An algebraic integration for Mahler measure

Matilde N. Lalín

University of Texas at Austin

mlalin@math.utexas.edu

http://www.ma.utexas.edu/users/mlalin

Mahler measure

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$m(P) = \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n$$

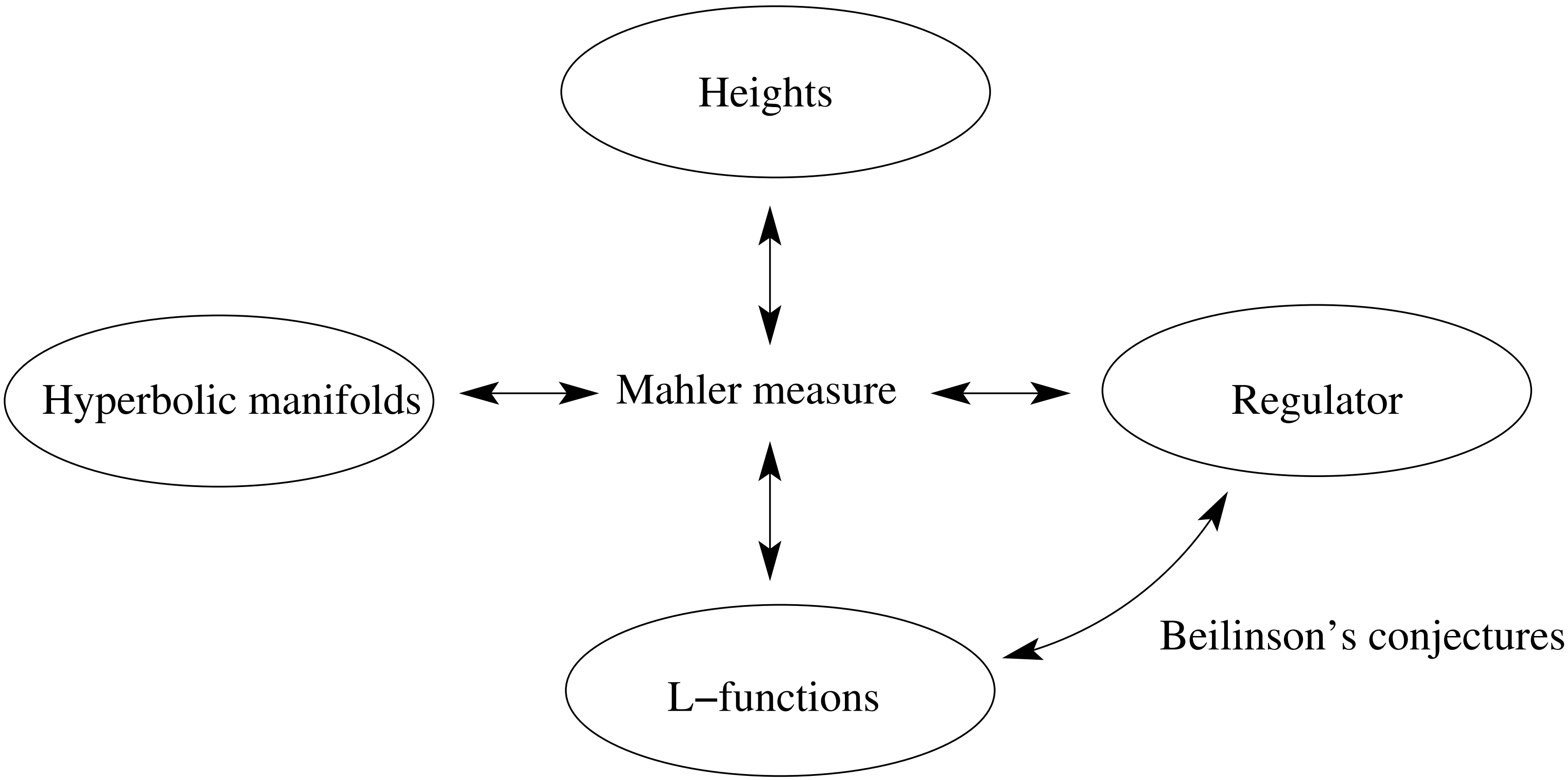
$$= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n}$$

Examples in several variables

Smyth (1981)

$$m(1+x+y) \;=\; \frac{3\sqrt{3}}{4\pi}\mathsf{L}(\chi_{-3},2) = \mathsf{L}'(\chi_{-3},-1)$$

$$m(1+x+y+z) \;=\; \frac{7}{2\pi^2}\zeta(3)$$


Heights
Hyperbolic manifolds
Mahler measure
Regulator
L–functions
Beilinson's conjectures

Examples in several variables

L (2003)

$$\pi^{2n} m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\cdots\left(\frac{1-x_{2n}}{1+x_{2n}}\right)z\right)$$

$$= \sum_{h=1}^{n} c_{n,h}\pi^{2n-2h}\zeta(2h+1)$$

$$= \pi^{2n}\sum_{h=1}^{n} c'_{n,h}\zeta'(-2h)$$

$$c_{n,h} = \frac{2^{n-2}}{(2n-1)!}s_{n-h}(1^2,\ldots,(n-1)^2)(2h)!(2^{2h+1}-1)$$

Examples

$$\pi^3 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\left(\frac{1-x_3}{1+x_3}\right)z\right)$$

$$= 24\mathsf{L}(\chi_{-4},4)+\pi^2\mathsf{L}(\chi_{-4},2)$$

$$\pi^4 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)z\right)$$

$$= 62\zeta(5)+\frac{14}{3}\pi^2\zeta(3)$$

$$\pi^4 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)=93\zeta(5)$$

Polylogarithms

The *kth polylogarithm* is

$$\mathrm{Li}_k(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^k} \qquad x \in \mathbb{C}, \quad |x| < 1$$

It has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$.

Zagier:

$$P_k(x) := \mathrm{Re}_k \left(\sum_{j=0}^{k} \frac{2^j B_j}{j!} (\log |x|)^j \mathrm{Li}_{k-j}(x) \right)$$

B_j is jth Bernoulli number, $\mathrm{Li}_0(x) \equiv -\frac{1}{2}$,

$\mathrm{Re}_k = \mathrm{Re}$ or Im if k is odd or even.

One-valued, real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$, continuous in $\mathbb{P}^1(\mathbb{C})$.

P_k satisfies lots of functional equations

$$P_k\left(\frac{1}{x}\right) = (-1)^{k-1} P_k(x) \qquad P_k(\bar{x}) = (-1)^{k-1} P_k(x)$$

Bloch–Wigner dilogarithm ($k = 2$)

$$D(x) := \operatorname{Im}(\operatorname{Li}_2(x)) + \arg(1-x)\log|x|$$

Five-term relation

$$D(x)+D(1-xy)+D(y)+D\left(\frac{1-y}{1-xy}\right)+D\left(\frac{1-x}{1-xy}\right)=0$$

Philosophy of Beilinson's conjectures

Global information from local information through L-functions

- Arithmetic-geometric object X (for instance, $X = \mathcal{O}_F$, F number field)

- L-function ($\mathsf{L}_X = \zeta_F$)

- Finitely-generated abelian group K ($K = \mathcal{O}_F^*$)

- Regulator map

  $$r : K \to \text{smooth differential forms}$$

  ($r = \log|\cdot|$)

Conjecture: special value of $\mathsf{L}_X \sim_{\mathbb{Q}^*} \int_\gamma r(\xi)$

(E.g. Dirichlet class number formula, F real quadratic, $\zeta_F'(0) \sim_{\mathbb{Q}^*} \log|\epsilon| \qquad \epsilon \in \mathcal{O}_F^*$)

An algebraic integration for Mahler measure

Deninger (1997) : General framework.

Rodriguez-Villegas (1997) : $P(x,y) \in \mathbb{C}[x,y]$

$$m(P) = m(P^*) - \frac{1}{2\pi}\int_\gamma \eta(x,y)$$

$$\eta(x,y) = \log|x| \,\mathrm{d}\arg y - \log|y| \,\mathrm{d}\arg x$$

$$\eta(x, 1-x) = \mathrm{d}D(x)$$

In $\bigwedge^2(\mathbb{C}(C)^*) \otimes \mathbb{Q}$,

$$x \wedge y = \sum_j r_j \;\; z_j \wedge (1 - z_j).$$

$\{x, y\} = 0$ in $K_2(\mathbb{C}(C)) \otimes \mathbb{Q}$.

$$\int_\gamma \eta(x,y) = \sum r_j \int_\gamma \eta(z_j, 1-z_j) = \sum r_j D(z_j)|_{\partial\gamma}.$$

Big picture

$$\cdots \to (K_3(\bar{\mathbb{Q}}) \supset) K_3(\partial\gamma) \to K_2(C, \partial\gamma) \to K_2(C) \to \cdots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

- $\eta(x,y)$ is exact, then $\{x,y\} \in K_3(\partial\gamma)$. We have $\partial\gamma \neq \emptyset$ and we use Stokes' Theorem.

  $\rightsquigarrow$ dilogarithms $\rightsquigarrow$ zeta function of a number field.

- $\partial\gamma = \emptyset$, then $\{x,y\} \in K_2(C)$. We have $\eta(x,y)$ is not exact.

  $\rightsquigarrow$ L-series of a curve.

We may get combinations of both situations.

The three-variable case

We follow Smyth's example

$$P(x,y,z) = (1-x)+(1-y)z \qquad S = \{P(x,y,z)=0\}$$

$$m(P) = m(1-y) + \frac{1}{(2\pi \mathrm{i})^3}\int_{\mathbb{T}^3} \log\left|z - \frac{1-x}{1-y}\right| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}\frac{\mathrm{d}z}{z}$$

$$= \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2} \log^+\left|\frac{1-x}{1-y}\right| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

$$= -\frac{1}{(2\pi)^2}\int_{\Gamma} \log|z| \frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

$$= -\frac{1}{(2\pi)^2}\int_{\Gamma} \eta(x,y,z)$$

$$\Gamma = S \cap \{|x|=|y|=1, |z|\geq 1\}$$

$$\eta(x,y,z) = \log|x| \left(\frac{1}{3}\mathrm{d}\log|y|\,\mathrm{d}\log|z| - \mathrm{d}\arg y\,\mathrm{d}\arg z\right)$$

$$+\log|y| \left(\frac{1}{3}\mathrm{d}\log|z|\,\mathrm{d}\log|x| - \mathrm{d}\arg z\,\mathrm{d}\arg x\right)$$

$$+\log|z| \left(\frac{1}{3}\mathrm{d}\log|x|\,\mathrm{d}\log|y| - \mathrm{d}\arg x\,\mathrm{d}\arg y\right)$$

$$\mathrm{d}\eta(x,y,z) = \mathrm{Re}\left(\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y} \wedge \frac{\mathrm{d}z}{z}\right)$$

Theorem 1

$$\eta(x, 1-x, y) = \mathrm{d}\,\omega(x,y)$$

where

$$\omega(x,y) = -D(x)\mathrm{d}\arg y$$

$$+\frac{1}{3}\log|y|(\log|1-x|\mathrm{d}\log|x| - \log|x|\mathrm{d}\log|1-x|)$$

$$\eta(x,y,z) = -\eta(x,1-x,y) - \eta(y,1-y,x)$$

$$m((1-x)+(1-y)z) = \frac{1}{4\pi^2}\int_\gamma \omega(x,y)+\omega(y,x)$$

Theorem 2

$$\omega(x,x) = \mathrm{d}P_3(x)$$

Want to apply Stokes' Theorem again.

Maillot: if $P \in \mathbb{R}[x,y,z]$,

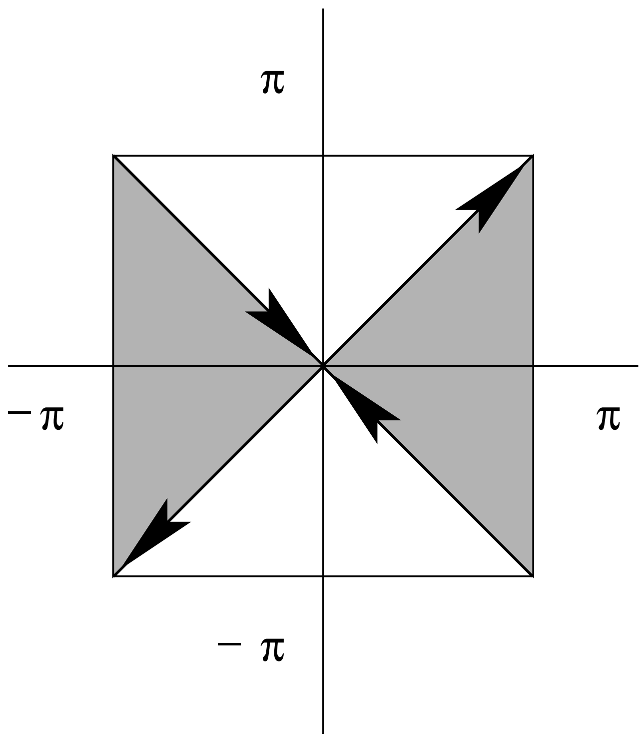
$\partial\Gamma = \gamma = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\} \cap \{|x| = |y| = 1\}$

ω defined in

$$C = \{P(x,y,z) = P(x^{-1},y^{-1},z^{-1}) = 0\}$$

$$\frac{(1-x)(1-x^{-1})}{(1-y)(1-y^{-1})} = 1$$

$$C = \{x = y\} \cup \{xy = 1\}$$



$$m(P) = \frac{1}{4\pi^2} 8(P_3(1) - P_3(-1)) = \frac{7}{2\pi^2}\zeta(3)$$

In general

$$m(P) = m(P^*) - \frac{1}{(2\pi)^2}\int_\Gamma \eta(x,y,z)$$

In $\bigwedge^3(\mathbb{C}(S)^*) \otimes \mathbb{Q}$,

$$x \wedge y \wedge z = \sum r_i \; x_i \wedge (1-x_i) \wedge y_i$$

$\{x,y,z\} = 0$ in $K_3^M(\mathbb{C}(S)) \otimes \mathbb{Q}$.

Then

$$\int_\Gamma \eta(x,y,z) = \sum r_i \int_\Gamma \eta(x_i, 1-x_i, y_i)$$

$$= \sum r_i \int_{\partial\Gamma} \omega(x_i, y_i)$$

Need

$$\{x\}_2 \otimes y = \sum r_i \{x_i\}_2 \otimes x_i$$

in $(\mathbf{B}_2(\mathbb{C}(C)) \otimes \mathbb{C}(C)^*)_{\mathbb{Q}}$, where

$$\mathbf{B}_2(F) := \mathbb{Z}[\mathbb{P}^1_F]/\text{five term relation}$$

Then

$$\int_\gamma \omega(x,y) = \sum r_i \int_\gamma \omega(x_i, x_i) = \sum r_i \, P_3(x_i)|_{\partial\gamma}$$

The condition is $\{x_i\}_2 \otimes y_i$ is 0 "in" $K_4^{\{3\}}(\mathbb{C}(C))_{\mathbb{Q}}$.

Big picture II

$$\cdots \to K_4(\partial\Gamma) \to K_3(S, \partial\Gamma) \to K_3(S) \to \cdots$$

$$\partial\Gamma = S \cap \mathbb{T}^3$$

$$\cdots \to (K_5(\bar{\mathbb{Q}}) \supset) K_5(\partial\gamma) \to K_4(C, \partial\gamma) \to K_4(C) \to \cdots$$

$$\partial\gamma = C \cap \mathbb{T}^2$$

In each step, we have the same two options as before.

More about regulators

$r_{\mathcal{D}} : K_{2j-i}^{\{j\}}(X) \to H_{\mathcal{D}}^{i}(X, \mathbb{R}(j))$

"Hard to compute"

Example:

$K_1^{\{1\}}(X) = \mathcal{O}_X^* \otimes \mathbb{Q} \stackrel{r_{\mathcal{D}} = \log|\cdot|}{\longrightarrow} H_{\mathcal{D}}^1(X, \mathbb{R}(1)).$

Special case $i = j$:

$H_{\mathcal{D}}^{i}(X, \mathbb{R}(i)) = \{\varphi \in \mathcal{A}^{i-1}(X)(i-1) \,|\, \mathrm{d}\varphi = \pi_{i-1}(\omega), \omega \in F^i(X)\} \,/\, \mathrm{d}\mathcal{A}^{i-2}(X)(i-1).$

$\mathcal{A}^i(X)(j)$ smooth i-forms with values in $(2\pi \mathrm{i})^j \mathbb{R}$.

$F^i(X)$ holomorphic i-forms with at most log singularities.

Polylogarithmic motivic complexes
(Goncharov)

"Explicit construction" of $r_{\mathcal{D}}$ and $K_{2j-i}^{\{j\}}(X)$.

F field, define $\mathcal{R}_n(F) \subset \mathbb{Z}[\mathbb{P}^1_F]$ and

$$\mathbf{B}_n(F) := \mathbb{Z}[\mathbb{P}^1_F]/\mathcal{R}_n(F)$$

$\mathcal{R}_1(F) := \langle \{x\} + \{y\} - \{xy\}, \quad x, y \in F^*, \{0\}, \{\infty\} \rangle$

$$\mathbb{Z}[\mathbb{P}^1_F] \xrightarrow{\delta_n} \begin{cases} \mathbf{B}_{n-1}(F) \otimes F^* & \text{if } n \geq 3 \\ \bigwedge^2 F^* & \text{if } n = 2 \end{cases}$$

$$\delta_n(\{x\}) = \begin{cases} \{x\}_{n-1} \otimes x & \text{if } n \geq 3 \\ (1-x) \wedge x & \text{if } n = 2 \\ 0 & \text{if } \{x\} = \{0\}, \{1\}, \{\infty\} \end{cases}$$

$$\mathcal{A}_n(F) := \ker \delta_n$$

$$\mathcal{R}_n(F) := \langle \alpha(0) - \alpha(1), \alpha(t) \in \mathcal{A}_n(F(t)) \rangle$$

Proposition 3

$$\delta_n(\mathcal{R}_n(F)) = 0$$

$\mathbf{B}_F(n)$:

$$\mathbf{B}_n(F) \xrightarrow{\delta} \mathbf{B}_{n-1}(F) \otimes F^* \xrightarrow{\delta} \ldots \xrightarrow{\delta} \mathbf{B}_2(F) \otimes \wedge^{n-2} F^* \xrightarrow{\delta} \wedge^n F^*$$

$$\delta : \{x\}_p \otimes \wedge_{i=1}^{n-p} y_i \to \delta_p(\{x\}_p) \wedge \wedge_{i=1}^{n-p} y_i.$$

$$\begin{aligned} H^1(\mathbf{B}_F(1)) &\cong K_1(F) \\ H^1(\mathbf{B}_F(2))_{\mathbb{Q}} &\cong K_3^{\text{ind}}(F)_{\mathbb{Q}} \\ H^2(\mathbf{B}_F(2)) &\cong K_2(F) \\ H^n(\mathbf{B}_F(n)) &\cong K_n^M(F) \end{aligned}$$

Conjecture 4

$$H^i(\mathbf{B}_F(n) \otimes \mathbb{Q}) \cong K_{2n-i}^{\{n\}}(F)_{\mathbb{Q}}$$

(Goncharov) X complex algebraic variety. There exist $\eta_n(m)$ inducing a homomorphism of complexes

$$\begin{array}{ccccccc} \mathbf{B}_n(\mathbb{C}(X)) & \xrightarrow{\delta} & \mathbf{B}_{n-1}(\mathbb{C}(X)) \otimes \mathbb{C}(X)^* & \xrightarrow{\delta} & \dots & \xrightarrow{\delta} & \bigwedge^n \mathbb{C}(X)^* \\ \downarrow \eta_n(1) & & \downarrow \eta_n(2) & & & & \downarrow \eta_n(n) \\ \mathcal{A}^0(X)(n-1) & \xrightarrow{\mathrm{d}} & \mathcal{A}^1(X)(n-1) & \xrightarrow{\mathrm{d}} & \dots & \xrightarrow{\mathrm{d}} & \mathcal{A}^{n-1}(X)(n-1) \end{array}$$

($\mathcal{A}^i(X)(j) =$ smooth i-forms with values in $(2\pi \mathrm{i})^j\mathbb{R}$) such that

- $\eta_n(1)(\{x\}_n) = \widehat{P}_n(x)$.

- $\mathrm{d}\eta_n(n)(x_1 \wedge \ldots \wedge x_n) = \pi_n\left(\frac{\mathrm{d}x_1}{x_1} \wedge \ldots \wedge \frac{\mathrm{d}x_n}{x_n}\right)$.

- $\eta_n(m)$ compatible with residues (residues are given by tame symbols).

Conjecture 5 *The image of*

$$\tilde{\eta_n}(i) : H^i(\mathbf{B}_X(n)) \to H^i_{\mathcal{D}}(X, \mathbb{R}(n))$$

coincides with the image of Beilinson's regulator.

Examples we studied

Smyth(1981):

$$\pi^2 m(1+x+y+z) = \frac{7}{2}\zeta(3)$$

Smyth(2002):

$$\pi^2 m(1+x+y^{-1}+(1+x+y)z) = \frac{14}{3}\zeta(3)$$

L (2003):

$$\pi^2 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)z\right) = 7\zeta(3)$$

$$\pi^2 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)x+\left(1-\left(\frac{1-x_1}{1+x_1}\right)\right)y\right)$$

$$= \frac{7}{2}\zeta(3) + \frac{\pi^2\log 2}{2}$$

Condon (2003):

$$\pi^2 m\left(z-\left(\frac{1-x}{1+x}\right)(1+y)\right)=\frac{28}{5}\zeta(3)$$

D'Andrea & L (2003):

$$\pi^2 m\left(z(1-xy)^{m+n}-(1-x)^m(1-y)^n\right)$$

$$=2n(P_3(\phi_1^m)-P_3(-\phi_2^m))+2m(P_3(\phi_2^n)-P_3(\phi_1^n))$$

$$\pi^2 m((1-x)(1-y)-(1-w)(1-z))=\frac{9}{2}\zeta(3)$$

L (2003):

$$\pi^2 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)x\right.$$

$$\left.+\left(1-\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\right)y\right)$$

$$=\frac{21}{4}\zeta(3)+\frac{\pi^2\log 2}{2}$$