Mahler measure under variations of the base group

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Mahler measure of several variable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{\mathrm{d}x_1}{x_1} \dots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

By Jensen's formula,

$$m\left(a \prod (x - \alpha_i)\right) = \log |a| + \sum \log \max\{1, |\alpha_i|\}.$$

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Examples in several variables

- Smyth (1981)

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = \frac{\mathrm{Vol}(\mathrm{Fig8})}{2\pi}$$

- Boyd, Deninger, Rodriguez-Villegas (1997)

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} - 1\right) \stackrel{?}{=} L'(E_1, 0)$$

E_1 elliptic curve, projective closure of $x + \frac{1}{x} + y + \frac{1}{y} - 1 = 0$.
(50 decimal places)

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A technique for reciprocal polynomials

Rodriguez-Villegas (1997)

$$P_\lambda(x,y) = 1 - \lambda P(x,y) \qquad P(x,y) = x + \frac{1}{x} + y + \frac{1}{y}$$

Reciprocal

$$m(P,\lambda) := m(P_\lambda)$$

$$m(P,\lambda) = \frac{1}{(2\pi \mathrm{i})^2} \int_{\mathbb{T}^2} \log|1 - \lambda P(x,y)| \frac{\mathrm{d}x}{x} \frac{\mathrm{d}y}{y}.$$

Note

$$|\lambda P(x,y)| < 1, \qquad \lambda \quad \text{small}, \quad x,y \in \mathbb{T}^2$$

$$\tilde{m}(P,\lambda) = \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2} \log(1-\lambda P(x,y))\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

$$= -\sum_{n=1}^{\infty}\frac{\lambda^n}{n}\frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2} P(x,y)^n\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y} = -\sum_{n=1}^{\infty}\frac{a_n\lambda^n}{n}$$

$$a_n := [P(x,y)^n]_0$$

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Let

$$u(P,\lambda) = \frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2} \frac{1}{1-\lambda P(x,y)}\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y} = \sum_{n=0}^{\infty} a_n \lambda^n$$

$$\frac{\mathrm{d}\tilde{m}(P,\lambda)}{\mathrm{d}\lambda} = -\frac{1}{(2\pi \mathrm{i})^2}\int_{\mathbb{T}^2} \frac{P(x,y)}{1-\lambda P(x,y)}\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y}$$

In the case $P = x + \frac{1}{x} + y + \frac{1}{y}$,

$$a_n = \begin{cases} \binom{2m}{m}^2 & n = 2m \\ \\ 0 & otherwise \end{cases}$$

Definition

Γ *finitely generated group with generators* $x_1, \ldots, x_l$

$$Q = Q(x_1, \ldots, x_l) = \sum_{g \in \Gamma} c_g g \in \mathbb{C}\Gamma,$$

$$Q^* = \sum_{g \in \Gamma} \overline{c_g} g^{-1} \in \mathbb{C}\Gamma \textit{ reciprocal.}$$

$P = P(x_1, \ldots, x_l) \in \mathbb{C}\Gamma$, $P = P^*$, $|\lambda|^{-1} >$ *length of* P,

$$m_\Gamma(P, \lambda) = -\sum_{n=1}^{\infty} \frac{a_n \lambda^n}{n},$$

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We also write

$$u_\Gamma(P,\lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$$

for the generating function of the a_n.

$Q(x_1,\ldots,x_l) \in \mathbb{C}\Gamma$

$$QQ^* = \frac{1}{\lambda}\left(1-(1-\lambda QQ^*)\right)$$

for λ real and positive and $1/\lambda$ larger than the length of QQ^*.

$$m_\Gamma(Q) = -\frac{\log\lambda}{2} - \sum_{n=1}^{\infty}\frac{b_n}{2n}, \qquad b_n = \left[(1-\lambda QQ^*)^n\right]_0 .$$

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Volume of hyperbolic knots

K knot: smooth embedding $S^1 \subset S^3$.

$$\Gamma = \pi_1(S^3 \setminus K) = \langle x_1, \dots, x_g \mid r_1, \dots, r_{g-1} \rangle$$

Derivation: mapping $\mathbb{C}\Gamma \to \mathbb{C}\Gamma$ (any group)

- $D(u + v) = Du + Dv$.
- $D(u \cdot v) = D(u)\epsilon(v) + uD(v)$

$$\epsilon : \mathbb{C}\Gamma \to \mathbb{C} \qquad \sum_g c_g g \to \sum_g c_g.$$

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Fox (1953) $\{x_1, \dots\}$ generators, there is $\frac{\partial}{\partial x_i}$ such that

$$\frac{\partial x_j}{\partial x_i} = \delta_{i,j}.$$

Back to knots,
Let

$$F = \begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \dots & \frac{\partial r_1}{\partial x_g} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_{g-1}}{\partial x_1} & \dots & \frac{\partial r_{g-1}}{\partial x_g} \end{pmatrix} \in M^{(g-1)\times g}(\mathbb{C}\Gamma)$$

Fox matrix.

Delete a column $F \rightsquigarrow A \in M^{(g-1)\times(g-1)}(\mathbb{C}\Gamma)$.

Theorem (Lück, 2002)

Suppose K is a hyperbolic knot. Then, for λ sufficiently small

$$\frac{1}{3\pi}\mathrm{Vol}(S^3 \setminus K) = -(g-1)\ln\lambda - \sum_{n=1}^{\infty}\frac{1}{n}\mathrm{tr}_{\mathbb{C}\Gamma}\left((1-\lambda AA^*)^n\right).$$

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Cayley Graphs

Γ of order m

$$\alpha : \Gamma \to \mathbb{C} \qquad \alpha(g) = \overline{\alpha(g^{-1})} \quad \forall g \in \Gamma$$

Weighted Cayley graph:

- Vertices $g_1, \ldots, g_m$.
- (directed) Edge between g_i and g_j has weight $\alpha(g_i^{-1} g_j)$.

Weighted adjacency matrix

$$A(\Gamma, \alpha) = \{\alpha(g_i^{-1} g_j)\}_{i,j}$$

Let $\chi_1, \dots, \chi_h$ be the irreducible characters of Γ of degrees $n_1, \dots, n_h$.

Theorem (Babai, 1979)

The spectrum of $A(\Gamma, \alpha)$ can be arranged as

$$\mathcal{S} = \{\sigma_{i,j} : i = 1, \dots, h; j = 1, \dots, n_i\}.$$

such that $\sigma_{i,j}$ has multiplicity n_i and

$$\sigma_{i,1}^t + \cdots + \sigma_{i,n_i}^t = \sum_{g_1,\dots,g_t \in \Gamma} \left(\prod_{s=1}^t \alpha(g_s) \right) \chi_i \left(\prod_{s=1}^t g_s \right).$$

The Mahler measure over finite groups

$$P = \sum_i (\delta_i S_i + \overline{\delta_i} S_i^{-1}) + \sum_j \eta_j T_j \in \mathbb{C}\Gamma$$

$\delta_i \in \mathbb{C}$, $\eta_j \in \mathbb{R}$, and $S_i, T_j \in \Gamma$,

$$a_n = \frac{\mathrm{tr}(A^n)}{|\Gamma|}$$

Theorem

For Γ finite

$$m_\Gamma(P,\lambda) = \frac{1}{|\Gamma|}\log\det(I - \lambda A),$$

A is the adjacency matrix of the Cayley graph (with weights) and $\frac{1}{\lambda} > \rho(A)$.

Analytic continuation for $m_\Gamma(P,\lambda)$ to $\mathbb{C} \setminus \mathrm{Spec}(A)$.

Finite Abelian Groups

$$\Gamma = \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_l\mathbb{Z}$$

Corollary

$$m_\Gamma(P,\lambda) = \frac{1}{|\Gamma|} \log \left(\prod_{j_1,\dots,j_l} \left(1 - \lambda P(\xi_{m_1}^{j_1},\dots,\xi_{m_l}^{j_l})\right) \right)$$

where ξ_k is a primitive root of unity.

Theorem

For small λ,

$$\lim_{m_1,\dots,m_l\to\infty} m_{\mathbb{Z}/m_1\mathbb{Z}\times\cdots\times\mathbb{Z}/m_l\mathbb{Z}}(P,\lambda) = m_{\mathbb{Z}^l}(P,\lambda).$$

Where the limit is with $m_1,\dots,m_l$ going to infinity independently.

Dihedral groups

$$\Gamma = D_m = \langle \rho, \sigma \mid \rho^m, \sigma^2, \sigma\rho\sigma\rho \rangle.$$

Theorem

Let $P \in \mathbb{C}[D_m]$ be reciprocal. Then

$$[P^n]_0 = \frac{1}{2m} \sum_{j=1}^{m} \left(P^n\left(\xi_m^j, 1\right) + P^n\left(\xi_m^j, -1\right)\right),$$

where P^n is expressed as a sum of monomials ρ^k, $\sigma\rho^k$ before being evaluated.

For $\Gamma = \mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle x, y \,|\, x^m, y^2, [x,y] \rangle$,

$$[P^n]_0 = \frac{1}{2m} \sum_{j=1}^{m} \left(P\left(\xi_m^j, 1\right)^n + P\left(\xi_m^j, -1\right)^n \right).$$

Compare D_m and $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ with $x = \rho$ and $y = \sigma$ in D_m.

Theorem

Let

$$P = \sum_{k=0}^{m-1} \alpha_k x^k + \sum_{k=0}^{m-1} \beta_k y x^k$$

with real coefficients and reciprocal in $\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ *(therefore it is also reciprocal in* D_m*). Then*

$$m_{\mathbb{Z}/m\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}(P, \lambda) = m_{D_m}(P, \lambda).$$

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$$m_{\mathbb{Z}/m\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}(P, \lambda) = m_{D_m}(P, \lambda).$$

Corollary

Let $P \in \mathbb{R}[\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}]$ *be reciprocal. Then*

$$m_{\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}(P,\lambda) = m_{D_\infty}(P,\lambda),$$

where $D_\infty = \langle \rho, \sigma \,|\, \sigma^2, \sigma\rho\sigma\rho \rangle$.

Quotient approximations of the Mahler measure

Γ_m are quotients of Γ:

Theorem

Let $P \in \Gamma$ reciprocal.

- *For* $\Gamma = D_\infty$, $\Gamma_m = D_m$,

$$\lim_{m\to\infty} m_{D_m}(P,\lambda) = m_{D_\infty}(P,\lambda).$$

- *For* $\Gamma = PSL_2(\mathbb{Z}) = \langle x, y \mid x^2, y^3\rangle$, $\Gamma_m = \langle x, y \mid x^2, y^3, (xy)^m\rangle$,

$$\lim_{m\to\infty} m_{\Gamma_m}(P,\lambda) = m_{PSL_2(\mathbb{Z})}(P,\lambda).$$

- *For* $\Gamma = \mathbb{Z} * \mathbb{Z} = \langle x, y\rangle$, $\Gamma_m = \langle x, y \mid [x,y]^m\rangle$,

$$\lim_{m\to\infty} m_{\Gamma_m}(P,\lambda) = m_{\mathbb{Z}*\mathbb{Z}}(P,\lambda).$$

Arbitrary number of variables

For $P_{1,l} = x_1 + x_1^{-1} + \cdots + x_l + x_l^{-1}$,

$$u_{\mathbb{F}_l}(P_{1,l}, \lambda) = g_{2l}(\lambda).$$

where

$$g_d(\lambda) = \frac{2(d-1)}{d - 2 + d\sqrt{1 - 4(d-1)\lambda^2}}.$$

is the generating function of the circuits of a d-regular tree (Bartholdi, 1999).

For $P_{2,l} = (1 + x_1 + \cdots + x_{l-1})\left(1 + x_1^{-1} + \cdots + x_{l-1}^{-1}\right)$,

$$u_{\mathbb{F}_{l-1}}(P_{2,l}, \lambda) = g_l(\lambda).$$

In particular,

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Abelian case.

For $P_{1,l} = x_1 + x_1^{-1} + \cdots + x_l + x_l^{-1}$,

$$\left[P_{1,l}^n\right]_0 = \sum_{a_1+\cdots+a_l=n} \frac{(2n)!}{(a_1!)^2 \ldots (a_l!)^2},$$

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$$\left[P_{1,l}^{2n}\right]_0 = \binom{2n}{n} \left[P_{2,l}^n\right]_0$$

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$$\left[P_{1,l}^{2n}\right]_0 = \binom{2n}{n} \left[P_{2,l}^n\right]_0$$

Now $P = x + x^{-1} + y + y^{-1}$.

$$u_{\mathbb{Z}\times\mathbb{Z}}(P,\lambda) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 \lambda^{2n} = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 16\lambda^2\right)$$

$$u_{\mathbb{Z}\times\mathbb{Z}/2\mathbb{Z}}(P,\lambda) = \sum_{n=0}^{\infty} \binom{4n}{2n} \lambda^{2n}$$

$$u_{\mathbb{Z}*\mathbb{Z}}(P,\lambda) = \frac{3}{1 + 2\sqrt{1 - 12\lambda^2}}$$

Recurrence relations $x + x^{-1} + y + y^{-1}$

Coefficients satisfy recurrence relations

$$\mathbb{Z} \times \mathbb{Z} : \quad n^2 a_{2n} - 4(2n-1)^2 a_{2n-2} = 0$$

$$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} : \quad n(2n-1) a_{2n} - 2(4n-1)(4n-3) a_{2n-2} = 0$$

$$\mathbb{Z} * \mathbb{Z} : \quad n a_{2n} - 2(14n-9) a_{2n-2} + 96(2n-3) a_{2n-4} = 0$$

- $\mathbb{Z}^l$
  Rodriguez - Villegas: $u(\lambda)$ periods of a differential in the curve defined by $1 = \lambda P(x,y)$. By Griffiths (1969)

  $$A_k(\lambda)u^{(k)} + A_{k-1}(\lambda)u^{(k-1)} + \cdots + A_0(\lambda)u = 0,$$

  Picard-Fuchs differential equation (A_j polynomials).
  $\Rightarrow$ Recurrence of the coefficients.
  Wilf and Zeilberger: a_n multisums, generating series is hypergeometric.
- This recurrence result extends to the case of Γ finitely generated abelian group.

- Finite groups :

$$a_n = \frac{\mathrm{tr}(A^n)}{|\Gamma|}$$

minimal polynomial of A.

- $\mathbb{F}_l$
  By Haiman (1993): $u(\lambda)$ is algebraic.
  Algebraic functions in non-commuting variables.

$P = x + x^{-1} + y + y^{-1}$

$$\Gamma = \left\langle x, y \,|\, x^2y = yx^2, y^2x = xy^2 \right\rangle$$

Domb (1960)

$$a_{2n} = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k}$$

Same as ordinary Mahler measure for

$$1 - \lambda \left(x + x^{-1} + z\left(y + y^{-1}\right)\right)\left(x + x^{-1} + z^{-1}\left(y + y^{-1}\right)\right)$$

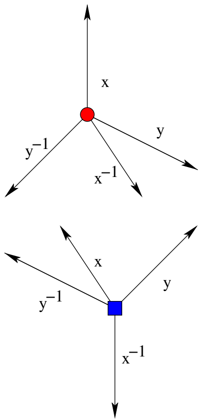
$$n^3 a_{2n} - 2(2n-1)(5n^2-5n+2)a_{2n-2} + 6(n-1)^3 a_{2n-4} = 0$$

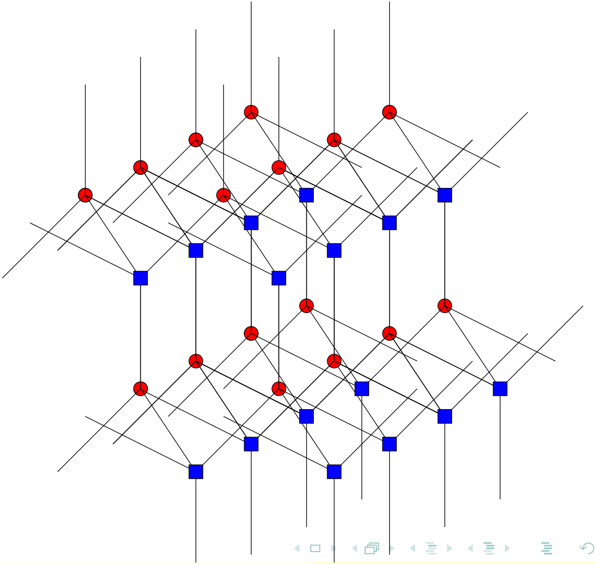
Rogers (2007)

$$1-\lambda\left(4+\left(x+x^{-1}\right)\left(y+y^{-1}\right)+\left(y+y^{-1}\right)\left(z+z^{-1}\right)+\left(z+z^{-1}\right)\left(x+x^{-1}\right)\right)$$

$${}_3F_2\left(\frac{1}{3},\frac{1}{2},\frac{2}{3};1,1;-\frac{108\lambda}{(1-16\lambda)^3}\right) = (1-16\lambda)\sum_{n=0}^{\infty} a_{2n}\lambda^n$$

The diamond lattice



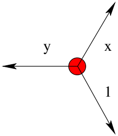


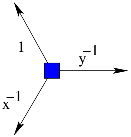
$$Q = (1 + x + y)\left(1 + x^{-1} + y^{-1}\right)$$

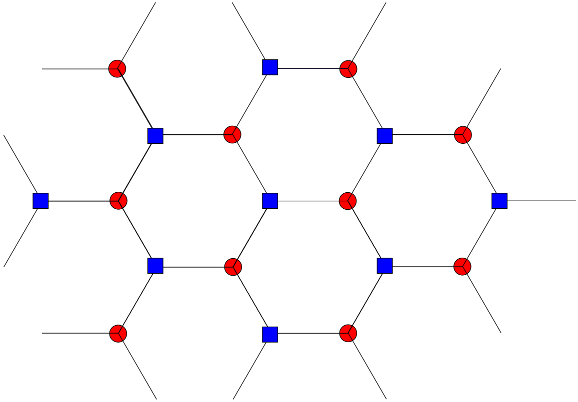
$$[Q^n]_0 = a_n$$

$$n^2 a_n - (10n^2 - 10n + 3)a_{n-1} + 9(n-1)^2 a_{n-2} = 0,$$

Honeycomb lattice $(1+x+y)\left(1+x^{-1}+y^{-1}\right)$







$$P = x + x^{-1} + y + y^{-1} + xy^{-1} + x^{-1}y$$

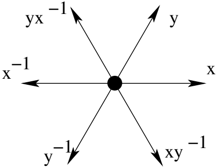
$$[P^n]_0 = b_n$$

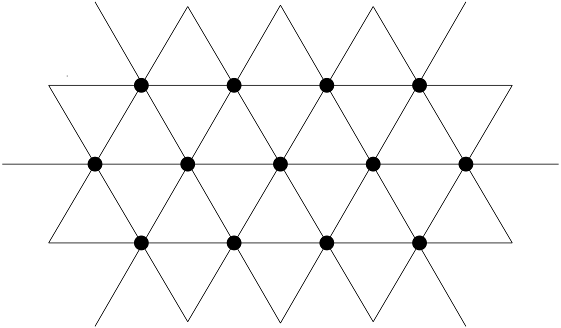
$$n^2 b_n - n(n-1)b_{n-1} - 24(n-1)^2 b_{n-2} - 36(n-2)(n-1)b_{n-3} = 0.$$

$$Q = 3 + P$$

$$b_n = \sum_{j=0}^{n} \binom{n}{j} (-3)^{n-j} a_j$$

Triangular lattice $x + x^{-1} + y + y^{-1} + xy^{-1} + x^{-1}y$





Further study: Tree entropy and Volume Conjecture

$m\left(P, \frac{1}{l^1(P)}\right)$ related to $h(G)$

where G is the Cayley graph and h is the tree entropy

$$h(G) := \log \deg_G(o) - \sum_{n=1}^{\infty} \frac{p_n(o, G)}{n},$$

- o fixed vertex
- $p_n(o, G)$ is the probability that a simple random walk started at o on G is again at o after n steps.

Lyons (2005)
G_n are finite graphs that tend to a fixed transitive infinite graph G, then

$$h(G) = \lim_{n\to\infty} \frac{\log \tau(G_n)}{|V(G_n)|},$$

where $\tau(G)$ is the complexity, i.e., the number of spanning trees.
Compare to

Conjecture ((Volume Conjecture) Kashaev, H. Murakami, J. Murakami (1997))

Let K be a hyperbolic knot, and $J_n(K,q)$ its normalized colored Jones polynomial. Then

$$\frac{1}{2\pi}\mathrm{Vol}(S^3 \setminus K) = \lim_{n\to\infty} \frac{\log\left|J_n\left(K, \mathrm{e}^{\frac{2\pi i}{n}}\right)\right|}{n}$$