Polylogarithms and Hyperbolic volumes

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1 The hyperbolic space

In this introduction we follow mainly Milnor [6].

Hyperbolic geometry is a non-Euclidean geometry, meaning that it starts with the negation of the parallel postulate of Euclidean geometry. The first rigorous works in the subject were due to Lobachevsky (1829), Bolyai (1832), and Gauss (late 1820's) .

There are several models for the hyperbolic space, but we will concentrate in the Half-space model of Beltrami (1868). Our space is given by

$$\mathbb{H}^n = \{(x_1, \dots, x_{n-1}, x_n) \,|\, x_i \in \mathbb{R}, x_n > 0\},$$

where the metric is given by

$$\mathrm{d}s^2 = \frac{\mathrm{d}x_1^2 + \dots + \mathrm{d}x_n^2}{x_n^2}.$$

From this, we can see that the volume element is given by

$$\mathrm{d}V = \frac{\mathrm{d}x_1 \dots \mathrm{d}x_n}{x_n^n}.$$

Notice that the boundary is given by

$$\partial\mathbb{H}^n = \{(x_1, \dots, x_{n-1}, 0)\} \cup \infty.$$

The geodesics of $\mathbb{H}^n$ are given by vertical lines and semicircles whose endpoints lie in $\{x_n = 0\}$ and intersect it orthogonally.

Poincaré (1882) studied the orientation preserving isometries of $\mathbb{H}^2$ which is a group that can be identified with the projective linear group

$$PSL(2,\mathbb{R}) = \left\{ \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in M(2,\mathbb{R}) \,\middle|\, ad - bc = 1 \right\} / \pm I.$$

Under the identification $z = x_1 + x_2 i$, the action is given by $z \to \frac{az+b}{cz+d}$.

Latter Poincaré (1883) also determined the orientation preserving isometries of $\mathbb{H}^3$, which is $PSL(2,\mathbb{C})$. To see this, it is convenient to interpret $\mathbb{H}^3$ as a subspace of the quaternions

$$\mathbb{H}^3 = \{z = x_1 + x_2 i + x_3 j \,|\, x_3 > 0\},$$

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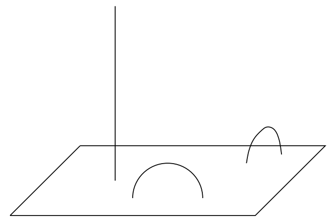

Figure 1: Geodesics in $\mathbb{H}^3$

(where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$).

Then the action is given again by $z \to (az+b)(cz+d)^{-1} = (az+b)(\bar{z}\bar{c}+\bar{d})|cz+d|^{-2}$.

The action of $PSL(2,\mathbb{C})$ can be also described by means of Poincaré extension. The action is clear in the boundary of the space. Now, given a point p in $\mathbb{H}^3$, it can be described as intersection of three hemispheres with equators in the boundary. The isometry moves the equators to three new equators in the boundary, which determine three new hemispheres which intersect in the image of p.

Poincaré was concerned with the study of discrete groups of hyperbolic isometries. Picard (1884) observed that the fundamental domain for the action of $PSL(2,\mathbb{Z}[i])$ in $\mathbb{H}^3$ had a finite volume. Humbert (1919) extended this result.

2 Volumes in $\mathbb{H}^3$

In this section we follow Milnor [6], and Thurston [10].

The basic function to express volume is the Lobachevsky function

$$\text{л}(\theta) = -\int_0^\theta \log|2\sin t|\mathrm{d}t. \tag{1}$$

We note that

$$\text{л}(\theta) = \frac{1}{2}\operatorname{Im}\left(\mathrm{Li}_2\left(e^{2i\theta}\right)\right),$$

where

$$\mathrm{Li}_2(z) = \sum_{n=1}^{\infty}\frac{z^n}{n^2}, \qquad |z| \leq 1, \tag{2}$$

is the dilogarithm. The name comes as an analogy to the formula

$$-\log(1-z) = \sum_{n=1}^{\infty}\frac{z^n}{n}.$$

It has a (multivalued) analytic continuation to $\mathbb{C}\setminus[1,\infty)$ via the formula

$$\mathrm{Li}_2(z) = -\int_0^z \log(1-x)\frac{\mathrm{d}x}{x}.$$

A horosphere centered at $z \in \partial\mathbb{H}^3$ is a hypersurface in $\mathbb{H}^3$ that is orthogonal to all the geodesics going through z.

Consider an ideal tetrahedron Δ in $\mathbb{H}^3$ (with vertices in $\partial\mathbb{H}^3$). For a small horosphere that is centered around a vertex, it intersects the simplex in a triangle whose angles are the

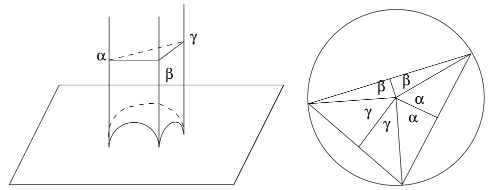

Figure 2: Ideal tetrahedron in $\mathbb{H}^3$

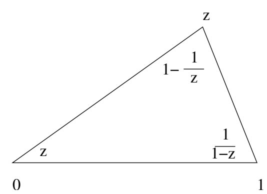

Figure 3: The shapes of an ideal tetrahedron

three dihedral angles along the edges meeting at the vertex. This triangle is defined up to similarity. Since the horosphere is isometric to an Euclidean plane, the sum of the angles is π. From this, dihedral angles at opposite edges must be the same, and the triangles cut by horospheres centered at each vertex are similar.

Theorem 1 *(Milnor [6, 10], after Lobachevsky)*
The volume of an ideal tetrahedron with dihedral angles α, β, and γ is given by

$$\mathrm{Vol}(\Delta) = \text{Л}(\alpha) + \text{Л}(\beta) + \text{Л}(\gamma). \tag{3}$$

The proof is achieved by moving a vertex to ∞ and using baricentric subdivision to get six simplices (orthoschemes) with three right dihedral angles, one vertex at ∞ and another at $\partial\mathbb{H}^3$.

We now consider the triangle with angles α, β, and γ. Similarity classes of triangles can be parameterized by the complex upper-half plane by sending two vertices to $0, 1$, and the third vertex is z. Moreover, the numbers z, $\frac{1}{1-z}$, and $1-\frac{1}{z}$ give all the same triangle (depending on the choice of the vertices that are sent to $0, 1$). To specify the number z (shape) one has to choose an edge, the dihedral angle at this edge will be $\arg(z)$. The tetrahedron will be denoted by $\Delta(z)$.

If the ideal vertices are z_1, z_2, z_3, z_4, the shape can be obtained by means of the cross ratio,

$$z = [z_1 : z_2 : z_3 : z_4] = \frac{(z_3 - z_2)(z_4 - z_1)}{(z_3 - z_1)(z_4 - z_2)}.$$

Now we consider the Bloch-Wigner modification of the dilogarithm,

$$D(z) = \mathrm{Im}(\mathrm{Li}_2(z) + \log|z| \log(1 - z)).$$

This function is continuous in $\mathbb{P}^1(\mathbb{C})$ and real-analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$. Moreover, it satisfies many functional equations such as

$$D(z) = -D(1 - z) = -D\left(\frac{1}{z}\right), \tag{4}$$

as well as

$$D(z) = -D(\bar{z}), \tag{5}$$

equations such as the five-term relation:

$$D(x) + D(1-xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0, \tag{6}$$

and

$$D(z) = \frac{1}{2}\left(D\left(\frac{z}{\bar{z}}\right) + D\left(\frac{1-z^{-1}}{1-\bar{z}^{-1}}\right) + D\left(\frac{(1-z)^{-1}}{(1-\bar{z})^{-1}}\right)\right). \tag{7}$$

The above equation can be combined with Milnor's theorem to prove that

$$\mathrm{Vol}(\Delta(z)) = D(z). \tag{8}$$

The five-term relation (6) has an interpretation in this context as well. If we fix five points in $\partial\mathbb{H}^3 \cong \mathbb{P}^1(\mathbb{C})$, then the sum of the signed volumes of the five possible tetrahedra must be zero:

$$\sum_{i=0}^{5}(-1)^i \mathrm{Vol}([z_1 : \cdots : \hat{z}_i : \cdots : z_5]) = 0,$$

and this is the content of the five-term relation.

2.1 Dedekind ζ-function

Let F be a number field (i.e. finite extension of $\mathbb{Q}$). Then $[F:\mathbb{Q}] = n = r_1 + 2r_2$, where r_1 is the number of real embeddings and r_2 the number of pairs of complex embeddings. Let $\tau_1, \dots, \tau_{r_1}$ the set of real embeddings and $\sigma_1, \dots, \sigma_{r_2}$ a set of complex embeddings (where we choose one for each pair of conjugate embeddings).

Recall that the Dedekind ζ-function of F is given by

$$\zeta_F(s) = \sum_{\mathfrak{A}\,\mathrm{ideal} \neq 0} \frac{1}{N(\mathfrak{A})^s}, \qquad \mathrm{Re}\, s > 1, \tag{9}$$

where $N(\mathfrak{A}) = |\mathcal{O}_F/\mathfrak{A}|$ is the norm of the ideal. The ζ-function can be also written as an Euler product,

$$\prod_{\mathfrak{P}\,\mathrm{prime}} \frac{1}{1 - N(\mathfrak{P})^{-s}}.$$

Theorem 2 *(Dirichlet's class number formula) $\zeta_F(s)$ extends to a meromorphic function for all complex s with only one simple pole at $s = 1$ with residue*

$$\lim_{s\to 1}(s-1)\zeta_F(s) = \frac{2^{r_1}(2\pi)^{r_2} h_F\, \mathrm{reg}_F}{\omega_F\sqrt{|D_F|}}, \tag{10}$$

where

- *h_F is the class number. The number of elements in the ideal class group, formed by classes of ideals in $\mathcal{O}_F$ with the equivalent relation $I \sim J$ iff $(a)I = (b)J$, $a, b \in \mathcal{O}_F$.*
- *ω_F is the number of roots of unity in F.*

- *D_F is the discriminant. Take an integral basis of $\mathcal{O}_F$, $\{\omega_1, \ldots, \omega_n\}$ and take the determinant of $\{\mathrm{Tr}(\omega_i\omega_j)\}$. It measures the size of $\mathcal{O}_F$.*

- *reg_F is the regulator. Take a fundamental system of units $\{u_1, \ldots, u_{r_1+r_2-1}\}$ (i.e., a basis for $\mathcal{O}_F^*$ modulo torsion), and consider the function*

$$L(u_i) = (\log|\tau_1 u_i|, \ldots, \log|\tau_{r_1} u_i|, 2\log|\sigma_1 u_i|, \ldots, 2\log|\sigma_{r_2-1} u_i|).$$

Then reg_F is the determinant of the matrix. I.e., the regulator is (up to a sign) the volume of fundamental domain for $L(\mathcal{O}_F^*)$.

Note that the functional equation

$$\xi_F(s) = \xi_F(1-s), \qquad \xi_F(s) = \left(\frac{|D_F|}{4^{r_2}\pi^n}\right)^{\frac{s}{2}} \Gamma\left(\frac{s}{2}\right)^{r_1} \Gamma(s)^{r_2} \zeta_F(s),$$

(where $\Gamma(s) = \int_0^\infty t^{z-1}e^{-t}\mathrm{d}t$) allows us to write

$$\lim_{s\to 0} s^{1-r_1-r_2}\zeta_F(s) = -\frac{h_F\,\mathrm{reg}_F}{\omega_F}.$$

This was observed by Lichtenbaum [5].

On the other hand, if F is totally real ($r_2 = 0$), then a theorem of Klingen and Siegel says

$$\zeta_F(2m) = r(m)\sqrt{|D_F|}\pi^{2mn}, \qquad m > 0,$$

where $r(m) \in \mathbb{Q}$. This result generalizes Euler's $\zeta(2m) = \frac{(-1)^{m-1}(2\pi)^{2m}B_m}{2(2m)!}$

2.2 Building manifolds

Now we follow Zagier – Gangl [12].

Here is a method of Bianchi and Humbert to construct hyperbolic three-manifolds with finite volume. Let $F = \mathbb{Q}\left(\sqrt{-d}\right)$ where $d \geq 1$ is a square-free integer (the discriminant is $-D_d$ where D_d equals d or $4d$ according to whether $d \equiv 3 (\mathrm{mod}\, 4)$ or otherwise). Then $PSL\,(2, \mathcal{O}_d)$, where $\mathcal{O}_d$ is the ring of integers of $\mathbb{Q}\left(\sqrt{-d}\right)$ is a discrete subgroup of $PSL(2,\mathbb{C})$. Let Γ a torsion-free subgroup of finite index of $PSL\,(2, \mathcal{O}_d)$. Then $\mathbb{H}^3/\Gamma$ is an oriented hyperbolic three-manifold.

For example, if $d = 3$, $\mathcal{O}_d = \mathbb{Z}[\omega]$ where $\omega = \frac{-1+\sqrt{-3}}{2}$. Riley [9] proved that there is a subgroup $\Gamma \subset PSL(2, \mathbb{Z}[\omega])$ of index 12, such that $\mathbb{H}^3/\Gamma$ is diffeomorphic to the complement of the Fig-8 knot.

Theorem 3 *(Essentially Humbert)*

$$\mathrm{Vol}\left(\mathbb{H}^3/PSL(2, \mathcal{O}_d)\right) = \frac{D_d\sqrt{D_d}}{4\pi^2}\zeta_{\mathbb{Q}(\sqrt{-d})}(2). \tag{11}$$

In the example that means

$$\mathrm{Vol}(S^3 \setminus \mathrm{Fig}-8) = 12\frac{3\sqrt{3}}{4\pi^2}\zeta_{\mathbb{Q}(\sqrt{-3})}(2) = 3D\left(\mathrm{e}^{\frac{2i\pi}{3}}\right) = 2D\left(\mathrm{e}^{\frac{i\pi}{3}}\right).$$

On the other hand, any hyperbolic 3-manifold can be triangulated into tetrahedra (ideal tetrahedra, after applying Dehn surgery on some of the cusps) and we can assume that the arguments are algebraic numbers. Combining the results from the previous section, we can express $\zeta_{\mathbb{Q}(\sqrt{-d})}$ as a combination of D evaluated in algebraic numbers.

For a general number field, let $[F:\mathbb{Q}] = r_1 + 2r_2$ with $r_2 = 1$, Zagier [11] takes Γ to be the group of units of an order in a quaternion algebra B over F that is ramified at all real places (i.e. $B \otimes_F F \cong$ Hamiltonian quaternions for each real completion). The quotient $M = \mathbb{H}^3/\Gamma$ is a compact manifold whose volume is a rational multiple of

$$\frac{\sqrt{|D_F|}}{\pi^{2(n-1)}}\zeta_F(2).$$

For $r_2 > 1$, Zagier [11] starts with B, defining Γ as before. Now the complex embeddings give a map $\sigma = (\sigma_i)_i : B \to M(2,\mathbb{C})^{r_2}$ such that $\sigma(\Gamma)$ is a discrete subgroup of $PSL(2,\mathbb{C})^{r_2}$ such that the quotient $M = \left(\mathbb{H}^3\right)^{r_2}/\Gamma$ is a compact $3r_2$-dimensional manifold whose volume is a rational multiple of

$$\frac{\sqrt{|D_F|}}{\pi^{2(r_1+r_2)}}\zeta_F(2).$$

M can be written as a union of $\Delta(z_1)\times\cdots\times\Delta(z_{r_2})$. The volume is then a sum of r_2-fold products of dilogarithms.

2.3 The Bloch group

As we said before, any complete hyperbolic 3-manifold can be triangulated into tetrahedra (ideal tetrahedra, if we allow to remove a finite number of closed geodesics to M, see Neumann–Yang [7]) and the volume can be written as

$$\mathrm{Vol}(M) = \sum_{j=1}^{J} D(z_j).$$

The parameters must satisfy

$$\sum_{j=1}^{J} z_j \wedge (1-z_j) = 0 \in \bigwedge^2 \mathbb{C}^*.$$

($\bigwedge^2 \mathbb{C}^*$ is the set of all formal linear combinations $x\wedge y$, $x, y \in \mathbb{C}$, such that $x \wedge x = 0$ and $x_1x_2 \wedge y = x_1 \wedge y + x_2 \wedge y$.) This condition is consequence of combinatorial restrictions for the triangulation (see [8]).

We may rephrase this condition by stating that $\mathrm{Vol}(M) = D(\xi_M)$, where $\xi_M \in \mathcal{A}(\bar{\mathbb{Q}})$, and

$$\mathcal{A}(F) = \left\{ \sum n_i[z_i] \in \mathbb{Z}[F] \,\middle|\, \sum n_i\left(z_i \wedge (1-z_i)\right) = 0 \right\}. \tag{12}$$

Changing the triangulations should not change the volume, and this is the case, due to the five-term relation (6).

Let

$$\mathcal{C}(F) = \left\{ [x] + [1-xy] + [y] + \left[\frac{1-y}{1-xy}\right] + \left[\frac{1-x}{1-xy}\right] \,\middle|\, x, y \in F, xy \neq 1 \right\}, \tag{13}$$

then the Bloch group is defined as

$$\mathcal{B}(F) = \mathcal{A}(F)/\mathcal{C}(F).$$

We have that $D : \mathcal{B}(\mathbb{C}) \to \mathbb{R}$ is a well-defined function, and that $\mathrm{Vol}(M) = D(\xi_M)$ for some $\xi_M \in \mathcal{B}(\bar{\mathbb{Q}})$ (here we use Bloch's result that $\mathcal{B}(\mathbb{C})_{\mathbb{Q}} = \mathcal{B}(\bar{\mathbb{Q}})_{\mathbb{Q}}$), independently of the triangulation (this was precisselly proved by Nuemann–Yang [7]). Therefore, for $r_2 = 1$, $\zeta_F(2)$ is equal to $\sqrt{|D|}\pi^{2(n-1)}D(\xi)$.

2.4 The K-theory connection

It turns out that all this construction has a parallel in terms of algebraic K-theory. To any ring R one can associate algebraic K-groups $K_0(R)$, $K_1(R) = R^*$, etc. The construction is not explicit and it is hard to describe what these groups are.

[*More detail: By theorems of Milnor, Moore, Suslin,*

$$K_n(F) \otimes \mathbb{Q} = \mathrm{Prim}\, H_n(GL_{2n-1}(F), \mathbb{Q}),$$

where $GL(R) = \cup_{n \geq 1} GL_n(R)$ *and* $\mathrm{Prim}\, H_n(G) = \{x \in H_n(G) \,|\, \Delta_*(x) = x \otimes 1 + 1 \otimes x\}$, *where* $\Delta_* : H_n(G) \to H_n(G \times G)$ *is the map induced by the diagonal* $\Delta : G \to G \times G$.]

Now for a number field, Borel proved that for $n \geq 2$, $K_n(F) \otimes \mathbb{Q} \simeq K_n(\mathcal{O}_F) \otimes \mathbb{Q}$ is free abelian and its rank is given by

$$\dim_{\mathbb{Q}}(K_n(F) \otimes \mathbb{Q}) = \begin{cases} 0 & n \geq 2\,\text{even}, \\ r_1 + r_2 & n \equiv 1 \bmod 4, \\ r_2 & n \equiv 3 \bmod 4. \end{cases}$$

Let $n_+ = r_1 + r_2$ and $n_- = r_2$. Then $n_\mp$ means that we take n_- or n_+ according to whether $n \equiv -1$ or $1 \bmod 4$.

Moreover, there is a map

$$\mathrm{reg}_m : K_{2m-1}(\mathbb{C}) \to \mathbb{R},$$

called regulator, such that the composition

$$K_{2m-1}(F) \to K_{2m-1}(\mathbb{R})^{r_1} \times K_{2m-1}(\mathbb{C})^{r_2} \to \mathbb{R}^{n_\mp}$$

maps the free part of $K_{2m-1}(F)$ isomorphically onto a cocompact lattice of $\mathbb{R}^{n_\mp}$ whose covolume is a rational multiple of $\sqrt{|D_F|}\zeta_F(m)/\pi^{kn_\pm}$.

[*More detail: There is a distinguished class, called the Borel class,*

$$B_n \in H^{2n-1}(GL_{2n-1}(\mathbb{C}), \mathbb{R}(n-1))$$

in the continuous cohomology. The pairing with this class is what provides the Borel regulator.]

In the case of $m = 2$, Bloch gave a map ϕ_F from $K_3(F)$ to $\mathcal{B}(F)$ and Suslin proved this map is an isomorphism (up to finite kernel and cokernel). This map is essentially D for $m = 2$.

$$\begin{tikzcd} K_3(F) \arrow[r, "\mathrm{reg}_2"] & \mathbb{R}^{r_2} \\ \mathcal{B}(F) \arrow[u, "\phi_F"] \arrow[ur, "(D\circ\sigma_1,\ldots,D\circ\sigma_{r_2})"'] & \end{tikzcd}$$

From that,

Theorem 4 *For a number field $[F:\mathbb{Q}] = r_1 + 2r_2$,*

- *$\mathcal{B}(F)$ is finitely generated of rank r_2.*
- *For a $\mathbb{Q}$-basis $\xi_1, \dots \xi_{r_2}$ of $\mathcal{B}(F) \otimes \mathbb{Q}$ and $\sigma_1, \dots, \sigma_{r_2}$ a set of complex embeddings of F into $\mathbb{C}$ (choosing one for each pair), then*

$$\zeta_F(2) \sim_{\mathbb{Q}^*} \sqrt{|D_F|}\pi^{2(r_1+r_2)} \det \left\{ D\left(\sigma_i\left(\xi_j\right)\right) \right\}_{1 \leq i,j \leq r_2}. \tag{14}$$

3 Zagier's conjecture

One wishes to generalize the above result to values $\zeta_F(m)$. The k-polylogarithm is defined as

$$\mathrm{Li}_k(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^k} \qquad |z| \leq 1. \tag{15}$$

As in the case of the dilogarithm, it has an analytic continuation to $\mathbb{C} \setminus [1, \infty)$, and one may consider the corrected function

$$\mathcal{L}_k(z) = \mathrm{Re}_k \left(\sum_{j=0}^{k-1} \frac{2^j B_j}{j!} \log^j |z| \mathrm{Li}_{k-j}(z) \right), \tag{16}$$

where Re_k is equal to Re or Im depending on whether k is odd or even, and B_j is the jth Bernoulli number. Like D, it is continuous in $\mathbb{P}^1(\mathbb{C})$ and real analytic in $\mathbb{P}^1(\mathbb{C}) \setminus \{0, 1, \infty\}$.

Then one proceeds to construct the generalized Bloch groups

$$\mathcal{B}_k(F) = \mathcal{A}_k(F)/\mathcal{C}_k(F).$$

Morally, $\mathcal{C}_k(F)$ will be the set of functional equations of the kth polylogarithm and $\mathcal{A}_k(F)$ is the set of allowed elements.

Here is Zagier's construction:

$$\mathcal{A}_k(F) := \{\xi \in \mathbb{Z}[F] \,|\, \iota_\phi(\xi) \in \mathcal{C}_{k-1}(F) \ \forall \phi \in \mathrm{Hom}(F^*, \mathbb{Z})\}\}, \tag{17}$$

where $\iota_\phi(\sum n_i[x_i]) = \sum n_i \phi(x_i)[x_i]$.

$$\mathcal{C}_k(F) := \left\{\xi \in \mathcal{A}_k(F) \,|\, \mathcal{L}_k(\sigma(\xi)) = 0 \ \forall \sigma \in \Sigma_F \right\}. \tag{18}$$

Then

Conjecture 5 *Let F be a number field. Let $n_+ = r_1 + r_2$, $n_- = r_2$, and $\mp = (-1)^{k-1}$. Then*

- *$\mathcal{B}_k(F)$ is finitely generated of rank $n_\mp$.*
- *For a $\mathbb{Q}$-basis $\xi_1, \dots \xi_{n_\mp}$ of $\mathcal{B}_k(F) \otimes \mathbb{Q}$ and take $\{\tau_1, \dots \tau_{r_1}, \sigma_1, \dots, \sigma_{r_2}\}$ or $\{\sigma_1, \dots, \sigma_{r_2}\}$ (according to $n_\mp$). Then*

$$\zeta_F(k) \sim_{\mathbb{Q}^*} \sqrt{|D_F|}\pi^{kn_\pm} \det \left\{ \mathcal{L}_k\left(\sigma_i\left(\xi_j\right)\right) \right\}_{1 \leq i,j \leq n_\mp}. \tag{19}$$

3.1 An example

This example is taken from [12].

For $F=\mathbb{Q}(\sqrt{5})$, $r_1=2$, $r_2=0$. Then [1] and $\left[\frac{1+\sqrt{5}}{2}\right]\in\mathcal{A}_3(F)$, form a basis for $\mathcal{B}_3(F)$ and

$$\left|\begin{matrix}\mathcal{L}_3(1) & \mathcal{L}_3(1)\\ \mathcal{L}_3\left(\frac{1+\sqrt{5}}{2}\right) & \mathcal{L}_3\left(\frac{1-\sqrt{5}}{2}\right)\end{matrix}\right| = \left|\begin{matrix}\zeta(3) & \zeta(3)\\ \frac{1}{10}\zeta(3)+\frac{25}{48}\sqrt{5}L(3,\chi_5) & \frac{1}{10}\zeta(3)-\frac{25}{48}\sqrt{5}L(3,\chi_5)\end{matrix}\right|$$
$$=-\frac{25}{24}\sqrt{5}\zeta(3)L(3,\chi_5)=-\frac{25}{24}\sqrt{5}\zeta_F(3).$$

3.2 More K-theory

The motivation for the conjectures is that $K_{2m-1}(F)$ and $\mathcal{B}_m(F)$ should be isomorphic and the regulator map should be given by $\mathcal{L}_m$. It is not known in general, but a regulator map and a map between $K_{2m-1}(F)$ and $\mathcal{B}_m(F)$ have been defined by de Jeu and Beilinson-Deligne. Goncharov [3] proved the surjectivity of this map for $m=3$, thus proving Zagier's conjecture for $\zeta_F(3)$.

Goncharov [3] defines a complex

$$G_F(k):\mathcal{G}_k(F)\xrightarrow{\partial}\mathcal{G}_{k-1}\otimes F^*\xrightarrow{\partial}\mathcal{G}_{k-2}\otimes\wedge^2F^*\xrightarrow{\partial}\cdots\xrightarrow{\partial}\mathcal{G}_2\otimes\wedge^{k-2}F^*\xrightarrow{\partial}\wedge^kF^*$$

where $\mathcal{G}_k(F)=\mathbb{Z}[F]/\mathcal{C}_k(F)$ and $\partial[x]\otimes x_1\wedge\cdots\wedge x_l=[x]\otimes x\wedge x_1\wedge\cdots\wedge x_l$. The cohomology of this complex at the first place is given by $H^1(G_F(k))\simeq\mathcal{B}_k(F)$.

Goncharov conjectures

$$H^i(G_F(k)\otimes\mathbb{Q})\simeq gr_k^{\gamma}K_{2k-i}(F)\otimes\mathbb{Q}$$

meaning that the cohomology is given by Adams filtration in K-theory. This sharpens Zagier's conjecture.

3.3 The rational factor

We may wonder about the rational factor in Zagier's conjecture.

Conjecture 6 *(Birch-Tate) Let F be a totally real field and w be the group of units in an algebraic closure of F, and G the Galois group. Let w_2 the subgroup that is fixed by the action of σ^2 for each $\sigma\in G$. Then*

$$\zeta_F(-1)^{r_1}=\left|\frac{K_2(\mathcal{O}_F)}{w_2}\right|.$$

This follows from Iwasawa main conjecture (up to a power of 2), proved by Mazur and Wiles.

Lichtenbaum [5] conjectured

Conjecture 7

$$\lim_{s\to-m}(s+m)^{-n\mp}|\zeta_F(s)|=\left|\frac{K_{2m}(\mathcal{O}_F)}{K_{2m+1}^{\mathrm{ind}}(\mathcal{O}_F)_{\mathrm{tor}}}\right|\mathrm{reg}_m(\xi).$$

up to a power of 2.

More can be said with Bloch-Kato conjectures but we will not go further.

4 Volumes in higher dimensions

An orthoscheme in $\mathbb{H}^n$ is a simplex bounded by hyperplanes $H_0, \ldots, H_n$ such that $H_i \perp H_j$ whenever $|i-j| > 1$. Then they have at most n non-right dihedral angles. If v_i is the vertex opposite to H_i, then at most v_0 and v_n can be at $\partial\mathbb{H}^n$. If both of them are, the orthoscheme is called doubly asymptotic.

Theorem 8 *(Schläfli's formula) $R \subset \mathbb{H}^n$ is an orthoscheme,*

$$\mathrm{dVol}_n(R) = \frac{1}{n-1}\sum_{j=1}^{n} \mathrm{Vol}_{n-2}(F_j)\mathrm{d}\alpha_j.$$

where $F_j = R \cap H_{j-1} \cap H_j$, α_j is the angle attached at F_j. and $\mathrm{Vol}_0(F_j) = 1$.

This formula reduces the volume of a hyperbolic $2m$-simplex to volumes in hyperbolic dimension $2m-1$ and lower.

For dimension 5, the volume can be expressed as sum of trilogarithmic expressions (Böhn, Müller, Kellerhals [4]).

A hyperbolic manifold is an orientable complete Riemannian manifold with constant sectional curvature -1. By Gauss-Bonnet theorem, if M is $2m$-dimensional, its volume is given by

$$\mathrm{Vol}(M) = -\frac{1}{2}\mathrm{Vol}(S^{2m})\chi(M).$$

Theorem 9 *(Goncharov, [2]) Any $(2m-1)$-dimensional hyperbolic manifold of finite volume M defines an element $\gamma(M) \in K_{2m-1}(\bar{\mathbb{Q}}) \otimes \mathbb{Q}$ such that*

$$\mathrm{Vol}(M) = \mathrm{reg}_m(\gamma(M)).$$

Conjecture 10 *(Goncharov, [2]) Let M be a $(2m-1)$-dimensional hyperbolic manifold, then there is a $\xi_M \in \mathcal{B}_m(\bar{\mathbb{Q}}) \otimes \bar{\mathbb{Q}}^*$ such that*

$$\mathrm{Vol}(M) = \mathcal{L}_m(\xi_M).$$

This conjecture is a consequence of combining Zagier's conjecture with Theorem 9. Goncharov proves this conjecture for $k = 3$ using Theorem 9.

The hyperbolic volumes $\mathrm{Vol}(M^{2m-1})$ are discrete for $m > 2$. According to a theorem of Wang, there is only a finite number of manifolds with volume less than c for any given $c \in \mathbb{R}$.

On the other hand, for $m = 2$, the volumes of hyperbolic 3-manifolds are nondiscrete, due to Thurston and Jorgensen.

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