

Higher Mahler measure and Lehmer's question

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Mahler measure for one-variable polynomials

Let $P(x)$ nonzero in $\mathbb{C}[x^\pm]$.

$$P(x) = a \prod_i (x - r_i)$$

The Mahler measure is defined by

$$M(P) = |a| \prod_{|r_i|>1} |r_i|, \quad m(P) = \log |a| + \sum_{|r_i|>1} \log |r_i|.$$

By Jensen's formula,

$$m(P) := \int_0^1 \log \left| P \left(e^{2\pi i \theta} \right) \right| d\theta.$$

Kronecker's Lemma

$P \in \mathbb{Z}[x]$, $P \neq 0$,

$$m(P) = 0 \Leftrightarrow P(x) = x^k \prod \Phi_{n_i}(x)$$

where Φ_{n_i} are cyclotomic polynomials.

Lehmer's question

Lehmer (1933):

Given $\epsilon > 0$, can we find a polynomial $P(x) \in \mathbb{Z}[x]$ such that $0 < m(P) < \epsilon$?

$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) \cong 0.162357612\dots$$

Is the above polynomial the best possible?.

$P \in \mathbb{C}[x]$ reciprocal iff

$$P(x) = \pm x^{\deg P} P(x^{-1}).$$

Theorem (Breusch, 1951, Smyth, 1971)

$P \in \mathbb{Z}[x]$ nonreciprocal,

$$m(P) \geq m(x^3 - x - 1) \cong 0.2811995743 \dots$$

Higher Mahler measure

For $k \in \mathbb{Z}_{\geq 0}$, the k -higher Mahler measure of P is

$$m_k(P) := \int_0^1 \log^k \left| P \left(e^{2\pi i \theta} \right) \right| d\theta.$$

$$k = 1 : \quad m_1(P) = m(P),$$

and

$$m_0(P) = 1.$$

The simplest examples

(Kurokawa, L., Ochiai, 2008)

$$m_2(x-1) = \frac{\zeta(2)}{2} = \frac{\pi^2}{12}.$$

$$m_3(x-1) = -\frac{3\zeta(3)}{2}.$$

$$m_4(x-1) = \frac{3\zeta(2)^2 + 21\zeta(4)}{4} = \frac{19\pi^4}{240}.$$

$$m_5(x-1) = -\frac{15\zeta(2)\zeta(3) + 45\zeta(5)}{2}.$$

$$m_6(x-1) = \frac{45}{2}\zeta(3)^2 + \frac{275}{1344}\pi^6.$$

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{m_k(x-1)}{k!} s^k &= \int_0^1 |e^{2\pi i\theta} - 1|^s d\theta \\ &= \frac{\Gamma(s+1)}{\Gamma\left(\frac{s}{2}+1\right)^2} = \exp\left(\sum_{k=2}^{\infty} \frac{(-1)^k(1-2^{1-k})\zeta(k)}{k} s^k\right). \end{aligned}$$

Transform the integral into a Beta function, then use the Weierstrass product of the Γ -function.

Lehmer's question for even higher measure

Theorem (L., Sinha, 2010)

If $P(x) \in \mathbb{Z}[x]$, then for any $h \geq 1$,

$$m_{2h}(P) \geq \begin{cases} \left(\frac{\pi^2}{12}\right)^h, & \text{if } P(x) \text{ is reciprocal,} \\ \left(\frac{\pi^2}{48}\right)^h, & \text{if } P(x) \text{ is non-reciprocal.} \end{cases}$$

- $m_2(P)$ for P reciprocal is minimized when P is a product of monomials and cyclotomic polynomials.
- $m_2(P) \geq \frac{\pi^2}{12}$ for P a product of monomials and cyclotomic polynomials.
- For P nonreciprocal, take $P(x)x^{\deg P}P(x^{-1})$.
- For $m_{2h}(P)$, use Hölder's inequality.

$$m_2(x^3 + x + 1) \simeq 0.3275495729\dots$$

Proposition

If $P(x)$ has the form

$$P(x) = \prod_{j=1}^n (x - e^{2\pi i \alpha_j}),$$

with $0 \leq \alpha_1 \leq \dots \leq \alpha_n < 1$, then

$$m_2(P) = \frac{\pi^2}{2} \sum_{1 \leq j, k \leq n} \left((\alpha_j - \alpha_k)^2 - |\alpha_j - \alpha_k| + \frac{1}{6} \right).$$

Multiple Mahler measure

Let $P_1, \dots, P_k \in \mathbb{C}[x^\pm]$ be non-zero Laurent polynomials. Their multiple higher Mahler measure is defined by

$$m(P_1, \dots, P_k) = \int_0^1 \log |P_1(e^{2\pi i\theta})| \cdots \log |P_k(e^{2\pi i\theta})| d\theta.$$

$$m_2 \left(\prod_i (x - r_i) \right) = \sum_{i,j} m(x - r_i, x - r_j).$$

Multiple Mahler measure for simple polynomials

Lemma (Kurokawa, L., Ochiai, 2008)

For $0 \leq \alpha \leq 1$

$$m(x - 1, x - e^{2\pi i \alpha}) = \frac{\pi^2}{2} \left(\alpha^2 - \alpha + \frac{1}{6} \right).$$

Examples

$$m(x - 1, x + 1) = -\frac{\pi^2}{24},$$

$$m(x - 1, x \pm i) = -\frac{\pi^2}{96},$$

$$m(x - 1, x - e^{2\pi i \alpha}) = 0 \Leftrightarrow \alpha = \frac{3 \pm \sqrt{3}}{6}.$$

m_2 of cyclotomic polynomials

Proposition (L., Sinha, 2010)

- For any two positive integers a and b ,

$$m(x^a - 1, x^b - 1) = \frac{\pi^2}{12} \frac{(a, b)^2}{ab}.$$

- For a positive integer n , let $\phi_n(x)$ denote the n -th cyclotomic polynomial and φ Euler's function. Then

$$m_2(\phi_n(x)) = \frac{\pi^2}{12} \frac{\varphi(n) 2^{r(n)}}{n},$$

where $r(n)$ denotes the number of distinct prime divisors of n .

Proof involves:

For any two positive coprime integers a and b ,

$$S(a, b) := \sum_{j=0}^{a-1} \sum_{k=0}^{b-1} \left| \frac{k}{b} - \frac{j}{a} \right| = \frac{2a^2b^2 - 3ab + a^2 + b^2 - 1}{6ab}.$$

m_2 of noncyclotomic polynomials

Lemma (Kurokawa, L., Ochiai, 2008)

For $a, b \in \mathbb{C}$,

$$m(1 - ax, 1 - bx) = \begin{cases} \frac{1}{2} \operatorname{Re} \operatorname{Li}_2(\bar{a}b) & \text{if } |a|, |b| \leq 1, \\ \frac{1}{2} \operatorname{Re} \operatorname{Li}_2(b/\bar{a}) & \text{if } |a| \geq 1, |b| \leq 1, \\ \frac{1}{2} \operatorname{Re} \operatorname{Li}_2(1/\bar{a}b) + \log |a| \log |b| & \text{if } |a|, |b| \geq 1, \end{cases}$$

where Li_2 is the dilogarithm function.

$$\operatorname{Li}_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2} \quad |z| < 1$$

m_2 of noncyclotomic polynomials

$P(x)$	$m(P)$	$m_2(P)$
$x^8 + x^5 - x^4 + x^3 + 1$	0.2473585132	1.0980813745
$x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1$	0.1623576120	1.7447964556
$x^{10} - x^6 + x^5 - x^4 + 1$	0.1958888214	1.2863292447
$x^{10} + x^7 + x^5 + x^3 + 1$	0.2073323581	1.2320444893
$x^{10} - x^8 + x^5 - x^2 + 1$	0.2320881973	1.1704950485
$x^{10} + x^8 + x^7 + x^5 + x^3 + x^2 + 1$	0.2368364616	1.1914083866
$x^{10} + x^9 - x^5 + x + 1$	0.2496548880	1.0309287773
$x^{12} + x^{11} + x^{10} - x^8 - x^7 - x^6 - x^5 - x^4 + x^2 + x + 1$	0.2052121880	1.4738375004
$x^{12} + x^{11} + x^{10} + x^9 - x^6 + x^3 + x^2 + x + 1$	0.2156970336	1.5143823478
$x^{12} + x^{11} - x^7 - x^6 - x^5 + x + 1$	0.2239804947	1.2059443050
$x^{12} + x^{10} + x^7 - x^6 + x^5 + x^2 + 1$	0.2345928411	1.2434560052
$x^{12} + x^{10} + x^9 + x^8 + 2x^7 + x^6 + 2x^5 + x^4 + x^3 + x^2 + 1$	0.2412336268	1.6324129051
$x^{14} + x^{11} - x^{10} - x^7 - x^4 + x^3 + 1$	0.1823436598	1.3885013172
$x^{14} - x^{12} + x^7 - x^2 + 1$	0.1844998024	1.3845721865
$x^{14} - x^{12} + x^{11} - x^9 + x^7 - x^5 + x^3 - x^2 + 1$	0.2272100851	1.4763006621
$x^{14} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + 1$	0.2351686174	1.4352060397
$x^{14} + x^{13} - x^8 - x^7 - x^6 + x + 1$	0.2368858459	1.2498299096
$x^{14} + x^{13} + x^{12} - x^9 - x^8 - x^7 - x^6 - x^5 + x^2 + x + 1$	0.2453300143	1.3362661982
$x^{14} + x^{13} - x^{11} - x^7 - x^3 + x + 1$	0.2469561884	1.3898540050

Known noncyclotomic $P \in \mathbb{Z}[x]$, $\deg P \leq 14$, $m(P) < 0.25$.

(Mossinghoff: <http://www.cecm.sfu.ca/~mjm/Lehmer/search/>)

Lemma (Kurokawa, L., Ochiai, 2008)

$$\begin{aligned} m(x-1, x-e^{2\pi i\alpha}, x-e^{2\pi i\beta}) &= -\frac{1}{4} \sum_{1 \leq k, l} \frac{\cos 2\pi((k+l)\beta - l\alpha)}{kl(k+l)} \\ &\quad -\frac{1}{4} \sum_{1 \leq k, m} \frac{\cos 2\pi((k+m)\alpha - m\beta)}{km(k+m)} \\ &\quad -\frac{1}{4} \sum_{1 \leq l, m} \frac{\cos 2\pi(l\alpha + m\beta)}{lm(l+m)}. \end{aligned}$$

Proposition

If $P(x)$ has the form

$$P(x) = \prod_{j=1}^n (x - e^{2\pi i \alpha_j}),$$

with $0 \leq \alpha_1 \leq \dots \leq \alpha_n < 1$, then

$$\begin{aligned} m_3(P) &= -\frac{3}{2}n^2\zeta(3) - 3n \sum_{1 \leq k < l \leq n} C_3(2\pi(\alpha_l - \alpha_k)) \\ &\quad - 3\pi \sum_{1 \leq k < l \leq n} S_2(2\pi(\alpha_l - \alpha_k))(n(\alpha_l - \alpha_k) - (l - k)), \end{aligned}$$

where

$$C_\ell(t) = \sum_{n=1}^{\infty} \frac{\cos(nt)}{n^\ell} \quad \text{and} \quad S_\ell(t) = \sum_{n=1}^{\infty} \frac{\sin(nt)}{n^\ell}.$$

Proposition

Let $(a, b, c) = 1$. For integers d, m , let $d_m = \frac{d}{(d, m)}$ and $m_d = \frac{m}{(d, m)}$. Denote by $[d_m^{-1}]_{m_d}$ the inverse of d_m modulo m_d .

$$\begin{aligned} m(x^a - 1, x^b - 1, x^c - 1) &= -\frac{abc}{4[a, b]^3} \zeta(3) \\ &+ \frac{\pi}{4c(a, b)} \sum_{\substack{h=1 \\ b_a \nmid ch}}^{\infty} \frac{\cot\left(\pi \frac{[a_b^{-1}]_{b_a} ch}{b_a}\right)}{h^2} \\ &+ \text{permutations} \end{aligned}$$

$$m_3 \left(\frac{x^n - 1}{x - 1} \right) = \frac{3}{2} \zeta(3) \left(\frac{-2 + 3n - n^3}{n^2} \right) + \frac{3\pi}{2} \sum_{\substack{j=1 \\ n \nmid j}}^{\infty} \frac{\cot \left(\pi \frac{j}{n} \right)}{j^2}.$$

Examples

$$m_3(x^2 + x + 1) = -\frac{10}{3} \zeta(3) + \frac{\sqrt{3}\pi}{2} L(2, \chi_{-3}).$$

$$m_3(x^3 + x^2 + x + 1) = -\frac{81}{16} \zeta(3) + \frac{3\pi}{2} L(2, \chi_{-4}).$$

Lehmer's question for odd higher measure

Theorem (L., Sinha, 2010)

Let $P_n(x) = \frac{x^n - 1}{x - 1}$. For $h \geq 1$ fixed,

$$\lim_{n \rightarrow \infty} m_{2h+1}(P_n) = 0.$$

Moreover, the sequence $m_{2h+1}(P_n)$ is nonconstant.

- $$\lim_{n \rightarrow \infty} m_{2h+1} \left(\frac{x^n - 1}{x - 1} \right) = m_{2h+1} \left(\frac{y - 1}{x - 1} \right) = 0.$$

- $$m_{2h+1} \left(\frac{x^n - 1}{x - 1} \right) = C(h) \frac{\log^{2h-1} n}{n} \left(1 + O(\log^{-1} n) \right).$$

(suggested by Soundararajan)

Summing up

- Lehmer's question FALSE for m_{2h} , $h \geq 1$.
- Lehmer's question TRUE for m_{2h+1} , $h \geq 1$.
- $m_k(P)$ is interesting for P cyclotomic and that answers Lehmer's question.

Further questions

- $m_3(P)$ for P noncyclotomic?
- Lehmer's question for P noncyclotomic.
- Bounds for $m_{2h}(P)$, $h \geq 2$.
- Explain zeta values!!!

Thank you!
Merci!