Dilogarithm, a cool function

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Definition 1 *The* Dilogarithm *is the function defined by the power series*

$$\mathrm{Li}_2(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^2} \qquad z \in \mathbb{C}, \quad |z| < 1 \tag{1}$$

The name comes from the analogy with

$$-\log(1-z) = \sum_{n=1}^{\infty} \frac{z^n}{n} \qquad z \in \mathbb{C}, \quad |z| < 1$$

The dilogarithm has an analytic continuation to $\mathbb{C} \setminus (1, \infty)$ given by

$$\mathrm{Li}_2(z) := -\int_0^z \log(1-t) \frac{\mathrm{d}t}{t}$$

This function has only eight computable special values: $0, \pm 1, \frac{1}{2}, \pm 1 \pm \frac{1-\sqrt{5}}{2}$. On the other hand, it satisfies several functional equations. Some of them are

$$\mathrm{Li}_2(x) = n \sum_{z^n = x} \mathrm{Li}_2(z) \qquad (n = 1, 2, 3, \ldots) \tag{2}$$

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(y) + \mathrm{Li}_2(z) = \frac{1}{2}\left(\mathrm{Li}_2\left(-\frac{xy}{z}\right) + \mathrm{Li}_2\left(-\frac{yz}{x}\right) + \mathrm{Li}_2\left(-\frac{zx}{y}\right)\right) \quad \text{for} \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$$

The dilogarithm jumps by $2\pi\mathrm{i}\log|z|$ as z crosses the cut by $(1, \infty)$. Hence, it is natural to consider $\mathrm{Li}_2(z) + \mathrm{i}\arg(1-z)\log|z|$. In fact,

Definition 2 *The* Bloch – Wigner Dilogarithm *is defined by*

$$D(z) := \mathrm{Im}(\mathrm{Li}_2(z)) + \arg(1-z)\log|z| \tag{3}$$

This function has several properties:

- $D(z)$ is real analytic in $\mathbb{C} \setminus \{0, 1\}$ and continuous in $\mathbb{C}$
- The formula

$$D(z) = \frac{1}{2}\left(D\left(\frac{z}{\bar{z}}\right) + D\left(\frac{1 - z^{-1}}{1 - \bar{z}^{-1}}\right) + D\left(\frac{(1-z)^{-1}}{(1-\bar{z})^{-1}}\right)\right) \tag{4}$$

  expresses $D(z)$ in terms of the real function in a single variable:

$$D(\mathrm{e}^{\mathrm{i}\theta}) = \mathrm{Im}(\mathrm{Li}_2(\mathrm{e}^{\mathrm{i}\theta})) = \sum_{n=1}^{\infty} \frac{\sin n\theta}{n^2}$$

- There are lots of functional equations. All of them have a version with the classical dilogarithm, but the equations with Li_2 typically include terms with logarithms.

$$D(z) = -D(\bar{z}) \qquad (\Rightarrow D(z) = 0 \quad \text{for} \quad z \in \mathbb{R}) \tag{5}$$

$$D(z) = D\left(1 - \frac{1}{z}\right) = D\left(\frac{1}{1-z}\right) = -D\left(\frac{1}{z}\right) = -D\left(\frac{z}{z-1}\right) = -D(1-z) \tag{6}$$

The five-term relation:

$$D(x) + D(1-xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0 \tag{7}$$

This functional equation becomes more clear if we think of D as a function on the cross-ratio of four complex numbers:

$$\tilde{D}(z_0, z_1, z_2, z_3) = D\left(\frac{z_0 - z_2}{z_0 - z_3}\frac{z_1 - z_3}{z_1 - z_2}\right) \qquad z_n \in \mathbb{P}^1(\mathbb{C}) \tag{8}$$

The five-term relation becomes

$$\sum_{n=0}^{4} (-1)^n \tilde{D}(z_0, \ldots, \hat{z}_n, \ldots, z_4) = 0 \tag{9}$$

For $f : \mathbb{P}^1(\mathbb{C}) \longrightarrow \mathbb{P}^1(\mathbb{C})$ a function of degree n and $a = f(z)$,

$$\sum_{f(z_i)=a_i} \tilde{D}(z, z_1, z_2, z_3) = n\tilde{D}(a, a_1, a_2, a_3) \tag{10}$$

We obtain an analogous to equation (2) with $f(z) = z^n$, $(a, a_1, a_2, a_3) = (0, \infty, z, 1)$.

Amazingly, we have the following:

Theorem 3 *(Bloch) If f is a measurable function on $\mathbb{P}^1(\mathbb{C})$ satisfying the five-term relation (9) then there exist a constant λ such that $f(z) = \lambda D(z)$.*

Theorem 4 *Any functional equation for $D(z)$ is a formal consequence of the five-term relation (9).*

Let us see some of the contexts in which this function occurs.

1. Mahler Measure

Definition 5 *Given a polynomial $P(x) = a_d x^d + a_{d-1}x^{d-1} + \ldots + a_0 = a_d \prod_{n=1}^{d}(x - \alpha_n)$ with complex coefficients, define the* Mahler measure *of P*

$$M(P) := |a_d| \prod_{n=1}^{d} \max\{1, |\alpha_n|\} \tag{11}$$

The logarithmic Mahler measure *is simply*

$$m(P) := \log M(P) = \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta})| \mathrm{d}\theta^1$$

For $P \in \mathbb{C}[x_1, \ldots, x_n]$*, the (logarithmic)* Mahler measure *is defined by*

$$m(P) := \int_0^1 \ldots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n \tag{12}$$

It is possible to prove that this integral is not singular and that $m(P)$ always exists.

For the several-variable case, it seems that there is no simpler formula than the integral defining the measure. However, many examples have been found relating the Mahler measure of polynomials in two variables to special values of L-functions in quadratic characters, L-functions on elliptic curves and dilogarithms.

Perhaps the simplest example is Maillot's formula:

Theorem 6 *(Maillot)*

$$\pi m(ax + by + c) = \begin{cases} D\left(\left|\frac{a}{b}\right| \mathrm{e}^{i\gamma}\right) + \alpha \log |a| + \beta \log |b| + \gamma \log |c| & \triangle \\ \\ \pi \log \max\{|a|, |b|, |c|\} & \text{not } \triangle \end{cases} \tag{13}$$

Here $\triangle$ stands for the fact of whether $|a|$, $|b|$, and $|c|$ are the lengths of the sides of a triangle, and α, β, and γ are the angles opposite to the sides of lengths $|a|$, $|b|$, and $|c|$ respectively.

2. Volume of Hyperbolic Ideal Tetrahedra

Consider the space $\mathbb{H}^3$ which can be represented as $\mathbb{C} \times \mathbb{R}_{\geq 0} \cup \{\infty\}$. In this space the geodesics are either vertical lines or semicircles in vertical planes with endpoints in $\mathbb{C} \times \{0\}$. An *ideal tetrahedron* is a tetrahedron whose vertices are all in $\mathbb{C} \times \{0\} \cup \{\infty\} = \mathbb{P}^1(\mathbb{C})$. Such a tetrahedron Δ with vertices z_0, z_1, z_2, z_3 has a hyperbolic volume equal to

$$\mathrm{Vol}(\Delta) = \tilde{D}(z_0, z_1, z_2, z_3) \tag{14}$$

The invariance of the formula by the action of $PSL_2(\mathbb{C})$ is in agreement with the fact that this is the group of isometries (preserving orientation) of $\mathbb{H}^3$.

These ideal tetrahedra are important because any completed oriented 3-manifold with finite volume can be decomposed in ideal tetrahedra. These decompositions provide more identities of the dilogarithm.

3. Borel's Theorem

Finally, there is theorem due to Borel, that relates the dilogarithm to the value at $s = 2$ of the zeta function of a number field.

Recall Dirichlet's class formula. Let F be a number field, $\mathcal{O}_F$ its ring of integers,

[1]Because of Jensen's equality $\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log \max\{1, |\alpha|\}$

Definition 7 *The zeta function of F is defined as*

$$\zeta_F(s) := \prod_{\mathfrak{p}\subset\mathcal{O}_F}\left(1-\frac{1}{(N\mathfrak{p})^s}\right)^{-1} = \sum_{\mathfrak{a}\subset\mathcal{O}_F}\frac{1}{N\mathfrak{a}} \tag{15}$$

Let r_1 be the number of real embeddings of F and r_2 be the number of pairs of complex embeddings. Let R_F be the regulator, h_F the class number, ω_F the number of roots of unity in F and Δ_F the discriminant.

Theorem 8 *(Dedekind formula)*

$$\lim_{s\to 1}(s-1)\zeta_F(s) = \frac{2^{r_1+r_2}\pi^{r_2}R_F h_F}{\omega_F|\Delta_F|^{1/2}} \tag{16}$$

Now $\mathcal{B}$ is a certain group, the Bloch group, which is constructed from F and that has rank equal to r_2.

Theorem 9 *(Borel) Let $\xi_1,\ldots,\xi_{r_2}$ be a $\mathbb{Q}$ - basis for $\mathcal{B}\otimes\mathbb{Q}$. Let $\sigma_1,\ldots,\sigma_{r_2}$ pairwise non-conjugate complex embeddings of F. Then*

$$\zeta_F(2) \sim_{\mathbb{Q}^\times} \frac{\pi^{2(r_1+r_2)}\det(D(\sigma_j(\xi_k)))}{|\Delta_F|^{1/2}} \tag{17}$$

The "$\sim_{\mathbb{Q}^\times}$" means up to a rational factor. Unfortunately, we do not know how big this factor is in general, and that makes it hard to compute this factor with numerical methods.

In the context of Borel's theorem, the dilogarithm plays an analogous role to the regulator in Dirichlet's class formula. The regulator is a determinant involving logarithms (and the volume of a lattice). In Borel's theorem it is replaced by a determinant of dilogarithms (and it is also the volume of a lattice).

References

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[2] D. Zagier, The Dilogarithm function in Geometry and Number Theory, *Number Theory and related topics, Tata Inst. Fund. Res. Stud. Math.* **12** Bombay (1988), 231 - 249.

[3] Notes of the course "*Topics in K-theory and L-functions* by F. Rodriguez Villegas". http://www.math.harvard.edu/ villegas/course.html