Higher Mahler measures

Matilde Lalín
(joint with N. Kurokawa (Tokyo Institute of Technology),
H. Ochiai (Nagoya University))

University of Alberta
mlalin@math.ualberta.ca
http://www.math.ualberta.ca/~mlalin

Workshop on Discovery and Experimentation in Number Theory,
September 2009

Mahler measure of multivariable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$m(P) := \int_0^1 \dots \int_0^1 \log \left| P\left(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n}\right) \right| d\theta_1 \dots d\theta_n$$

By Jensen's formula,

$$m\left(a \prod (x - \alpha_i)\right) = \log |a| + \sum \log \max\{1, |\alpha_i|\}.$$

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Higher Mahler measure

For $k \in \mathbb{Z}_{\geq 0}$, the k-higher Mahler measure of P is

$$m_k(P) := \int_0^1 \cdots \int_0^1 \log^k \left| P\left(e^{2\pi i\theta_1}, \ldots, e^{2\pi i\theta_n}\right)\right| d\theta_1 \cdots d\theta_n.$$

$$k = 1: \qquad m_1(P) = m(P),$$

and

$$m_0(P) = 1.$$

The simplest example

$$
\begin{aligned}
m_2(1-x) &= \frac{\zeta(2)}{2} = \frac{\pi^2}{12}.\\
m_3(1-x) &= -\frac{3\zeta(3)}{2}.\\
m_4(1-x) &= \frac{3\zeta(2)^2 + 21\zeta(4)}{4} = \frac{19\pi^4}{240}.\\
m_5(1-x) &= -\frac{15\zeta(2)\zeta(3) + 45\zeta(5)}{2}.\\
m_6(1-x) &= \frac{45}{2}\zeta(3)^2 + \frac{275}{1344}\pi^6.
\end{aligned}
$$

Zeta Mahler measure

Akatsuka (2007):

$$Z(s,P) = \int_0^1 \dots \int_0^1 \left| P\left(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \dots, \mathrm{e}^{2\pi \mathrm{i}\theta_n}\right)\right|^s \mathrm{d}\theta_1 \dots \mathrm{d}\theta_n.$$

$$Z(s,P) = \sum_{k=0}^{\infty} \frac{m_k(P) s^k}{k!}.$$

Akatsuka (2009): $Z(s, x+k)$, $Z(s, x+x^{-1}+k)$

An example

Theorem

$$Z(s, x-1) \quad = \quad \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^k (1-2^{1-k})\zeta(k)}{k} s^k\right)$$

around $s = 0$.

$$
\begin{aligned}
Z(s, x-1) &= \int_0^1 \left|1 - \mathrm{e}^{2\pi \mathrm{i}\theta}\right|^s \mathrm{d}\theta = \int_0^1 (2\sin \pi\theta)^s \mathrm{d}\theta \\
&= 2^{s+1} \int_0^{1/2} (\sin \pi\theta)^s \mathrm{d}\theta.
\end{aligned}
$$

$t = \sin^2 \pi\theta$:

$$
= \frac{2^s}{\pi} \int_0^1 t^{\frac{s-1}{2}} (1-t)^{-1/2} \mathrm{d}t.
$$

$$
\begin{aligned}
&= \frac{2^s}{\pi} B\left(\frac{s+1}{2}, \frac{1}{2}\right) \\
&= \frac{2^s}{\pi} \frac{\Gamma\left(\frac{s+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{s}{2}+1\right)} = \frac{\Gamma(s+1)}{\Gamma\left(\frac{s}{2}+1\right)^2}
\end{aligned}
$$

Weierstrass product:

$$\Gamma(s+1)^{-1} = \mathrm{e}^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) \mathrm{e}^{-\frac{s}{n}}$$

yields

$$\begin{aligned} Z(s, x-1) &= \frac{\Gamma(s+1)}{\Gamma\left(\frac{s}{2}+1\right)^2} = \prod_{n=1}^{\infty} \frac{\left(1+\frac{s}{2n}\right)^2}{1+\frac{s}{n}} \\ &= \exp\left(\sum_{n=1}^{\infty} \left(2 \log\left(1 + \frac{s}{2n}\right) - \log\left(1 + \frac{s}{n}\right)\right)\right) \\ &= \exp\left(\sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \sum_{n=1}^{\infty} \left(2\left(\frac{1}{2n}\right)^k - \frac{1}{n^k}\right) s^k\right) \\ &= \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^k (1-2^{1-k})\zeta(k)}{k} s^k\right). \end{aligned}$$

$$Z(s, x-1) = \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^k(1-2^{1-k})\zeta(k)}{k} s^k\right)$$

$$\begin{aligned}
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\end{aligned}$$

$$\ldots$$

Examples in two variables

Theorem

$$\begin{aligned} m_2(1+x+y) &= \frac{\mathrm{i}}{\pi}(\mathrm{Li}_{2,1}(\omega,\bar\omega) - \mathrm{Li}_{2,1}(\bar\omega,\omega) \\ &\quad -\mathrm{Li}_{1,1,1}(1,\omega,\bar\omega) + \mathrm{Li}_{1,1,1}(1,\bar\omega,\omega)) + \frac{\pi^2}{9} \\ &= \frac{5\pi^2}{54} \end{aligned}$$

$$\mathrm{Li}_{2,1}(x,y) = \sum_{0<m<n} \frac{x^m y^n}{m^2 n}$$

Smyth (1981)

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2)$$

Theorem

$$m_2(1+x+y(1-x)) = \frac{4\mathrm{i}}{\pi}(\mathrm{Li}_{2,1}(-\mathrm{i},-\mathrm{i}) - \mathrm{Li}_{2,1}(\mathrm{i},\mathrm{i}))$$
$$+\frac{6\mathrm{i}}{\pi}(\mathrm{Li}_{2,1}(\mathrm{i},-\mathrm{i}) - \mathrm{Li}_{2,1}(-\mathrm{i},\mathrm{i}))$$
$$+\frac{\mathrm{i}}{\pi}(\mathrm{Li}_{2,1}(1,-\mathrm{i}) - \mathrm{Li}_{2,1}(1,\mathrm{i})) - \frac{7\zeta(2)}{16} + \frac{\log 2}{\pi}\mathrm{L}(\chi_{-4},2)$$

Smyth (1981)

$$m(1+x+y(1-x)) = \frac{2}{\pi}\mathrm{L}(\chi_{-4},2)$$

Multiple Mahler measure

Let $P_1, \dots, P_k \in \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$ be non-zero Laurent polynomials. Their multiple higher Mahler measure is defined by

$$\begin{aligned} m(P_1, \dots, P_k) &= \int_0^1 \cdots \int_0^1 \log\left|P_1\left(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_n}\right)\right| \\ &\quad \cdots \log\left|P_k\left(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_n}\right)\right| d\theta_1 \cdots d\theta_n \end{aligned}$$

$$m(P_1) \cdots m(P_k) = m(P_1, \dots, P_k)$$

when the variables of P_j's are algebraically independent.

Multiple Mahler measure for simple polynomials

Theorem

For $0 \leq \alpha \leq 1$

$$m(1-x, 1-\mathrm{e}^{2\pi \mathrm{i}\alpha}x) = \frac{\pi^2}{2}\left(\alpha^2 - \alpha + \frac{1}{6}\right).$$

Examples

$$\begin{aligned} m(1-x, 1+x) &= -\frac{\pi^2}{24}, \\ m(1-x, 1\pm \mathrm{i}x) &= -\frac{\pi^2}{96}, \\ m(1-x, 1-\mathrm{e}^{2\pi \mathrm{i}\alpha}x) &= 0 \Leftrightarrow \alpha = \frac{3\pm\sqrt{3}}{6}. \end{aligned}$$

Cyclotomic polynomials and Lehmer's question

$$m_2(x^n) = 0$$

$$m_2(P) \geq \frac{\pi^2}{12} \sim 0.822467033...$$

if P product of cyclotomic polynomials and x^n, $P \neq x^n$.

$$m_2(P) \geq L^2 \sim 0.0263599941...$$

assuming Lehmer's question for the usual Mahler measure, $P \neq x^n$.

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Jensen's formula for multiple Mahler measure

$$m(1-\alpha x) = \begin{cases} 0 & \text{if } |\alpha| \le 1, \\ \log|\alpha| & \text{if } |\alpha| \ge 1. \end{cases}$$

$$m(1-\alpha x, 1-\beta x) = \begin{cases} \frac{1}{2}\operatorname{Re}\operatorname{Li}_2\left(\alpha\bar{\beta}\right) & \text{if } |\alpha|, |\beta| \le 1, \\ \frac{1}{2}\operatorname{Re}\operatorname{Li}_2\left(\frac{\alpha\beta}{|\alpha|^2}\right) & \text{if } |\alpha| \ge 1, |\beta| \le \\ \frac{1}{2}\operatorname{Re}\operatorname{Li}_2\left(\frac{\alpha\bar{\beta}}{|\alpha\beta|^2}\right) + \log|\alpha|\log|\beta| & \text{if } |\alpha|, |\beta| \ge 1. \end{cases}$$

Higher zeta Mahler measure

$$Z(s_1, \dots, s_k; P_1, \dots, P_k) = \int_0^1 \cdots \int_0^1 \left|P_1\left(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_n}\right)\right|^{s_1} \cdots \left|P_k\left(e^{2\pi i\theta_1}, \cdots, e^{2\pi i\theta_n}\right)\right|^{s_k} d\theta_1 \cdots d\theta_n$$

The Taylor coefficients yield multiple higher Mahler measure.

Theorem

- $$Z(s,t;x-1,x+1) = \frac{\Gamma(s+1)\Gamma(t+1)}{\Gamma\left(\frac{s}{2}+1\right)\Gamma\left(\frac{t}{2}+1\right)\Gamma\left(\frac{s+t}{2}+1\right)}$$
  $$= \exp\left(\sum_{k=2}^{\infty}\frac{(-1)^k}{k}\zeta(k)\left((1-2^{-k})(s^k+t^k)-2^{-k}(s+t)^k\right)\right)$$

- $$m(\underbrace{x-1,\ldots,x-1}_{k},\underbrace{x+1,\ldots,x+1}_{l})$$
  belongs to $\mathbb{Q}[\pi^2,\zeta(3),\zeta(5),\zeta(7),\ldots]$ *for integers* $k,l\geq 0$.

$$
\begin{aligned}
m(x-1, x+1) &= -\frac{\zeta(2)}{4} = -\frac{\pi^2}{24}, \\
m(x-1, x-1, x+1) &= 2\frac{\zeta(3)}{8} = \frac{\zeta(3)}{4}, \\
m(x-1, x+1, x+1) &= 2\frac{\zeta(3)}{8} = \frac{\zeta(3)}{4}. \\
m_3(x-1) &= -\frac{3\zeta(3)}{2}.
\end{aligned}
$$

Properties of zeta Mahler measures

- $\lambda > 0$,

$$Z(s, \lambda P) = \lambda^s Z(s, P)$$

- $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \quad P = P^*, \quad |\lambda| \leq 1/||P||_\infty$,

$$Z(s, 1 + \lambda P) = \sum_{k=0}^{\infty} \binom{s}{k} Z(k, P) \lambda^k,$$

$$m(1 + \lambda P) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} Z(k, P) \lambda^k.$$

More generally,

$$m_j(1 + \lambda P) = j! \sum_{0 < k_1 < \cdots < k_j} \frac{(-1)^{k_j - j}}{k_1 \dots k_j} Z(k_j, P) \lambda^{k_j}.$$

The case $P = x + y + c$

Let $c \geq 2$.

$$Z(s, x+y+c) = c^s \sum_{j=0}^{\infty} \binom{s/2}{j}^2 \frac{1}{c^{2j}} \binom{2j}{j}.$$

$$= c^s {}_3F_2\left(\begin{matrix} -\frac{s}{2}, -\frac{s}{2}, \frac{1}{2} \\ 1, 1 \end{matrix} \middle| \frac{4}{c^2} \right),$$

where the generalized hypergeometric series ${}_3F_2$ is defined by

$${}_3F_2\left(\begin{matrix} a_1, a_2, a_3 \\ b_1, b_2 \end{matrix} \middle| z \right) = \sum_{j=0}^{\infty} \frac{(a_1)_j (a_2)_j (a_3)_j}{(b_1)_j (b_2)_j j!} z^j,$$

with the Pochhammer symbol defined by
$(a)_j = a(a+1)\cdots(a+j-1)$.

In particular, we obtain the special values

- $$m_2(x+y+2) = \frac{\zeta(2)}{2},$$
- $$m_3(x+y+2) = \frac{9}{2}(\log 2)\zeta(2) - \frac{15}{4}\zeta(3).$$

Further questions

Why do we get such numbers?
Is there an explanation in terms of regulators?

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