Higher Mahler measures

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Pierce (1918) $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_{i}(x - \alpha_i)$$

$$\Delta_n = \prod_{i}(\alpha_i^n - 1)$$

$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$
Lehmer (1933)

\[ \frac{\Delta_{n+1}}{\Delta_n} \]

\[
\lim_{n \to \infty} \frac{|\alpha^{n+1} - 1|}{|\alpha^n - 1|} = \begin{cases} 
|\alpha| & \text{if } |\alpha| > 1 \\
1 & \text{if } |\alpha| < 1
\end{cases}
\]

For

\[ P(x) = a \prod_i (x - \alpha_i) \]

\[ M(P) = |a| \prod_i \max\{1, |\alpha_i|\} \]

\[ m(P) = \log M(P) = \log |a| + \sum_i \log^+ |\alpha_i| \]
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\]
Kronecker’s Lemma

\[ P \in \mathbb{Z}[x], \ P \neq 0, \]

\[ m(P) = 0 \iff P(x) = x^n \prod \phi_i(x) \]
Lehmer's Question

\[ m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) = 0.162357612 \ldots \]

Lehmer(1933) Does there exist \( C > 0 \) such that \( P(x) \in \mathbb{Z}[x] \)

\[ m(P) = 0 \quad \text{or} \quad m(P) > C ?? \]

\[ \sqrt{\Delta_{379}} = 1, 794, 327, 140, 357 \]
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Mahler measure of multivariable polynomials

\[ P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}], \text{ the (logarithmic) Mahler measure is:} \]

\[ m(P) = \int_0^1 \cdots \int_0^1 \log |P(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n})| \, d\theta_1 \cdots d\theta_n \]

Jensen’s formula:

\[ \int_0^1 \log |e^{2\pi i \theta} - \alpha| \, d\theta = \log^+ |\alpha| \]

recovers one-variable case.
$P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the (logarithmic) **Mahler measure** is:

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**Jensen’s formula:**

$$\int_0^1 \log |e^{2\pi i \theta} - \alpha| \, d\theta = \log^+ |\alpha|$$

recovers one-variable case.
\begin{itemize}
  \item $m(P) \geq 0$ if $P$ has integral coefficients.
  \item $m(P \cdot Q) = m(P) + m(Q)$
  \item $\alpha$ algebraic number, and $P_{\alpha}$ minimal polynomial over $\mathbb{Q}$,
    
    $$m(P_{\alpha}) = [\mathbb{Q}(\alpha) : \mathbb{Q}] \ h(\alpha)$$

  where $h$ is the logarithmic Weil height.
\end{itemize}
$P \in \mathbb{C}[x_1, \ldots, x_n]$

$$\lim_{k_2 \to \infty} \ldots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \ldots, x^{k_n})) = m(P(x_1, x_2, \ldots, x_n))$$
Jensen’s formula $\rightarrow$ simple expression in one-variable case.

Several-variable case?
Examples in several variables

Smyth (1981)

\[
m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)
\]

\[
m(1 + x + y + z) = \frac{7}{2\pi^2} \zeta(3)
\]
More examples in several variables

  \[ \pi^2 m \left( z - \left( \frac{1 - x}{1 + x} \right) (1 + y) \right) = \frac{28}{5} \zeta(3) \]

  \[ \pi^2 m \left( \text{Res}_t(x + yt + t^2, z + wt + t^2) \right) \]
  \[ = \pi^2 m \left( z(1 - xy)^2 - (1 - x)(1 - y) \right) = 25\sqrt{5}L(\chi_5, 3) \]

  \[ \pi^2 m(x^2 + 1 + (x + 1)y + (x - 1)z) = \pi L(\chi_{-4}, 2) + \frac{21}{8} \zeta(3) \]
L. (2006)

\[ π^3 m \left(1 + x + \left(\frac{1 - x_1}{1 + x_1}\right)(1 + y)z\right) = 24L(\chi_{-4}, 4) \]

\[ π^4 m \left(1 + x + \left(\frac{1 - x_1}{1 + x_1}\right)\left(\frac{1 - x_2}{1 + x_2}\right)(1 + y)z\right) = 93ζ(5) \]

Known formulas for

\[ π^{n+2} m \left(1 + x + \left(\frac{1 - x_1}{1 + x_1}\right)\ldots\left(\frac{1 - x_n}{1 + x_n}\right)(1 + y)z\right) \]
The relationship with regulators

Deninger (1997)

\[ m(P) = \text{easy term} + \frac{1}{(2i\pi)^{n-1}} \int_{\Gamma} \eta_n(x_1, \ldots, x_n) \]

where

\[ \Gamma = \{ P(x_1, \ldots, x_n) = 0 \} \cap \{|x_1| = \cdots = |x_{n-1}| = 1, |x_n| \geq 1\} \]

\( \eta_n(x_1, \ldots, x_n) \) is a \( \mathbb{R}(n-1) \)-valued smooth \( n-1 \)-form in \( \{ P = 0 \} \).
Regulators

Encode special values of $L$-functions.

Example: Dirichlet class number formula

$$\lim_{s \to 1} (s - 1)\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2}h_F\text{reg}_F}{\omega_F \sqrt{|D_F|}},$$

$$\lim_{s \to 0} s^{1-r_1-r_2}\zeta_F(s) = -\frac{h_F\text{reg}_F}{\omega_F}.$$

- Explains all the known cases involving $\zeta$, $L(\chi)$ using $\text{Li}_k(x)$ and Borel’s Theorem in $K$-theory.

$$\text{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}$$

- It is constructive (no need of “happy idea” integrals).

- Conjecture for $n$-variables using Goncharov’s regulator currents. Provides motivation for Goncharov’s construction.

- Key use of Jensen’s formula

$$m(x - \alpha) = \log^+ |\alpha|$$
The $k$-higher Mahler measure of $P$ is defined by

$$m_k(P) = \int_0^1 \cdots \int_0^1 \log^k \left| P\left( e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n}\right) \right| \, d\theta_1 \cdots d\theta_n.$$ 

$k = 1 : \quad m_1(P) = m(P),$ 

and

$$m_0(P) = 1.$$
The simplest example

\[ m_2(1 - x) = \frac{\zeta(2)}{2} = \frac{\pi^2}{12}. \]
\[ m_3(1 - x) = -\frac{3\zeta(3)}{2}. \]
\[ m_4(1 - x) = \frac{3\zeta(2)^2 + 21\zeta(4)}{4} = \frac{19\pi^4}{240}. \]
\[ m_5(1 - x) = -\frac{15\zeta(2)\zeta(3) + 45\zeta(5)}{2}. \]
\[ m_6(1 - x) = \frac{45}{2} \zeta(3)^2 + \frac{275}{1344} \pi^6. \]
An example in two variables

**Theorem**

\[ m_2(1 + x + y) = \frac{7\pi^2}{54} = \frac{7}{9}\zeta(2) \]

Smyth (1981)

\[ m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) \]
Zeta Mahler measure

\[ Z(s, P) = \int_0^1 \ldots \int_0^1 |P(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n})|^s \, d\theta_1 \ldots d\theta_n. \]

\[ Z(s, P) = \sum_{k=0}^{\infty} \frac{m_k(P)s^k}{k!}. \]
An example

**Theorem**

\[
Z(s, x - 1) = \exp \left( \sum_{k=2}^{\infty} \frac{(-1)^k (1 - 2^{1-k}) \zeta(k)}{k} s^k \right)
\]

Akatsuka (2007): \( Z(s, x - c) \)
\[ Z(s, x - 1) = \int_0^1 |1 - e^{2\pi i \theta}|^s \, d\theta = \int_0^1 (2 \sin \pi \theta)^s \, d\theta \]

\[ = 2^{s+1} \int_0^{1/2} (\sin \pi \theta)^s \, d\theta. \]

For \( t = \sin^2 \pi \theta \):

\[ = \frac{2^s}{\pi} \int_0^1 t^{s-1/2} (1 - t)^{-1/2} \, dt. \]

\[ = \frac{2^s}{\pi} B \left( \frac{s+1}{2}, \frac{1}{2} \right) \]

\[ = \frac{2^s}{\pi} \frac{\Gamma \left( \frac{s+1}{2} \right) \Gamma \left( \frac{1}{2} \right)}{\Gamma \left( \frac{s}{2} + 1 \right)}. \]
Some properties of the Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1}e^{-t}dt$$

$$\Gamma(s + 1) = s\Gamma(s) \quad \Gamma(n + 1) = n!$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \quad \Gamma\left(\frac{s + 1}{2}\right) \Gamma\left(\frac{s}{2}\right) = \Gamma(s)2^{1-s}\sqrt{\pi}$$

$$Z(s, x - 1) = \frac{2^s}{\pi} \frac{\Gamma\left(\frac{s+1}{2}\right) \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2} + 1\right)} = \frac{\Gamma(s + 1)}{\Gamma\left(\frac{s}{2} + 1\right)^2}$$
Weierstrass product:

\[ \Gamma(s + 1)^{-1} = e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-\frac{s}{n}} \]

yields

\[ Z(s, x - 1) = \frac{\Gamma(s + 1)}{\Gamma\left(\frac{s}{2} + 1\right)^2} = \prod_{n=1}^{\infty} \left(1 + \frac{s}{2n}\right)^2 \frac{1}{1 + \frac{s}{n}} \]

\[ = \exp \left( \sum_{n=1}^{\infty} \left(2 \log \left(1 + \frac{s}{2n}\right) - \log \left(1 + \frac{s}{n}\right)\right) \right) \]

\[ = \exp \left( \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=1}^{\infty} \left(2 \left(\frac{1}{2n}\right)^k - \frac{1}{n^k}\right) s^k \right) \]

\[ = \exp \left( \sum_{k=2}^{\infty} \frac{(-1)^k(1 - 2^{1-k})\zeta(k)}{k} s^k \right). \]
\[ Z(s, x - 1) = \exp \left( \sum_{k=2}^{\infty} \frac{(-1)^k (1 - 2^{1-k}) \zeta(k)}{k} s^k \right) \]

\[ m_2(x - 1) = \frac{\zeta(2)}{2} = \frac{\pi^2}{12}. \]

\[ m_3(x - 1) = -\frac{3\zeta(3)}{2}. \]

\[ m_4(x - 1) = \frac{3\zeta(2)^2 + 21\zeta(4)}{4} = \frac{19\pi^4}{240}. \]

\[ m_5(x - 1) = -\frac{15\zeta(2)\zeta(3) + 45\zeta(5)}{2}. \]

\[ \ldots \]
Let $P_1, \ldots, P_k \in \mathbb{C}[x_1^\pm, \ldots, x_n^\pm]$ be non-zero Laurent polynomials. Their multiple higher Mahler measure is defined by

$$m(P_1, \ldots, P_k) = \int_0^1 \cdots \int_0^1 \log \left| P_1(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n}) \right| \cdots \log \left| P_k(e^{2\pi i \theta_1}, \ldots, e^{2\pi i \theta_n}) \right| d\theta_1 \cdots d\theta_n$$

$$m(P_1) \cdots m(P_k) = m(P_1, \ldots, P_k)$$

when the variables of $P_j$'s are algebraically independent.
**Theorem**

For $0 \leq \alpha \leq 1$

$$m(1 - x, 1 - e^{2\pi i \alpha} x) = \frac{\pi^2}{2} \left( \alpha^2 - \alpha + \frac{1}{6} \right).$$

**Examples**

- $m(1 - x, 1 + x) = -\frac{\pi^2}{24}$,
- $m(1 - x, 1 \pm ix) = -\frac{\pi^2}{96}$,
- $m(1 - x, 1 - e^{2\pi i \alpha} x) = 0 \iff \alpha = \frac{3 \pm \sqrt{3}}{6}$. 

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Jensen’s formula for multiple Mahler measure

\[ m(1-\alpha x, 1-\beta x) = \begin{cases} 
\frac{1}{2} \text{Re Li}_2 (\alpha \bar{\beta}) & \text{if } |\alpha|, |\beta| \leq 1, \\
\frac{1}{2} \text{Re Li}_2 \left( \frac{\alpha \beta}{|\alpha|^2} \right) & \text{if } |\alpha| \geq 1, |\beta| \leq 1, \\
\frac{1}{2} \text{Re Li}_2 \left( \frac{\alpha \bar{\beta}}{|\alpha \beta|^2} \right) + \log |\alpha| \log |\beta| & \text{if } |\alpha|, |\beta| \geq 1.
\end{cases} \]

Crucial for

\[ m_2(1 + x + y) \]
Theorem

\[
Z(s, t; x - 1, x + 1) = \exp \left( \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \zeta(k) \left( (1 - 2^{-k})(s^k + t^k) - 2^{-k}(s + t)^k \right) \right)
\]

\[
m(x - 1, \ldots, x - 1, x + 1, \ldots, x + 1)_{k \, l}
\]

belongs to \( \mathbb{Q}[\pi^2, \zeta(3), \zeta(5), \zeta(7), \ldots] \) for integers \( k, l \geq 0 \).
\begin{align*}
m(x - 1, x + 1) &= -\frac{\zeta(2)}{4} = -\frac{\pi^2}{24}, \\
m(x - 1, x - 1, x + 1) &= 2 \frac{\zeta(3)}{8} = \frac{\zeta(3)}{4}, \\
m(x - 1, x + 1, x + 1) &= 2 \frac{\zeta(3)}{8} = \frac{\zeta(3)}{4}.
\end{align*}

\[m_3(x - 1) = -\frac{3\zeta(3)}{2}.\]
Properties of zeta Mahler measures

- $\lambda > 0$,

\[ Z(s, \lambda P) = \lambda^s Z(s, P) \]

- $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, $P = P^*$, $|\lambda| \leq 1/||P||_{\infty}$,

\[ Z(s, 1 + \lambda P) = \sum_{k=0}^{\infty} \binom{s}{k} Z(k, P) \lambda^k, \]

\[ m(1 + \lambda P) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} Z(k, P) \lambda^k. \]

More generally,

\[ m_j(1 + \lambda P) = j! \sum_{0 < k_1 < \ldots < k_j} \frac{(-1)^{k_j-j}}{k_1 \ldots k_j} Z(k_j, P) \lambda^{k_j}. \]
The case $P = x + y + c$

Let $c \geq 2$.

\[
Z(s, x + y + c) = c^s \sum_{j=0}^{\infty} \binom{s/2}{j} \frac{1}{c^{2j}} \binom{2j}{j}.
\]

\[
= c^s \, _3F_2\left(\frac{-s}{2}, \frac{-s}{2}, \frac{1}{2} \bigg| \frac{4}{c^2}\right),
\]

where the generalized hypergeometric series $\, _3F_2$ is defined by

\[
\, _3F_2\left(\begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2 \end{array} \bigg| z \right) = \sum_{j=0}^{\infty} \frac{(a_1)_j(a_2)_j(a_3)_j}{(b_1)_j(b_2)_j j!} z^j,
\]

with the Pochhammer symbol defined by $(a)_j = a(a + 1) \cdots (a + j - 1)$. 
\[ Z(s, x + y + c) = \frac{Z(s, (x + y + c)(x^{-1} + y^{-1} + c))}{c^s Z\left(\frac{s}{2}, \left(1 + \frac{1}{c}(x + y)\right)\left(1 + \frac{1}{c}(x^{-1} + y^{-1})\right)\right)} \]

\[ = \frac{c^s}{(2\pi i)^2} \int_{|y|=1} \int_{|x|=1} \left(1 + \frac{x + y}{c}\right)^{s/2} \left(1 + \frac{x^{-1} + y^{-1}}{c}\right)^{s/2} \frac{dx}{x} \frac{dy}{y} \]

\[ = c^s \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \binom{s/2}{j} \binom{s/2}{k} \frac{1}{(2\pi i)^2} \int_{|y|=1} \int_{|x|=1} \left(\frac{x + y}{c}\right)^j \left(\frac{x^{-1} + y^{-1}}{c}\right)^k \]

\[ = c^s \sum_{j=0}^{\infty} \binom{s/2}{j}^2 \frac{1}{c^{2j}} \binom{2j}{j}. \]
In particular, we obtain the special values

\[ m_2(x + y + 2) = \frac{\zeta(2)}{2}, \]

\[ m_3(x + y + 2) = \frac{9}{2}(\log 2)\zeta(2) - \frac{15}{4}\zeta(3). \]

Proof uses

\[ \sum_{k=1}^{\infty} \binom{2k}{k} \frac{t^k}{k^2} = 2\text{Li}_2 \left( \frac{1 - \sqrt{1 - 4t}}{2} \right) - \log^2 \left( \frac{1 + \sqrt{1 - 4t}}{2} \right). \]
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