On certain examples of Mahler Measures as Multiple Polylogarithms

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1. Mahler Measure

Definition 1 *Given a polynomial* $P(x) = a_d x^d + a_{d-1}x^{d-1} + \ldots + a_0 = a_d \prod_{n=1}^{d}(x - \alpha_n)$ *with complex coefficients, define the* Mahler measure *of* P *as*

$$M(P) := |a_d| \prod_{n=1}^{d} \max\{1, |\alpha_n|\} \tag{1}$$

For $P \in \mathbb{C}[x_1, \ldots, x_n]$*, the (logarithmic)* Mahler measure *is defined by*

$$\begin{aligned} m(P) \quad &:= \quad \int_0^1 \ldots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n & (2) \\ &= \quad \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \ldots \frac{\mathrm{d}x_n}{x_n} & (3) \end{aligned}$$

It is possible to prove that this integral is not singular and that $m(P)$ always exists.

Because of Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log^+ |\alpha|^1 \tag{4}$$

we have

$$m(P) = \log M(P) \tag{5}$$

in the one variable case. We use this equality to define $M(P)$ for polynomials in several variables.

Proposition 2 *For* $P, Q \in \mathbb{C}[x_1, \ldots, x_n]$

$$m(P \cdot Q) = m(P) + m(Q) \tag{6}$$

Proposition 3 *Let* $P \in \mathbb{C}[x_1, \ldots, x_n]$ *such that* $a_{m_1,\ldots,m_n}$ *is the coefficient of* $x_1^{m_1} \ldots x_n^{m_n}$ *and* P *has degree* d_i *in* x_i*. Then*

$$|\, a_{m_1,\ldots,m_n}| \leq \binom{d_1}{m_1} \ldots \binom{d_n}{m_n} M(P) \tag{7}$$

$$M(P) \leq L(P) \leq 2^{d_1 + \ldots + d_n} M(P) \tag{8}$$

where $L(P)$ *is the length of the polynomial, the sum of the absolute values of the coefficients.*

It is also true that $m(P) \geq 0$ if P has integral coefficients.

Let us see some examples

[1] $\log^+ x = \log \max\{1, x\}$ for $x \in \mathbb{R}_{\geq 0}$

Examples 4

- By Kronecker's Lemma, $P \in \mathbb{Z}[x]$, $P \neq 0$, then $m(P) = 0$ if and only if P is the product of powers of x and cyclotomic polynomials.

- Lehmer:

  $m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) = \log(1.176280818\ldots) = 0.162357612\ldots$

  So far, no one has been able to find a polynomial with integral coefficients whose logarithmic Mahler measure is greater than zero and smaller than the Mahler measure of this example found by Lehmer in the 30's.

  The following questions are still open: Is there a lower bound for the Mahler measure of polynomials in one variable with integral coefficients? Does this degree 10 polynomial reach the lower bound?

- Smyth:
$$m(x + y + 1) = \frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3}, 2) = \mathrm{L}'(\chi_{-3}, -1) \tag{9}$$
  where
$$\mathrm{L}(\chi_{-3}, s) = \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s} \quad \text{and} \quad \chi_{-3}(n) = \begin{cases} 1 & \text{if} \quad n \equiv 1 \bmod 3 \\ -1 & \text{if} \quad n \equiv -1 \bmod 3 \\ 0 & \text{if} \quad n \equiv 0 \bmod 3 \end{cases}$$

- Smyth:
$$m(x + y + z + 1) = \frac{7}{2\pi^2}\zeta(3) \tag{10}$$

Let us also mention the following result:

Theorem 5 *For $P \in \mathbb{C}[x_1, \ldots, x_n]$*
$$\lim_{k_2 \to \infty} \ldots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \ldots x^{k_n})) = m(P(x_1, \ldots x_n)) \tag{11}$$

In particular Lehmer's problem in several variables reduces to the one variable case.

2. Dilogarithm

Definition 6 *The* Dilogarithm *is the function defined by the power series*
$$\mathrm{Li}_2(x) := \sum_{n=1}^{\infty} \frac{x^n}{n^2} \qquad x \in \mathbb{C}, \quad |x| < 1 \tag{12}$$

This function can be continued analytically to $\mathbb{C} \setminus (1, \infty)$ via the integral
$$\mathrm{Li}_2(x) := -\int_0^x \log(1 - t)\frac{\mathrm{d}t}{t} = -\int_0^1 \int_0^t \frac{\mathrm{d}s}{s - \frac{1}{x}} \frac{\mathrm{d}t}{t}$$

The dilogarithm has a jump of $2\pi\mathrm{i}\log|x|$ when it crosses the line in $(1, \infty)$. We consider the following modified version:

Definition 7 *The* Bloch – Wigner Dilogarithm *is defined by*

$$D(x) := \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1-x)\log|x| \tag{13}$$

This function turns out to be real analytic in $\mathbb{C} \setminus \{0,1\}$ and continuous in $\mathbb{C}$. Besides,

$$D(x) = -D(\bar{x}) \qquad (\Rightarrow D(x) = 0 \quad \text{for} \quad x \in \mathbb{R}) \tag{14}$$

It satisfies lots of functional equations, but they all turn out to be formal consequences of the following five-term relation:

$$D(x) + D(1-xy) + D(y) + D\left(\frac{1-y}{1-xy}\right) + D\left(\frac{1-x}{1-xy}\right) = 0 \tag{15}$$

In the limit cases (taking $y = 0$ and $y = x^{-1}$), this equation gives the following useful consequences:

$$D(z) = D\left(1 - \frac{1}{z}\right) = D\left(\frac{1}{1-z}\right) = -D\left(\frac{1}{z}\right) = -D\left(\frac{z}{z-1}\right) = -D(1-z) \tag{16}$$

The Dilogarithm appears as the Mahler measure of certain polynomials in two variables. Perhaps the simplest example is Maillot's:

$$\pi m(ax+by+c) = \begin{cases} D\left(\left|\frac{a}{b}\right| \mathrm{e}^{i\gamma}\right) + \alpha\log|a| + \beta\log|b| + \gamma\log|c| & \triangle \\ \\ \pi\log\max\{|a|,|b|,|c|\} & \text{not } \triangle \end{cases} \tag{17}$$

Here $\triangle$ stands for the fact of whether $|a|$, $|b|$, and $|c|$ are the lengths of the sides of a triangle, and α, β, and γ are the angles opposite to the sides of lengths $|a|$, $|b|$, and $|c|$ respectively.

3. Polylogarithms

In analogy with the dilogarithm, we have the polylogarithms

Definition 8 Multiple polylogarithms *are defined as the power series*

$$\mathrm{Li}_{k_1,\ldots,k_m}(x_1,\ldots,x_m) := \sum_{0<n_1<n_2<\ldots<n_m} \frac{x_1^{n_1}x_2^{n_2}\ldots x_m^{n_m}}{n_1^{k_1}n_2^{k_2}\ldots n_m^{k_m}} \tag{18}$$

which are convergent for $|x_i| < 1$. The weight of a polylogarithm function is the number $w = k_1 + \ldots + k_m$ and its length is the number m.

Definition 9 Hyperlogarithms *are defined as the iterated integrals*

$$\mathrm{I}_{k_1,\ldots,k_m}(a_1 : \ldots : a_m : a_{m+1}) :=$$

$$\int_0^{a_{m+1}} \underbrace{\frac{\mathrm{d}t}{t-a_1} \circ \frac{\mathrm{d}t}{t} \circ \ldots \circ \frac{\mathrm{d}t}{t}}_{k_1} \circ \underbrace{\frac{\mathrm{d}t}{t-a_2} \circ \frac{\mathrm{d}t}{t} \circ \ldots \circ \frac{\mathrm{d}t}{t}}_{k_2} \circ \ldots \circ \underbrace{\frac{\mathrm{d}t}{t-a_m} \circ \frac{\mathrm{d}t}{t} \circ \ldots \circ \frac{\mathrm{d}t}{t}}_{k_m}$$

where k_i are integers, a_i are complex numbers, and

$$\int_0^{b_{l+1}} \frac{\mathrm{d}t}{t-b_1} \circ \ldots \circ \frac{\mathrm{d}t}{t-b_l} = \int_{0\le t_1\le\ldots\le t_l\le b_{l+1}} \frac{\mathrm{d}t_1}{t_1-b_1} \cdots \frac{\mathrm{d}t_l}{t_l-b_l}$$

The value of the integral above only depends on the homotopy class of the path connecting 0 and a_{m+1} on $\mathbb{C} \setminus \{a_1, \dots, a_m\}$.

It is easy to see that,

$$\begin{aligned} \mathrm{I}_{k_1,\dots,k_m}(a_1 : \dots : a_m : a_{m+1}) &= (-1)^m \mathrm{Li}_{k_1,\dots,k_m}\left(\frac{a_2}{a_1}, \frac{a_3}{a_2}, \dots, \frac{a_m}{a_{m-1}}, \frac{a_{m+1}}{a_m}\right) \\ \mathrm{Li}_{k_1,\dots,k_m}(x_1, \dots, x_m) &= (-1)^m \mathrm{I}_{k_1,\dots,k_m}\left(\frac{1}{x_1, \dots, x_m} : \dots : \frac{1}{x_m} : 1\right) \end{aligned}$$

which gives an analytic continuation to multiple polylogarithms.

4. Examples of higher weight

We have obtained examples of polynomials in several variables whose Mahler measures depend on polylogarithms. The first column of the table shows the polynomials. Here α is a complex number different from zero. The second column indicates the values of the first column for the case $\alpha = 1$.

$\pi m((1+y) + \alpha(1-y)x)$	$2\mathrm{L}(\chi_{-4}, 2)$
$\pi^2 m((1+w)(1+y) + \alpha(1-w)(1-y)x)$	$7\zeta(3)$
$\pi^3 m((1+v)(1+w)(1+y) + \alpha(1-v)(1-w)(1-y)x)$	$7\pi\zeta(3) + 4\sum_{0\le j<k} \frac{(-1)^j}{(2j+1)^2 k^2}$
$\pi^2 m((1+x) + \alpha(y+z))$	$\frac{7}{2}\zeta(3)$
$\pi^3 m((1+w)(1+x) + \alpha(1-w)(y+z))$	$2\pi^2 \mathrm{L}(\chi_{-4}, 2) + 8\sum_{0\le j<k} \frac{(-1)^{j+k+1}}{(2j+1)^3 k}$
$\pi^4 m((1+v)(1+w)(1+x) + \alpha(1-v)(1-w)(y+z))$	$7\pi^2\zeta(3) + 8\sum_{0\le j<k} \frac{1}{(2j+1)^3 k^2}$
$\pi^2 m((1+w)(1+y) + (1-w)(x-y))$	$\frac{7}{2}\zeta(3) + \frac{\pi^2}{2}\log 2$

Here χ_{-4} is the real odd character of conductor 4, i.e.

$$\chi_{-4}(n) = \begin{cases} 1 & \text{if} \quad n \equiv 1 \bmod 4 \\ -1 & \text{if} \quad n \equiv -1 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

Let us observe that all the presented formulae share a common feature. If we assign weight 1 to any Mahler measure and to π, then all the formulae are homogeneous, meaning all the monomials have the same weight, and this weight is equal to the number of variables of the corresponding polynomial.

The idea behind those computations is the following.

1. Let $P_\alpha \in \mathbb{C}[x_1, \dots, x_n]$ whose coefficients depend on a parameter $\alpha \in \mathbb{C}$. For instance, start with $P_\alpha(x) = 1 + \alpha x$, whose Mahler measure is $\log^+ |\alpha|$.

2. We replace α by $\alpha\frac{1-y}{1+y}$ and obtain a polynomial $\tilde{P}_\alpha \in \mathbb{C}[x_1, \dots, x_n, y]$. In the example, $\tilde{P}_\alpha(x, y) = 1 + y + \alpha(1-y)x$.

3. The Mahler measure of the second polynomial is a certain integral of the Mahler measure of the first polynomial.

$$m(\tilde{P}_\alpha) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} m\left(P_{\alpha \frac{1-y}{1+y}}\right) \frac{\mathrm{d}y}{y}$$

4. If the Mahler measure depends only on the absolute value of $|\alpha|$, we can make a change of variables $u = \left|\alpha \frac{1-y}{1+y}\right|$ (to be precise, first write $y = \mathrm{e}^{\mathrm{i}\theta}$ and then make $u = |\alpha| \tan\left(\frac{\theta}{2}\right)$).We obtain,

$$m(\tilde{P}_\alpha) = \frac{2}{\pi} \int_0^\infty m\left(P_u\right) \frac{|\alpha|\,\mathrm{d}u}{u^2 + |\alpha|^2} = \frac{\mathrm{i}}{\pi} \int_0^\infty m\left(P_u\right) \left(\frac{1}{u + \mathrm{i}|\alpha|} - \frac{1}{u - \mathrm{i}|\alpha|}\right) \mathrm{d}u$$

In the example,

$$\begin{aligned} m(1 + y + \alpha(1 - y)x) &= \frac{\mathrm{i}}{\pi} \int_0^\infty \log^+ u \left(\frac{1}{u + \mathrm{i}|\alpha|} - \frac{1}{u - \mathrm{i}|\alpha|}\right) \mathrm{d}u \\ &= \frac{\mathrm{i}}{\pi} \int_0^1 \int_s^1 \frac{\mathrm{d}t}{t} \left(\frac{1}{s + \frac{\mathrm{i}}{|\alpha|}} - \frac{1}{s - \frac{\mathrm{i}}{|\alpha|}}\right) \mathrm{d}s \end{aligned}$$

$$= \frac{\mathrm{i}}{\pi}\left(\mathrm{I}_2\left(-\frac{\mathrm{i}}{|\alpha|} : 1\right) - \mathrm{I}_2\left(\frac{\mathrm{i}}{|\alpha|} : 1\right)\right) = -\frac{\mathrm{i}}{\pi}(\mathrm{Li}_2(\mathrm{i}|\alpha|) - \mathrm{Li}_2(-\mathrm{i}|\alpha|))$$

5. Newton Polytope

Definition 10 *Given $P \in \mathbb{C}[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$, write*

$$P(\mathbf{x}) = \sum_{\mathbf{m} \in J} a_{\mathbf{m}} \mathbf{x}^{\mathbf{m}}$$

where $J \subset \mathbb{Z}^n$ is a finite set, $\mathbf{x}^{\mathbf{m}} = x_1^{m_1} \ldots x_n^{m_n}$ and $a_{\mathbf{m}} \neq 0$ for $\mathbf{m} \in J$. We define the Newton polytope to be

$$\Delta(P) = \text{convex hull}\,(J) \tag{19}$$

A face $\tau < \Delta$ is a subset of the form $\Delta \cap H$ where H is a hyperplane H in $\mathbb{R}^n$, such that $\Delta \subset H^+$ or H^-. A facet is a face of dimension one less than the dimension of Δ.

Definition 11 *Let $\tau = \Delta \cap H$ a face of the Newton polytope, then*

$$P_\tau(\mathbf{x}) = \sum_{\mathbf{m} \in J \cap H} a_{\mathbf{m}} \mathbf{x}^{\mathbf{m}} \tag{20}$$

Proposition 12

$$m(P) \geq m(P_\tau) \tag{21}$$

We will need the following

Definition 13 *We say that $P \in \mathbb{C}[x, y, x^{-1}, y^{-1}]$ is* tempered *if the roots of P_τ are roots of the unity (for τ facets of Δ) . If the P_τ have integral coefficients, this is equivalent to ask that $m(P_\tau) = 0$ by Kronecker's Lemma.*

6. Regulator

Given a smooth projective curve C and x, y rational functions ($x, y \in \mathbb{C}(C)$), define

$$\eta(x, y) = \log|x| \mathrm{d}\arg(y) - \log|y| \mathrm{d}\arg(x) \tag{22}$$

Here

$$\mathrm{d}\arg x = \mathrm{Im}\left(\frac{\mathrm{d}x}{x}\right) \tag{23}$$

is well defined in $\mathbb{C}$ in spite of the fact that arg is not. η is a 1-form in $C \setminus S$, where S is the set of zeros and poles of x and y. It is also closed, because of

$$\mathrm{d}\eta = \mathrm{Im}\left(\frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y}\right) = 0$$

Let $P \in \mathbb{C}[x, y]$. Write

$$P(x, y) = a_d(x)y^d + \ldots + a_0(x)$$

$$P(x, y) = a_d(x) \prod_{n=1}^{d} (y - \alpha_n(x))$$

Then by Jensen's formula,

$$m(P) = m(a_d) + \frac{1}{2\pi \mathrm{i}} \sum_{n=1}^{d} \int_{\mathbb{T}^1} \log^+ |\alpha_n(x)| \frac{\mathrm{d}x}{x} = m(a_d) - \frac{1}{2\pi} \int_\gamma \eta(x, y) \tag{24}$$

Here γ is the union of paths in $C = \{P(x, y) = 0\}$ where $|x| = 1$ and $|y| \geq 1$. Also note that $\partial\gamma = \{(x, y) \in \mathbb{C}^2 \mid |x| = |y| = 1, P(x, y) = 0\}$

We want to arrive to one of these two situations:

1. η is exact, and $\partial\gamma \neq 0$. In this case we can integrate using Stokes Theorem.
2. η is not exact and $\partial\gamma = 0$. In this case we can compute the integral by using Residue's Theorem.

We will associate η with an element in $H^1(C \setminus S, \mathbb{R})$ in the following way. Given $[\gamma] \in H_1(C \setminus S, \mathbb{Z})$,

$$[\gamma] \to \int_\gamma \eta \tag{25}$$

(we identify $H^1(C \setminus S, \mathbb{R})$ with $H_1(C \setminus S, \mathbb{Z})'$).

Given $s \in C$, it induces a valuation in $\mathbb{C}(C)$: for $f \in \mathbb{C}(C)$, $v_s(f)$ is the order of f at s. We have the following,

Proposition 14 *For $s \in C$*

$$\oint_s \eta(x,y) = \log |(x,y)_s| \tag{26}$$

where

$$(x,y)_s = (-1)^{v_s(x)v_s(y)} \left. \frac{x^{v_s(y)}}{y^{v_s(x)}} \right|_s \tag{27}$$

is the tame symbol.

Corollary 15 *η extends to all of C if and only if $|(x,y)_s| = 1$ for every s. In particular, if $(x,y)_s \in \mu_\infty$, then η extends to all of C.*

We have the following,

Theorem 16 *For $C = \mathbb{P}^1$,*

$$\eta(t, 1-t) = \mathrm{d}D(t) \tag{28}$$

Using $x : C \longrightarrow \mathbb{P}^1$, we can extend the above result to any C.

We can put all of these together:

Theorem 17 *The function*

$$\eta : \mathbb{Q}(C)^* \times \mathbb{Q}(C)^* \longrightarrow H^1(C, \mathbb{R})$$

satisfies the following properties

1. $\eta(x,y) = -\eta(y,x)$
2. $\eta(x_1x_2, y) = \eta(x_1, y) + \eta(x_2, y)$
3. $\eta(x, 1-x) = 0$

then, it is a symbol, and can be factored through $K_2(\mathbb{Q}(C))$.

If we want $\eta(x,y)$ to be an exact form, it has to be trivial in $H^1(C, \mathbb{R})$. One way to guarantee this, is that $\{x,y\}$ is trivial in $K_2(\mathbb{Q}(C)) \otimes \mathbb{Q}$. Since the tame symbols are morphisms on $K_2(\mathbb{Q}(C))$, we need them to be torsion elements, so they have to be roots of the unit. This condition is easy to verify in practise according to the following:

Theorem 18 *$(x,y)_s \in \mu_\infty$ if and only if P is tempered.*

In general, if

$$x \wedge y = \sum_j r_j z_j \wedge (1 - z_j)$$

in $\bigwedge^2(\mathbb{Q}(C)^*) \otimes \mathbb{Q}$, then

$$\eta(x,y) = \mathrm{d}\left(\sum_j r_j D(z_j)\right) = \mathrm{d}D\left(\sum_j r_j [z_j]\right)$$

We have $\gamma \subset C$ such that

$$\partial\gamma = \sum_k \epsilon_k [w_k] \quad \epsilon_k = \pm 1$$

where $w_k \in C(\mathbb{C})$, $|x(w_k)| = |y(w_k)| = 1$. Then

$$2\pi m(P) = D(\xi) \quad \text{for } \xi = \sum_k \sum_j r_j [z_j(w_k)]$$

In order to interpret Smyth's case, take $P(x, y) = x + y - 1$. Writing $x = \mathrm{e}^{2\pi\mathrm{i}\theta}$, the path of integration becomes

$$\gamma(\theta) = 1 - e^{2\pi\mathrm{i}\theta}, \quad \theta \in [1/6\,; 5/6] \quad \Rightarrow \quad \partial\gamma = [\xi_6] - [\bar{\xi}_6]$$

$$2\pi m(x + y - 1) = D(\xi_6) - D(\bar{\xi}_6)$$

The following Theorem helps relating the Mahler measure to special values of zeta functions.

Theorem 19 *(Borel, Suslim, etc) Let F be a number field, $r_2 = 1$, and*

$$\xi = \sum_j n_j [a_j] \quad a_j \in F \setminus \{0, 1\}$$

If

$$\sum_j n_j a_j \wedge (1 - a_j) = 0 \quad \text{in } \bigwedge{}^2 (F^*) \otimes \mathbb{Q}$$

Then,

$$D(\sigma(\xi)) \sim_{\mathbb{Q}^*} \frac{|\Delta_F|^{3/2}}{\pi^{2(n-1)}} \zeta_F(2) \tag{29}$$

where σ is any of the two complex embeddings.

If ξ comes from the Mahler measure, the condition is satisfied if and only if

$$x(w_k) \wedge y(w_k) = 0$$

7. Hyperbolic Geometry

Consider the space $\mathbb{H}^3$ which can be represented as $\mathbb{C} \times \mathbb{R}_{\geq 0} \cup \{\infty\}$. In this space the geodesics are either vertical lines or semicircles in vertical planes with endpoints in $\mathbb{C} \times \{0\}$.

An *ideal tetrahedron* is a tetrahedron whose vertices are all in $\mathbb{C} \times \{0\} \cup \{\infty\} = \mathbb{P}^1(\mathbb{C})$. Such a tetrahedron Δ with vertices z_0, z_1, z_2, z_3 has a hyperbolic volume equal to

$$\mathrm{Vol}(\Delta) = D((z_0 : z_1 : z_2 : z_3)) \tag{30}$$

Where $(z_0 : z_1 : z_2 : z_3) = \frac{z_0 - z_2}{z_0 - z_3} \frac{z_1 - z_3}{z_1 - z_2}$ is the cross ratio. The invariance of the formula by the action of $PSL_2(\mathbb{C})$ is in agreement with the fact that this is the group of isometries (preserving orientation) of $\mathbb{H}^3$.

These ideal tetrahedra are important because any completed oriented 3-manifold with finite volume can be decomposed in ideal tetrahedra. In such a decomposition, the parameters z_i of the ideal tetrahedra $(\infty, 0, 1, z_i)$ are constrained to the following condition:

$$\sum_{i=1}^{n} z_i \wedge (1 - z_i) = 0 \tag{31}$$

These decompositions provide more identities of the dilogarithm.

More generally, if P is the A-polynomial of a compact, oriented, hyperbolic 3-manifold with one cusp, then

$$x \wedge y = \sum_{i=1}^{n} z_i \wedge (1 - z_i) \quad \Rightarrow \quad \eta(x, y) = \mathrm{dVol} \tag{32}$$

The A-polynomial can be deduced from the gluing equations without imposing the completing equations. The completing equations will impose the condition $x = y = 1$ and we will recover formula (31).

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