Higher Mahler measure and Lehmer's question

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Mahler measure for one-variable polynomials

Pierce (1918): $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - r_i)$$

$$\Delta_n = \prod_i (r_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Lehmer (1933):

$$\lim_{n\to\infty} \frac{|r^{n+1}-1|}{|r^n-1|} = \begin{cases} |r| & \text{if } |r|>1 \\ 1 & \text{if } |r|<1 \end{cases}$$

For

$$P(x) = a\prod_i (x-r_i)$$

$$M(P) = |a| \prod_{|r_i|>1} |r_i|, \qquad m(P) = \log|a| + \sum_{|r_i|>1} \log|r_i|.$$

By Jensen's formula,

$$m(P) := \int_0^1 \log\left|P\left(\mathrm{e}^{2\pi\mathrm{i}\theta}\right)\right| \mathrm{d}\theta.$$

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Kronecker's Lemma

$P \in \mathbb{Z}[x]$, $P \neq 0$,

$$m(P) = 0 \Leftrightarrow P(x) = x^k \prod \Phi_{n_i}(x)$$

where Φ_{n_i} are cyclotomic polynomials.

Lehmer's question

Lehmer (1933):

Given $\epsilon > 0$, can we find a polynomial $P(x) \in \mathbb{Z}[x]$ such that $0 < m(P) < \epsilon$?

$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) \cong 0.162357612\ldots$$

Is the above polynomial the best possible?.

$$\sqrt{\Delta_{379}} = 1,794,327,140,357$$

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$P \in \mathbb{C}[x]$ reciprocal iff

$$P(x) = \pm x^{\deg P} P\left(x^{-1}\right).$$

Theorem (Smyth, 1971)

$P \in \mathbb{Z}[x]$ *nonreciprocal,*

$$m(P) \geq m(x^3 - x - 1) \cong 0.2811995743\ldots.$$

Higher Mahler measure

For $k \in \mathbb{Z}_{\geq 0}$, the k-higher Mahler measure of P is

$$m_k(P) := \int_0^1 \log^k \left| P\left(\mathrm{e}^{2\pi \mathrm{i}\theta}\right)\right| \mathrm{d}\theta.$$

$$k = 1 : \qquad m_1(P) = m(P),$$

and

$$m_0(P) = 1.$$

The simplest examples

(Kurokawa, L., Ochiai)

$$m_2(x-1) = \frac{\zeta(2)}{2} = \frac{\pi^2}{12}.$$

$$m_3(x-1) = -\frac{3\zeta(3)}{2}.$$

$$m_4(x-1) = \frac{3\zeta(2)^2 + 21\zeta(4)}{4} = \frac{19\pi^4}{240}.$$

$$m_5(x-1) = -\frac{15\zeta(2)\zeta(3) + 45\zeta(5)}{2}.$$

$$m_6(x-1) = \frac{45}{2}\zeta(3)^2 + \frac{275}{1344}\pi^6.$$

$$\sum_{k=0}^{\infty} \frac{m_k(x-1)}{k!} s^k = \int_0^1 \left| e^{2\pi i\theta} - 1 \right|^s d\theta$$

$$= \frac{\Gamma(s+1)}{\Gamma\left(\frac{s}{2}+1\right)^2} = \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^k (1-2^{1-k})\zeta(k)}{k} s^k \right).$$

Transform the integral into a Beta function, then use the Weierstrass product of the Γ-function.

Lehmer's question for even higher measure

Theorem

If $P(x) \in \mathbb{Z}[x]$, then for any $h \geq 1$,

$$m_{2h}(P) \geq \begin{cases} \left(\frac{\pi^2}{12}\right)^h, & \text{if } P(x) \text{ is reciprocal,} \\ \left(\frac{\pi^2}{48}\right)^h, & \text{if } P(x) \text{ is non-reciprocal.} \end{cases}$$

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- $m_2(P)$ for P reciprocal is minimized when P is a product of monomials and cyclotomic polynomials.
- $m_2(P) \geq \frac{\pi^2}{12}$ for P a product of monomials and cyclotomic polynomials.
- For P nonreciprocal, take $P(x)x^{\deg P}P\left(x^{-1}\right)$.
- For $m_{2h}(P)$, use Hölder's inequality.

Multiple Mahler measure

Let $P_1, \ldots, P_k \in \mathbb{C}[x^{\pm}]$ be non-zero Laurent polynomials. Their multiple higher Mahler measure is defined by

$$m(P_1, \ldots, P_k) = \int_0^1 \log \left| P_1 \left(\mathrm{e}^{2\pi \mathrm{i}\theta} \right) \right| \cdots \log \left| P_k \left(\mathrm{e}^{2\pi \mathrm{i}\theta} \right) \right| \mathrm{d}\theta.$$

$$m_2 \left(\prod_i (x - r_i) \right) = \sum_{i,j} m(x - r_i, x - r_j).$$

Multiple Mahler measure for simple polynomials

Lemma (Kurokawa, L., Ochiai)

For $0 \leq \alpha \leq 1$

$$m(x-1, x-e^{2\pi i\alpha}) = \frac{\pi^2}{2}\left(\alpha^2 - \alpha + \frac{1}{6}\right).$$

Examples

$$\begin{aligned} m(x-1, x+1) &= -\frac{\pi^2}{24}, \\ m(x-1, x\pm i) &= -\frac{\pi^2}{96}, \\ m(x-1, x-e^{2\pi i\alpha}) &= 0 \Leftrightarrow \alpha = \frac{3\pm\sqrt{3}}{6}. \end{aligned}$$

m_2 of cyclotomic polynomials

Proposition

- *For any two positive integers a and b,*

$$m(x^a - 1, x^b - 1) = \frac{\pi^2}{12}\frac{(a,b)^2}{ab}.$$

- *For a positive integer n, let $\phi_n(x)$ denote the n-th cyclotomic polynomial and φ Euler's function. Then*

$$m_2(\phi_n(x)) = \frac{\pi^2}{12}\frac{\varphi(n)2^{r(n)}}{n},$$

where $r(n)$ denotes the number of distinct prime divisors of n.

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m_2 of noncyclotomic polynomials

$P(x)$	$m(P)$	$m_2(P)$
$x^8+x^5-x^4+x^3+1$	0.2473585132	1.0980813745
$x^{10}+x^9-x^7-x^6-x^5-x^4-x^3+x+1$	0.1623576120	1.7447964556
$x^{10}-x^6+x^5-x^4+1$	0.1958888214	1.2863292447
$x^{10}+x^7+x^5+x^3+1$	0.2073323581	1.2320444893
$x^{10}-x^8+x^5-x^2+1$	0.2320881973	1.1704950485
$x^{10}+x^8+x^7+x^5+x^3+x^2+1$	0.2368364616	1.1914083866
$x^{10}+x^9-x^5+x+1$	0.2496548880	1.0309287773
$x^{12}+x^{11}+x^{10}-x^8-x^7-x^6-x^5-x^4+x^2+x+1$	0.2052121880	1.4738375004
$x^{12}+x^{11}+x^{10}+x^9-x^6+x^3+x^2+x+1$	0.2156970336	1.5143823478
$x^{12}+x^{11}-x^7-x^6-x^5+x+1$	0.2239804947	1.2059443050
$x^{12}+x^{10}+x^7-x^6+x^5+x^2+1$	0.2345928411	1.2434560052
$x^{12}+x^{10}+x^9+x^8+2x^7+x^6+2x^5+x^4+x^3+x^2+1$	0.2412336268	1.6324129051
$x^{14}+x^{11}-x^{10}-x^7-x^4+x^3+1$	0.1823436598	1.3885013172
$x^{14}-x^{12}+x^7-x^2+1$	0.1844998024	1.3845721865
$x^{14}-x^{12}+x^{11}-x^9+x^7-x^5+x^3-x^2+1$	0.2272100851	1.4763006621
$x^{14}+x^{11}+x^{10}+x^9+x^8+x^7+x^6+x^5+x^4+x^3+1$	0.2351686174	1.4352060397
$x^{14}+x^{13}-x^8-x^7-x^6+x+1$	0.2368858459	1.2498299096
$x^{14}+x^{13}+x^{12}-x^9-x^8-x^7-x^6-x^5+x^2+x+1$	0.2453300143	1.3362661982
$x^{14}+x^{13}-x^{11}-x^7-x^3+x+1$	0.2469561884	1.3898540050

Known noncyclotomic $P \in \mathbb{Z}[x]$, $\deg P \leq 14$, $m(P) < 0.25$.

(Mossinghoff: `http://www.cecm.sfu.ca/~mjm/Lehmer/search/`)

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Lemma (Kurokawa, L., Ochiai)

$$m(x-1, x-\mathrm{e}^{2\pi i\alpha}, x-\mathrm{e}^{2\pi i\beta}) = -\frac{1}{4}\sum_{1\leq k,l}\frac{\cos 2\pi((k+l)\beta - l\alpha)}{kl(k+l)}$$
$$-\frac{1}{4}\sum_{1\leq k,m}\frac{\cos 2\pi((k+m)\alpha - m\beta)}{km(k+m)}$$
$$-\frac{1}{4}\sum_{1\leq l,m}\frac{\cos 2\pi(l\alpha + m\beta)}{lm(l+m)}.$$

Proposition

If $P(x)$ has the form

$$P(x) = \prod_{j=1}^{n}(x - \mathrm{e}^{2\pi i \alpha_j}),$$

with $0 \leq \alpha_1 \leq \cdots \leq \alpha_n < 1$, *then*

$$\begin{aligned} m_3(P) &= -\frac{3}{2}n^2\zeta(3) - 3n \sum_{1\leq k<l\leq n} C_3(2\pi(\alpha_l - \alpha_k)) \\ &\quad -3\pi \sum_{1\leq k<l\leq n} S_2(2\pi(\alpha_l - \alpha_k))\,(n(\alpha_l - \alpha_k) - (l-k))\,, \end{aligned}$$

where

$$C_\ell(t) = \sum_{n=1}^{\infty} \frac{\cos(nt)}{n^\ell} \quad \text{and} \quad S_\ell(t) = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n^\ell}.$$

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$$m_3\left(\frac{x^n-1}{x-1}\right)=\frac{3}{2}\zeta(3)\left(\frac{-2+3n-n^3}{n^2}\right)+\frac{3\pi}{2}\sum_{\substack{j=1\\ n\nmid j}}^{\infty}\frac{\cot\left(\pi\frac{j}{n}\right)}{j^2}.$$

Examples

$$m_3\left(x^2+x+1\right)=-\frac{10}{3}\zeta(3)+\frac{\sqrt{3}\pi}{2}L(2,\chi_{-3}).$$

$$m_3\left(x^3+x^2+x+1\right)=-\frac{81}{16}\zeta(3)+\frac{3\pi}{2}L(2,\chi_{-4}).$$

Lehmer's question for odd higher measure

Theorem

Let $P_n(x) = \frac{x^n-1}{x-1}$. *For* $h \geq 1$ *fixed,*

$$\lim_{n\to\infty} m_{2h+1}(P_n) = 0.$$

If we know that the sequence $m_{2h+1}(P_n)$ is nonconstant (a fact that is very reasonable to expect), then we obtain in this way a positive answer for Lehmer's question for m_{2h+1}.

Theorem (essentially Boyd, Lawton)

Let $P(x_1,\dots,x_n)\in\mathbb{C}[x_1,\dots,x_n]$ *and* $\mathbf{r}=(r_1,\dots,r_n)$, $r_i\in\mathbb{Z}_{>0}$. *Define* $P_{\mathbf{r}}(x)$ *as*

$$P_{\mathbf{r}}(x)=P(x^{r_1},\dots,x^{r_n}),$$

and let

$$q(\mathbf{r})=\min\left\{H(\mathbf{s}):\ \mathbf{s}=(s_1,\dots,s_n)\in\mathbb{Z}^n,\ \mathbf{s}\neq(0,\dots,0),\sum_{j=1}^n s_jr_j=0\right\},$$

Then

$$\lim_{q(\mathbf{r})\to\infty} m(P_{1\mathbf{r}},\dots,P_{l\mathbf{r}})=m(P_1,\dots,P_l).$$

$$\lim_{n\to\infty} m_{2h+1}\left(\frac{x^n-1}{x-1}\right) = m_{2h+1}\left(\frac{y-1}{x-1}\right) = 0.$$

Summing up

- Lehmer's question FALSE for m_{2h}, $h \geq 1$.
- Lehmer's question "TRUE" for m_{2h+1}, $h \geq 1$.
- $m_k(P)$ is interesting for P cyclotomic and that answers Lehmer's question.

Further questions

- $m_3(P)$ for P noncyclotomic?
- Lehmer's question for P noncyclotomic.
- Bounds for $m_{2h}(P)$, $h \geq 2$.
- Explain zeta values!!!

Thank you!

Merci!