. Assume $R = \mathbb{Z}_3$, $\overline{R} = \mathbb{Z}_{12}$ and $\phi : R \to \overline{R}$ is defined by $\phi(x) = 4x$, for all $x \in R$ (bars are removed).

   (a) Show that ϕ is well defined.

   (b) Verify that ϕ is a ring homomorphism.

   (c) Find Ker (ϕ).

   (d) Is $\phi(R^*) \subseteq \overline{R}^*$? Justify your answer.

7. Let F be a field and let

$$A = \{f(x) \in F[x] \mid f(1) = 0\} \subset F[x].$$

(a) Show that $A \neq \emptyset$

(b) Show that A is an ideal of $F[x]$

(c) Show that $A = (h(x)) = h(x)F[x]$ where $h(x) = x - 1$ and $(h(x))$ denotes the principal ideal generated by $h(x)$.

8. Consider the polynomial ring $R = \mathbb{Z}_2[x]$.

(a) List all polynomials of degree 3 in $\mathbb{Z}_2[x]$.

(b) Find all maximal ideals of $R = \mathbb{Z}_2[x]$ generated by polynomials of degree 3.

(c) Explain how to construct a field of 25 elements.