Some aspects of the multivariable Mahler Measure

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Mahler measure for one-variable polynomials

Pierce (1918): $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - \alpha_i)$$

$$\Delta_n = \prod_i (\alpha_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Lehmer (1933):

$$\lim_{n\to\infty} \frac{|\alpha^{n+1} - 1|}{|\alpha^n - 1|} = \begin{cases} |\alpha| & \text{if } |\alpha| > 1 \\ 1 & \text{if } |\alpha| < 1 \end{cases}$$

For

$$P(x) = a \prod_i (x - \alpha_i)$$

$$M(P) = |a| \prod_i \max\{1, |\alpha_i|\}$$

$$m(P) = \log M(P) = \log |a| + \sum_i \log^+ |\alpha_i|$$

Kronecker's Lemma

$P \in \mathbb{Z}[x]$, $P \neq 0$,

$$m(P) = 0 \Leftrightarrow P(x) = x^k \prod \Phi_{n_i}(x)$$

where Φ_{n_i} are cyclotomic polynomials

Lehmer (1933)

$$m(x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1)$$

$$=\log(1.176280818\dots)=0.162357612\dots$$

$$\sqrt{\Delta_{379}}=1,794,327,140,357$$

Does there exist $C>0$, for all $P(x)\in\mathbb{Z}[x]$

$$m(P)=0 \quad \text{or} \quad m(P)>C??$$

Is the above polynomial the best possible?

Lehmer's question

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Mahler measure of several variable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \cdots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \ldots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log^+ |\alpha|$$

recovers one-variable case.

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Properties

- α algebraic number, and P_α minimal polynomial over $\mathbb{Q}$,

$$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]\, h(\alpha)$$

  where h is the logarithmic Weil height.
- $m(P) \geq 0$ if P has integral coefficients.
- $m(P \cdot Q) = m(P) + m(Q)$

Boyd & Lawton Theorem

$P \in \mathbb{C}[x_1, \dots, x_n]$

$$\lim_{k_2 \to \infty} \dots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \dots, x^{k_n})) = m(P(x_1, x_2, \dots, x_n))$$

Jensen's formula $\longrightarrow$ simple expression in one-variable case.

Several-variable case?

Examples in several variables

Smyth (1981)

- $$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2) = \mathrm{L}'(\chi_{-3},-1)$$

- $$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

$$\mathrm{L}(\chi_{-3},s) = \sum_{n=1}^{\infty}\frac{\chi_{-3}(n)}{n^s} \qquad \chi_{-3}(n) = \begin{cases} 1 & n \equiv 1 \bmod 3 \\ -1 & n \equiv -1 \bmod 3 \\ 0 & n \equiv 0 \bmod 3 \end{cases}$$

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

Boyd, Deninger, Rodriguez-Villegas (1997)

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-k\right) \stackrel{?}{=} \frac{\mathrm{L}'(E_k,0)}{B_k} \qquad k\in\mathbb{N}, \quad k\neq 4$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\right) = 2\mathrm{L}'(\chi_{-4},-1)$$

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}-4\sqrt{2}\right) = \mathrm{L}'(A,0)$$

$$A: y^2 = x^3 - 44x + 112$$

Mahler measure and hyperbolic volumes

Cassaigne – Maillot (2000) for $a, b, c \in \mathbb{C}^*$,

$$\pi m(a + bx + cy)$$

$$= \begin{cases} D\left(\left|\frac{a}{b}\right| e^{i\gamma}\right) + \alpha \log|a| + \beta \log|b| + \gamma \log|c| & \triangle \\ \pi \log \max\{|a|, |b|, |c|\} & \text{not } \triangle \end{cases}$$

Bloch–Wigner dilogarithm ($k = 2$)

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1 - x) \log|x|$$

Five-term relation

$$D(x) + D(1 - xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0$$

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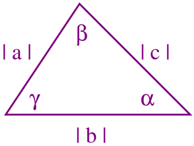
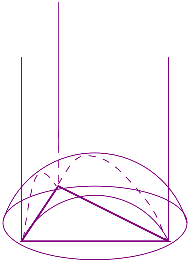
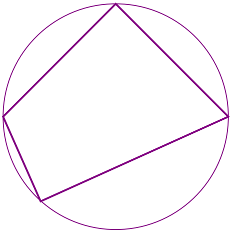
Ideal tetrahedron:

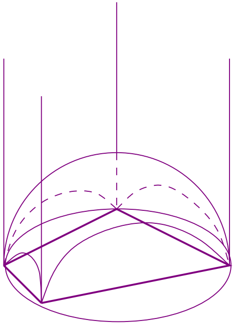




Vandervelde (2003)

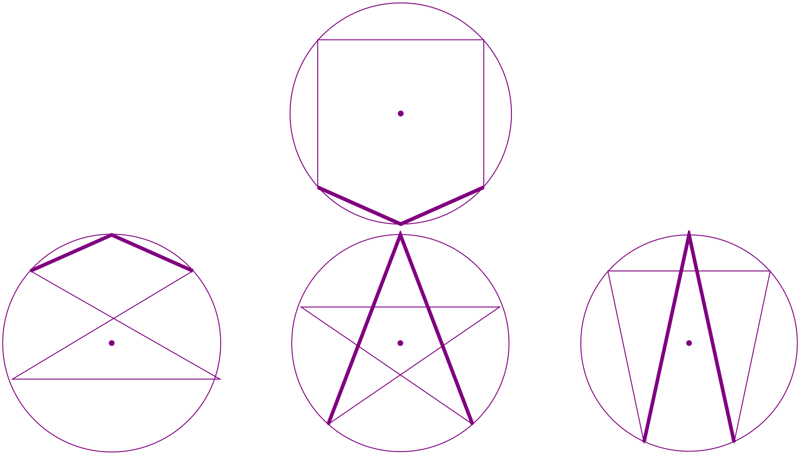
$$y = \frac{bx+d}{ax+c} \qquad \text{quadrilateral}$$





L(2004)

$$y = \frac{x^n - 1}{t(x^m - 1)} = \frac{x^{n-1} + \cdots + 1}{t(x^{m-1} + \cdots + 1)} \quad \text{polyhedral}$$



a b c

M orientable, complete, one-cusped, hyperbolic manifold.

$$M = \bigcup_{j=1}^{k} \Delta(z_j)$$

$$\mathrm{Vol}(M) = \sum_{j=1}^{k} D(z_j)$$

Boyd (2000)

$$\pi m(A) = \sum V_i$$

A is the A-polynomial.

$V_0 = \mathrm{Vol}(M)$ and the other V_i are pseudovolumes.

- $\mathrm{Im}\, z_i > 0 \rightsquigarrow$ geometric solution to the Gluing and Completeness equations.
- Other solutions $\rightsquigarrow$ pseudovolumes.

The connection with regulators

Deninger (1997), Rodriguez-Villegas (1997),
$P(x,y) \in \mathbb{C}[x,y]$

$$m(P) = m(P^*) - \frac{1}{2\pi}\int_\gamma \eta(x,y)$$

$$\eta(x,y) = \log|x|\,\mathrm{d}\arg y - \log|y|\,\mathrm{d}\arg x$$

$$\gamma = \{P(x,y)=0\} \cap \{|x|=1, |y|\geq 1\}$$

Philosophy of Beilinson's conjectures

Global information from local information through L-functions

- Arithmetic-geometric object X
- L-function
- Finitely-generated abelian group K
- Regulator map $\mathrm{reg}: K \to \mathbb{R}$

$$(K \text{ rank } 1) \qquad \mathrm{L}'_X(0) \sim_{\mathbb{Q}^*} \mathrm{reg}(\xi)$$

$$m(P) = m(P^*) - \frac{1}{2\pi}\int_\gamma \eta(x,y)$$

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- $\eta(x,y)$ is exact and $\partial\gamma \neq \emptyset$
  $\rightsquigarrow$ dilogarithms, zeta function
- $\partial\gamma = \emptyset$, and $\eta(x,y)$ is not exact,
  $\rightsquigarrow$ L-series of a curve

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Examples in three variables

- Smyth (2002):

$$\pi^2 m(1 + x + y^{-1} + (1 + x + y)z) = \frac{14}{3}\zeta(3)$$

- Condon (2003):

$$\pi^2 m\left(z - \left(\frac{1-x}{1+x}\right)(1+y)\right) = \frac{28}{5}\zeta(3)$$

- D'Andrea & L. (2003):

$$\pi^2 m\left(z(1-xy)^2-(1-x)(1-y)\right)=\frac{4\sqrt{5}\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\zeta(3)}$$

- Boyd & L. (2005):

$$m(x^2+1+(x+1)y+(x-1)z)=\frac{\mathrm{L}(\chi_{-4},2)}{\pi}+\frac{21}{8\pi^2}\zeta(3)$$

$$m(x^2+x+1+(x+1)y+z)=\frac{\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2)+\frac{19}{6\pi^2}\zeta(3)$$

More examples in several variables

L. (2005)

$$\pi^{2n} m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_{2n}}{1+x_{2n}}\right)z\right)$$
$$=\sum_{h=1}^{n} c_{n,h}\pi^{2n-2h}\zeta(2h+1)$$

$$c_{n,h}=\frac{2^{n-2}}{(2n-1)!}s_{n-h}(1^2,\dots,(n-1)^2)(2h)!(2^{2h+1}-1)$$

Examples

$$\pi^3 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\left(\frac{1-x_3}{1+x_3}\right)z\right)$$

$$= 24\mathrm{L}(\chi_{-4},4)+\pi^2\mathrm{L}(\chi_{-4},2)$$

$$\pi^4 m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)z\right)$$

$$=62\zeta(5)+\frac{14}{3}\pi^2\zeta(3)$$

$$\pi^4 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)=93\zeta(5)$$

$x + \frac{1}{x} + y + \frac{1}{y} - k$

Rodriguez-Villegas (1997)

$$P_k(x,y) = k - P(x,y) \qquad P(x,y) = x + \frac{1}{x} + y + \frac{1}{y}$$

$$m(k) := m(P_k)$$

Observe

$$m(k) = \log k + \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log |1 - \lambda P(x,y)| \frac{dx}{x} \frac{dy}{y}$$

where $\lambda = \frac{1}{k}$.

$$1 - \lambda P(x,y) > 0, \qquad \lambda \quad \text{small}, \quad x, y \in \mathbb{T}^2$$

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Let

$$u(\lambda) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \frac{1}{1-\lambda P(x,y)} \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}$$

$$= \sum_{n=0}^{\infty} \lambda^n \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} P(x,y)^n \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y} = \sum_{n=0}^{\infty} a_n \lambda^n$$

Where

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$$= \log k - \int_0^{\lambda} (u(t) - 1) \frac{\mathrm{d}t}{t} = \log k - \sum_{n=1}^{\infty} \frac{a_n \lambda^n}{n}$$

In this case,

$$a_n = 0 \qquad n \quad \text{odd}$$

$$a_{2m} = \binom{2m}{m}^2$$

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u, u', u'' are periods of a holomorphic differential in the curve defined by $1 - \lambda P(x, y)$.

Then $u(\lambda)$ satisfies a differential equation

$$A(\lambda)u'' + B(\lambda)u' + C(\lambda)u = 0$$

A, B, C polynomials in λ.

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A, B, C polynomials in λ.

In the example, $P_k(x,y) = k - x - \frac{1}{x} - y - \frac{1}{y}$,

$$u(\lambda) = \sum_{m=0}^{\infty} \binom{2m}{m}^2 \lambda^{2m}$$

$$v(\mu) = \sum_{m=0}^{\infty} \binom{2m}{m}^2 \mu^{m}$$

Then

$$\mu(16\mu - 1)v'' + (32\mu - 1)v' + 4v = 0.$$

Also

$$(n+1)^2 a_{2n+2} - 4(2n+1)^2 a_{2n} = 0.$$

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When x and y do not commute!

Take $P_k(x,y) = k - P(x,y) \in \mathbb{C}[\mathbb{F}_{x,y}] = \mathbb{C}[\mathbb{Z} * \mathbb{Z}]$.

Define

$$m_{nc}(k) = \log k - \sum_{n=1}^{\infty} \frac{a_n^{nc} \lambda^n}{n}$$

where

$$a_n^{nc} = [P(x,y)^n]_0^{nc}$$

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The analogous example,

$$P_k(x,y) = k - x - x^{-1} - y - y^{-1} \in \mathbb{C}[\mathbb{F}_{x,y}]$$

$$u_{nc}(\lambda) = 1 + 4\lambda^2 + 28\lambda^4 + 232\lambda^6 + \dots$$

$$u_{nc}(\lambda) = v_{nc}(\mu) \qquad \mu = \lambda^2$$

Then

$$v_{nc}(\mu) = \frac{3}{1 + 2\sqrt{1 - 12\mu}}$$

Counting circuits in a 4-regular tree.
In particular,

$$(12\mu - 1)(16\mu - 1)v''_{nc} + 2(240\mu - 19)v'_{nc} + 96v_{nc} = 0$$

$$(n+2)a^{nc}_{2n+4} - 2(14n+19)a^{nc}_{2n+2} + 96(2n+1)a^{nc}_{2n} = 0$$

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Another example,

$$P(x,y) = (1+x+y)\left(1+x^{-1}+y^{-1}\right)$$

$$u_{nc}(\lambda) = 1 + 3\lambda + 15\lambda^2 + 87\lambda^3 + 543\lambda^4 + \dots$$

$$u_{nc}(\lambda) = \frac{4}{1+3\sqrt{1-8\lambda}}$$

Counting circuits in a 3-regular tree.

$$(8\lambda-1)(9\lambda-1)u''_{nc} + 2(90\lambda-11)u'_{nc} + 36u_{nc} = 0$$

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$$u_{nc}(\lambda) = 1 + 4\lambda + 32\lambda^2 + 304\lambda^3 + 3136\lambda^4 + \dots$$

Conjecture (RV)

$$16u_{nc}^3\lambda = (u_{nc}-1)(u_{nc}+1)^2$$