Higher Mahler measures

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Pierce (1918) $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - \alpha_i)$$

$$\Delta_n = \prod_i (\alpha_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Lehmer (1933)

$$\frac{\Delta_{n+1}}{\Delta_n}$$

$$\lim_{n\to\infty} \frac{|\alpha^{n+1}-1|}{|\alpha^n-1|} = \begin{cases} |\alpha| & \text{if } |\alpha|>1 \\ 1 & \text{if } |\alpha|<1 \end{cases}$$

For

$$P(x) = a\prod_i (x-\alpha_i)$$

$$M(P) = |a|\prod_i \max\{1,|\alpha_i|\}$$

$$m(P) = \log M(P) = \log|a| + \sum_i \log^+|\alpha_i|$$

Lehmer (1933)

$$\frac{\Delta_{n+1}}{\Delta_n}$$

$$\lim_{n\to\infty} \frac{|\alpha^{n+1}-1|}{|\alpha^n - 1|} = \begin{cases} |\alpha| & \text{if } |\alpha| > 1 \\ 1 & \text{if } |\alpha| < 1 \end{cases}$$

For

$$P(x) = a \prod_i (x - \alpha_i)$$

$$M(P) = |a| \prod_i \max\{1, |\alpha_i|\}$$

$$m(P) = \log M(P) = \log |a| + \sum_i \log^+ |\alpha_i|$$

Kronecker's Lemma

$P \in \mathbb{Z}[x]$, $P \neq 0$,

$$m(P) = 0 \Leftrightarrow P(x) = x^n \prod \phi_i(x)$$

Lehmer's Question

$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1)$$

$$= 0.162357612\ldots$$

Lehmer(1933) Does there exist $C > 0$ such that $P(x) \in \mathbb{Z}[x]$

$$m(P) = 0 \quad \text{or} \quad m(P) > C??$$

$$\sqrt{\Delta_{379}} = 1,794,327,140,357$$

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Mahler measure of multivariable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$m(P) = \int_0^1 \dots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n$$

Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log^+ |\alpha|$$

recovers one-variable case.

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Properties

- $m(P) \geq 0$ if P has integral coefficients.
- $m(P \cdot Q) = m(P) + m(Q)$
- α algebraic number, and P_α minimal polynomial over $\mathbb{Q}$,

$$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]\, h(\alpha)$$

where h is the logarithmic Weil height.

Boyd & Lawton Theorem

$P \in \mathbb{C}[x_1, \dots, x_n]$

$$\lim_{k_2 \to \infty} \dots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \dots, x^{k_n})) = m(P(x_1, x_2, \dots, x_n))$$

Jensen's formula $\longrightarrow$ simple expression in one-variable case.

Several-variable case?

Smyth (1981)

- $$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2) = \mathrm{L}'(\chi_{-3},-1)$$

- $$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

More examples in several variables

- Condon (2003)

$$\pi^2 m\left(z-\left(\frac{1-x}{1+x}\right)(1+y)\right)=\frac{28}{5}\zeta(3)$$

- D'Andrea & L. (2007)

$$\pi^2 m\left(\mathrm{Res}_t(x+yt+t^2, z+wt+t^2)\right)$$

$$=\pi^2 m\left(z(1-xy)^2-(1-x)(1-y)\right)=25\sqrt{5}\mathrm{L}(\chi_5,3)$$

- Boyd & L. (2005)

$$\pi^2 m(x^2+1+(x+1)y+(x-1)z)=\pi\mathrm{L}(\chi_{-4},2)+\frac{21}{8}\zeta(3)$$

- L. (2006)

$$\pi^3 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right) = 24\mathrm{L}(\chi_{-4},4)$$

- $$\pi^4 m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)=93\zeta(5)$$

- Known formulas for

$$\pi^{n+2} m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_n}{1+x_n}\right)(1+y)z\right)$$

The relationship with regulators

Deninger (1997)

$$m(P) = \text{easy term} + \frac{1}{(2i\pi)^{n-1}} \int_\Gamma \eta_n(x_1, \dots, x_n)$$

where

$$\Gamma = \{P(x_1, \dots, x_n) = 0\} \cap \{|x_1| = \cdots = |x_{n-1}| = 1, |x_n| \geq 1\}$$

$\eta_n(x_1, \dots, x_n)$ is a $\mathbb{R}(n-1)$-valued smooth $n-1$-form in $\{P = 0\}$.

Encode special values of L-functions.

Example: Dirichlet class number formula

$$\lim_{s\to 1}(s-1)\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2} h_F \operatorname{reg}_F}{\omega_F\sqrt{|D_F|}},$$

$$\lim_{s\to 0} s^{1-r_1-r_2}\zeta_F(s) = -\frac{h_F \operatorname{reg}_F}{\omega_F}.$$

An algebraic integration for Mahler measure

Rodriguez-Villegas (1997), L. (2007)

- Explains all the known cases involving ζ, $\mathrm{L}(\chi)$ using $\mathrm{Li}_k(x)$ and Borel's Theorem in K-theory.

$$\mathrm{Li}_k(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^k}$$

- It is constructive (no need of "happy idea" integrals).
- Conjecture for n-variables using Goncharov's regulator currents. Provides motivation for Goncharov's construction.
- Key use of Jensen's formula

$$m(x-\alpha) = \log^+ |\alpha|$$

The k-higher Mahler measure of P is defined by

$$m_k(P) = \int_0^1 \dots \int_0^1 \log^k \left| P\left(\mathrm{e}^{2\pi\mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi\mathrm{i}\theta_n}\right)\right| \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n.$$

$$k = 1: \qquad m_1(P) = m(P),$$

and

$$m_0(P) = 1.$$

The simplest example

$$
\begin{aligned}
m_2(1-x) &= \frac{\zeta(2)}{2} = \frac{\pi^2}{12}. \\
m_3(1-x) &= -\frac{3\zeta(3)}{2}. \\
m_4(1-x) &= \frac{3\zeta(2)^2 + 21\zeta(4)}{4} = \frac{19\pi^4}{240}. \\
m_5(1-x) &= -\frac{15\zeta(2)\zeta(3) + 45\zeta(5)}{2}. \\
m_6(1-x) &= \frac{45}{2}\zeta(3)^2 + \frac{275}{1344}\pi^6.
\end{aligned}
$$

An example in two variables

Theorem

$$m_2(1+x+y) = \frac{7\pi^2}{54} = \frac{7}{9}\zeta(2)$$

Smyth (1981)

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2)$$

Zeta Mahler measure

$$Z(s,P) = \int_0^1 \dots \int_0^1 \left| P\left(\mathrm{e}^{2\pi\mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi\mathrm{i}\theta_n}\right) \right|^s \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n.$$

$$Z(s,P) = \sum_{k=0}^{\infty} \frac{m_k(P) s^k}{k!}.$$

An example

Theorem

$$Z(s, x-1) = \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^k(1-2^{1-k})\zeta(k)}{k} s^k\right)$$

Akatsuka (2007): $Z(s, x-c)$

$$
\begin{aligned}
Z(s,x-1) &= \int_0^1 \left|1-\mathrm{e}^{2\pi \mathrm{i}\theta}\right|^s \mathrm{d}\theta = \int_0^1 (2\sin\pi\theta)^s \mathrm{d}\theta \\
&= 2^{s+1}\int_0^{1/2} (\sin\pi\theta)^s \mathrm{d}\theta.
\end{aligned}
$$

$t = \sin^2 \pi\theta$:

$$
= \frac{2^s}{\pi}\int_0^1 t^{\frac{s-1}{2}}(1-t)^{-1/2}\mathrm{d}t.
$$

$$
\begin{aligned}
&= \frac{2^s}{\pi} B\left(\frac{s+1}{2}, \frac{1}{2}\right) \\
&= \frac{2^s}{\pi}\frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)}.
\end{aligned}
$$

Some properties of the Gamma function

$$\Gamma(s) = \int_0^\infty t^{s-1} \mathrm{e}^{-t} \mathrm{d}t$$

$$\Gamma(s+1) = s\Gamma(s) \quad \Gamma(n+1) = n!$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \qquad \Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{s}{2}\right) = \Gamma(s)2^{1-s}\sqrt{\pi}$$

$$Z(s, x-1) = \frac{2^s}{\pi} \frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} = \frac{\Gamma(s+1)}{\Gamma\left(\frac{s}{2}+1\right)^2}$$

Weierstrass product:

$$\Gamma(s+1)^{-1} = \mathrm{e}^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) \mathrm{e}^{-\frac{s}{n}}$$

yields

$$\begin{aligned}
Z(s, x-1) &= \frac{\Gamma(s+1)}{\Gamma\left(\frac{s}{2}+1\right)^2} = \prod_{n=1}^{\infty} \frac{\left(1+\frac{s}{2n}\right)^2}{1+\frac{s}{n}} \\
&= \exp\left(\sum_{n=1}^{\infty} \left(2\log\left(1+\frac{s}{2n}\right) - \log\left(1+\frac{s}{n}\right)\right)\right) \\
&= \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=1}^{\infty} \left(2\left(\frac{1}{2n}\right)^k - \frac{1}{n^k}\right) s^k\right) \\
&= \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^k(1-2^{1-k})\zeta(k)}{k} s^k\right).
\end{aligned}$$

$$Z(s,x-1)=\exp\left(\sum_{k=2}^{\infty}\frac{(-1)^k(1-2^{1-k})\zeta(k)}{k}s^k\right)$$

$$\begin{aligned}
m_2(x-1) &= \frac{\zeta(2)}{2}=\frac{\pi^2}{12}.\\
m_3(x-1) &= -\frac{3\zeta(3)}{2}.\\
m_4(x-1) &= \frac{3\zeta(2)^2+21\zeta(4)}{4}=\frac{19\pi^4}{240}.\\
m_5(x-1) &= -\frac{15\zeta(2)\zeta(3)+45\zeta(5)}{2}.
\end{aligned}$$

$\dots$

Multiple Mahler measure

Let $P_1, \dots, P_k \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ be non-zero Laurent polynomials. Their multiple higher Mahler measure is defined by

$$
\begin{aligned}
m(P_1, \dots, P_k) &= \int_0^1 \cdots \int_0^1 \log \left| P_1(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \cdots, \mathrm{e}^{2\pi \mathrm{i}\theta_n}) \right| \\
&\quad \cdots \log \left| P_k(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \cdots, \mathrm{e}^{2\pi \mathrm{i}\theta_n}) \right| \mathrm{d}\theta_1 \cdots \mathrm{d}\theta_n
\end{aligned}
$$

$$
m(P_1) \cdots m(P_k) = m(P_1, \dots, P_k)
$$

when the variables of P_j's are algebraically independent.

Higher Mahler measure for several linear polynomials

Theorem

For $0 \leq \alpha \leq 1$

$$m(1-x, 1-e^{2\pi i\alpha}x) = \frac{\pi^2}{2}\left(\alpha^2 - \alpha + \frac{1}{6}\right).$$

Examples

$$\begin{aligned} m(1-x, 1+x) &= -\frac{\pi^2}{24}, \\ m(1-x, 1 \pm ix) &= -\frac{\pi^2}{96}, \\ m(1-x, 1-e^{2\pi i\alpha}x) &= 0 \Leftrightarrow \alpha = \frac{3 \pm \sqrt{3}}{6}. \end{aligned}$$

Jensen's formula for multiple Mahler measure

$$m(1-\alpha x, 1-\beta x) = \begin{cases} \frac{1}{2}\,\mathrm{Re}\,\mathrm{Li}_2\left(\alpha\bar{\beta}\right) & \text{if } |\alpha|, |\beta| \leq 1, \\ \frac{1}{2}\,\mathrm{Re}\,\mathrm{Li}_2\left(\frac{\alpha\beta}{|\alpha|^2}\right) & \text{if } |\alpha| \geq 1, |\beta| \leq 1, \\ \frac{1}{2}\,\mathrm{Re}\,\mathrm{Li}_2\left(\frac{\alpha\bar{\beta}}{|\alpha\beta|^2}\right) + \log|\alpha|\log|\beta| & \text{if } |\alpha|, |\beta| \geq 1. \end{cases}$$

Crucial for

$$m_2(1+x+y)$$

Multiple zeta Mahler measure

Theorem

- $$Z(s,t;x-1,x+1)$$
$$=\exp\left(\sum_{k=2}^{\infty}\frac{(-1)^k}{k}\zeta(k)\left((1-2^{-k})(s^k+t^k)-2^{-k}(s+t)^k\right)\right)$$

- $$m(\underbrace{x-1,\dots,x-1}_{k},\underbrace{x+1,\dots,x+1}_{l})$$

belongs to $\mathbb{Q}[\pi^2,\zeta(3),\zeta(5),\zeta(7),\dots]$ *for integers* $k,l\geq 0$.

$$
\begin{aligned}
m(x-1,x+1) &= -\frac{\zeta(2)}{4} = -\frac{\pi^2}{24},\\
m(x-1,x-1,x+1) &= 2\frac{\zeta(3)}{8} = \frac{\zeta(3)}{4},\\
m(x-1,x+1,x+1) &= 2\frac{\zeta(3)}{8} = \frac{\zeta(3)}{4}.\\
m_3(x-1) &= -\frac{3\zeta(3)}{2}.
\end{aligned}
$$

- $\lambda > 0$,

$$Z(s, \lambda P) = \lambda^s Z(s, P)$$

- $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \quad P = P^*, \quad |\lambda| \leq 1/||P||_\infty$,

$$\begin{aligned} Z(s, 1+\lambda P) &= \sum_{k=0}^{\infty} \binom{s}{k} Z(k, P) \lambda^k, \\ m(1+\lambda P) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} Z(k, P) \lambda^k. \end{aligned}$$

More generally,

$$m_j(1+\lambda P) = j! \sum_{0<k_1<\cdots<k_j} \frac{(-1)^{k_j - j}}{k_1 \dots k_j} Z(k_j, P) \lambda^{k_j}.$$

Let $c \geq 2$.

$$\begin{aligned} Z(s, x+y+c) &= c^s \sum_{j=0}^{\infty} \binom{s/2}{j}^2 \frac{1}{c^{2j}} \binom{2j}{j}. \\ &= c^s {}_3F_2\left(\begin{array}{c} -\frac{s}{2}, -\frac{s}{2}, \frac{1}{2} \\ 1, 1 \end{array} \middle| \frac{4}{c^2} \right), \end{aligned}$$

where the generalized hypergeometric series ${}_3F_2$ is defined by

$${}_3F_2\left(\begin{array}{c} a_1, a_2, a_3 \\ b_1, b_2 \end{array} \middle| z \right) = \sum_{j=0}^{\infty} \frac{(a_1)_j (a_2)_j (a_3)_j}{(b_1)_j (b_2)_j j!} z^j,$$

with the Pochhammer symbol defined by $(a)_j = a(a+1)\cdots(a+j-1)$.

$$
\begin{aligned}
Z(s, x+y+c) &= Z\left(\frac{s}{2}, (x+y+c)\left(x^{-1}+y^{-1}+c\right)\right) \\
&= c^s Z\left(\frac{s}{2}, \left(1+\frac{1}{c}(x+y)\right)\left(1+\frac{1}{c}\left(x^{-1}+y^{-1}\right)\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{c^s}{(2\pi\mathrm{i})^2}\int_{|y|=1}\int_{|x|=1}\left(1+\frac{x+y}{c}\right)^{s/2}\left(1+\frac{x^{-1}+y^{-1}}{c}\right)^{s/2}\frac{\mathrm{d}x}{x}\frac{\mathrm{d}y}{y} \\
&= c^s\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\binom{s/2}{j}\binom{s/2}{k}\frac{1}{(2\pi\mathrm{i})^2}\int_{|y|=1}\int_{|x|=1}\left(\frac{x+y}{c}\right)^j\left(\frac{x^{-1}+y^{-1}}{c}\right)^k \\
&= c^s\sum_{j=0}^{\infty}\binom{s/2}{j}^2\frac{1}{c^{2j}}\binom{2j}{j}.
\end{aligned}
$$

In particular, we obtain the special values

- $$m_2(x+y+2) = \frac{\zeta(2)}{2},$$
- $$m_3(x+y+2) = \frac{9}{2}(\log 2)\zeta(2) - \frac{15}{4}\zeta(3).$$

Proof uses

$$\sum_{k=1}^{\infty} \binom{2k}{k} \frac{t^k}{k^2} = 2\mathrm{Li}_2\left(\frac{1-\sqrt{1-4t}}{2}\right) - \log^2\left(\frac{1+\sqrt{1-4t}}{2}\right).$$

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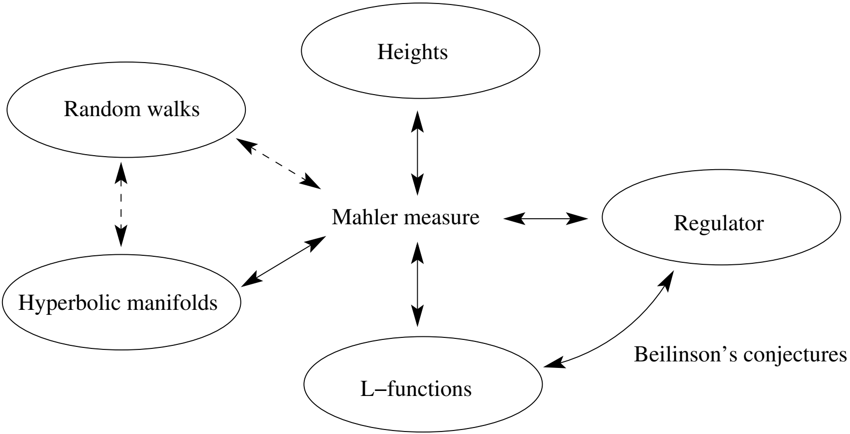
Heights
Random walks
Mahler measure
Regulator
Hyperbolic manifolds
L–functions
Beilinson's conjectures