Functional equations for Mahler measures of genus-one curves

(joint with Mat Rogers, University of British Columbia)

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Mahler measure of one-variable polynomials

Pierce (1918): $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - \alpha_i)$$

$$\Delta_n = \prod_i (\alpha_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Lehmer (1933):

$$\lim_{n\to\infty} \frac{|\alpha^{n+1} - 1|}{|\alpha^n - 1|} = \begin{cases} |\alpha| & \text{if } |\alpha| > 1 \\ 1 & \text{if } |\alpha| < 1 \end{cases}$$

For

$$P(x) = a \prod_i (x - \alpha_i)$$

$$M(P) = |a| \prod_i \max\{1, |\alpha_i|\}$$

$$m(P) = \log M(P) = \log |a| + \sum_i \log^+ |\alpha_i|$$

- Kronecker's Lemma: $P \in \mathbb{Z}[x]$, $P \neq 0$,

$$m(P) = 0 \Leftrightarrow P(x) = x^n \prod \phi_i(x)$$

- Lehmer's Question (1933): Does there exist $C > 0$ such that $P(x) \in \mathbb{Z}[x]$

$$m(P) = 0 \quad \text{or} \quad m(P) > C??$$

Is

$$m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1)$$

$$= 0.162357612\ldots$$

the best possible?

$$\sqrt{\Delta_{379}} = 1,794,327,140,357$$

Mahler measure of multivariable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \ldots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \ldots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

Jensen's formula:

$$\int_0^1 \log |\mathrm{e}^{2\pi \mathrm{i}\theta} - \alpha| \mathrm{d}\theta = \log^+ |\alpha|$$

recovers one-variable case.

- $m(P \cdot Q) = m(P) + m(Q)$

- $m(P) \geq 0$ if P has integral coefficients.

- $\alpha \in \bar{\mathbb{Q}}$, P_α is its minimal polynomial over $\mathbb{Q}$, then

$$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}]\, h(\alpha),$$

  where h is the logarithmic Weil height.

- Boyd–Lawton: $P \in \mathbb{C}[x_1, \ldots, x_n]$

$$\lim_{k_2 \to \infty} \ldots \lim_{k_n \to \infty} m(P(x, x^{k_2}, \ldots, x^{k_n}))$$

$$= m(P(x_1, \ldots, x_n))$$

Examples

Smyth (1981)

$$m(1+x+y)=\frac{3\sqrt{3}}{4\pi}\mathrm{L}(\chi_{-3},2)=\mathrm{L}'(\chi_{-3},-1)$$

Smyth(1981)

$$m(1+x+y+z)=\frac{7}{2\pi^2}\zeta(3)$$

D'Andrea & L (2003)

$$m\left(z(1-xy)^2-(1-x)(1-y)\right)=\frac{4\sqrt{5}\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\zeta(3)\pi^2}$$

Boyd & L (2005)

$$m(x^2+1+(x+1)y+(x-1)z)=\frac{\mathrm{L}(\chi_{-4},2)}{\pi}+\frac{21}{8\pi^2}\zeta(3)$$

Examples in several variables

L (2003)

$$m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)\left(\frac{1-x_3}{1+x_3}\right)z\right)$$

$$=\frac{24}{\pi^3}\mathrm{L}(\chi_{-4},4)+\frac{\mathrm{L}(\chi_{-4},2)}{\pi}$$

$$m\left(1+\left(\frac{1-x_1}{1+x_1}\right)\dots\left(\frac{1-x_4}{1+x_4}\right)z\right)$$

$$=\frac{62}{\pi^4}\zeta(5)+\frac{14}{3\pi^2}\zeta(3)$$

$$m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)(1+y)z\right)=\frac{24}{\pi^3}\mathrm{L}(\chi_{-4},4)$$

$$m\left(1+x+\left(\frac{1-x_1}{1+x_1}\right)\left(\frac{1-x_2}{1+x_2}\right)(1+y)z\right)=\frac{93}{\pi^4}\zeta(5)$$

The measures of a family of genus-one curves

Boyd, Deninger, Rodriguez-Villegas 1997-1998

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}+k\right) \stackrel{?}{=} \frac{\mathrm{L}'(E_k,0)}{s_k} \quad k \in \mathbb{N} \neq 0,4$$

E_k determined by $x+\frac{1}{x}+y+\frac{1}{y}=0$.

L-functions $\leftarrow$ Beilinson's conjectures

$k=4\sqrt{2}$ (CM case)

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}+4\sqrt{2}\right)=\mathrm{L}'(E_{4\sqrt{2}},0)$$

$k=3\sqrt{2}$ (modular curve $X_0(24)$)

$$m\left(x+\frac{1}{x}+y+\frac{1}{y}+3\sqrt{2}\right)=q\mathrm{L}'(E_{3\sqrt{2}},0)$$

$$q \in \mathbb{Q}^*, \qquad q \stackrel{?}{=} \frac{5}{2}$$

$$m(k) := m\left(x + \frac{1}{x} + y + \frac{1}{y} + k\right)$$

Theorem 1 *(Rodriguez-Villegas)*

$$m(k) = \mathrm{Re}\left(\frac{16 y_\mu}{\pi^2} \sum_{m,n}{}' \frac{\chi_{-4}(m)}{(m + n4\mu)^2 (m + n4\bar{\mu})}\right)$$

$$= \mathrm{Re}\left(-\pi \mathrm{i} \mu + 2 \sum_{n=1}^{\infty} \sum_{d|n} \chi_{-4}(d) d^2 \frac{q^n}{n}\right)$$

where $j(E_k) = j\left(-\frac{1}{4\mu}\right)$

$$q = \mathrm{e}^{2\pi \mathrm{i} \mu} = q\left(\frac{16}{k^2}\right) = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, 1 - \frac{16}{k^2}\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, \frac{16}{k^2}\right)}\right)$$

and y_μ *is the imaginary part of* μ.

Theorem 2 *For $h \in \mathbb{R}^*$,*

$$m(4h^2) + m\left(\frac{4}{h^2}\right) = 2m\left(2\left(h + \frac{1}{h}\right)\right).$$

(also due to Kurokawa & Ochiai)

For $h \in \mathbb{R}^*$, $|h| < 1$,

$$m\left(2\left(h + \frac{1}{h}\right)\right) + m\left(2\left(\mathrm{i}h + \frac{1}{\mathrm{i}h}\right)\right) = m\left(\frac{4}{h^2}\right).$$

Corollary 3

$$m(8) = 4m(2) = \frac{8}{5}m(3\sqrt{2})$$

The elliptic regulator

F field. Matsumoto Theorem:

$$K_2(F) = \langle \{a,b\}, a,b \in F\rangle \,/\, \langle \text{bilinear}, \{a, 1-a\}\rangle$$

$K_2(E) \otimes \mathbb{Q}$ subgroup of $K_2(\mathbb{Q}(E)) \otimes \mathbb{Q}$ determined by kernels of tame symbols.

$x, y \in \mathbb{C}(E)$

$$\eta(x,y) := \log|x| \mathrm{d} \arg y - \log|y| \mathrm{d} \arg x$$

1-form on $E(\mathbb{C}) \setminus S$ for any loop $\gamma \in E(\mathbb{C}) \setminus S$

$$(\gamma, \eta(x,y)) = \frac{1}{2\pi} \int_\gamma \eta(x,y)$$

The regulator map (Beilinson, Bloch):

$$r : K_2(E) \otimes \mathbb{Q} \to H^1(E, \mathbb{R})$$

$$\{x, y\} \to \left\{ \gamma \to \int_\gamma \eta(x, y) \right\}$$

for $\gamma \in H_1(E, \mathbb{Z})$.

$H^1(E, \mathbb{R})$ dual of $H_1(E, \mathbb{Z})$.

Follows from $\eta(x, 1 - x) = \mathrm{d}D(x)$,

$$D(x) = \mathrm{Im}(\mathrm{Li}_2(x)) + \arg(1 - x) \log |x|$$

is the Bloch-Wigner dilogarithm

We may think of $\gamma \in H_1(E, \mathbb{Z})^-$.

$E(\mathbb{C}) \cong \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z} \cong \mathbb{C}^*/q^{\mathbb{Z}}$ where z mod $\Lambda = \mathbb{Z}+\tau\mathbb{Z}$ is identified with $\mathrm{e}^{2\mathrm{i}\pi z}$.

Bloch regulator function given by a Kronecker-Eisenstein series

$$R_\tau\left(\mathrm{e}^{2\pi\mathrm{i}\alpha}\right) = \frac{y_\tau^2}{\pi} \sum_{m,n\in\mathbb{Z}}' \frac{\mathrm{e}^{2\pi\mathrm{i}(bn-am)}}{(m\tau+n)^2(m\bar{\tau}+n)}$$

if $\alpha = a + b\tau$ and y_τ is the imaginary part of τ.

Elliptic dilogarithm

$$D_\tau(z) := \sum_{n\in\mathbb{Z}} D(zq^n)$$

Regulator function given by

$$R_\tau = D_\tau - \mathrm{i}J_\tau$$

$$\mathbb{Z}[E(\mathbb{C})]^- = \mathbb{Z}[E(\mathbb{C})]/\sim \qquad [-P] \sim -[P].$$

R_τ is an odd function,

$$\mathbb{Z}[E(\mathbb{C})]^- \to \mathbb{C}.$$

$$(x) = \sum m_i(a_i), \qquad (y) = \sum n_j(b_j).$$

$$\mathbb{C}(E)^* \otimes \mathbb{C}(E)^* \to \mathbb{Z}[E(\mathbb{C})]^-$$

$$(x) \diamond (y) = \sum m_i n_j (a_i - b_j).$$

Theorem 4 *(Beilinson) $X/\mathbb{C}$ smooth algebraic curve, x, y non-constant functions in $\mathbb{C}(X)$, $\omega \in \Omega^1$*

$$\int_{E(\mathbb{C})} \eta(x,y) \wedge \bar{\omega} = R_\tau((x) \diamond (y))$$

Corollary 5 *(after an idea of Deninger) x, y are non-constant functions in $\mathbb{C}(E)$ with trivial tame symbols*

$$-r\{x,y\} = -\int_\gamma \eta(x,y) = \mathrm{Im}\left(\frac{\Omega}{y_\tau \Omega_0} R_\tau\left((x) \diamond (y)\right)\right)$$

where Ω_0 is the real period and $\Omega = \int_\gamma \omega$.

The relation with Mahler measures

Deninger

$$m(k) \sim_{\mathbb{Z}} \frac{1}{2\pi} r(\{x,y\})(\gamma)$$

In the example,

$$yP_k(x,y) = (y - y_{(1)}(x))(y - y_{(2)}(x)),$$

$$m(k) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} (\log^+ |y_{(1)}(x)| + \log^+ |y_{(2)}(x)|) \frac{\mathrm{d}x}{x}.$$

By Jensen's formula respect to y.

$$m(k) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} \log |y| \frac{\mathrm{d}x}{x} = -\frac{1}{2\pi} \int_{\mathbb{T}^1} \eta(x,y),$$

$\mathbb{T}^1 \in H_1(E, \mathbb{Z})$.

Modularity for the regulator

Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$ and let $\tau' = \frac{\alpha\tau+\beta}{\gamma\tau+\delta}$, such that

$$\begin{pmatrix} b' \\ a' \end{pmatrix} = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix}$$

Then:

$$R_{\tau'}\left(\mathrm{e}^{2\pi\mathrm{i}(a'+b'\tau')}\right) = \frac{1}{\gamma\bar{\tau}+\delta} R_\tau \left(\mathrm{e}^{2\pi\mathrm{i}(a+b\tau)}\right).$$

Functional equations for the regulator

From

$$J(z) = p \sum_{x^p = z} J(x)$$

Let p prime,

$$(1+\chi_{-4}(p)p^2) J_{4\tau}\left(\mathrm{e}^{2\pi \mathrm{i}\tau}\right) = \sum_{j=0}^{p-1} p J_{\frac{4(\tau+j)}{p}}\left(\mathrm{e}^{\frac{2\pi \mathrm{i}(\tau+j)}{p}}\right)$$

$$+\chi_{-4}(p) J_{4p\tau}\left(\mathrm{e}^{2\pi \mathrm{i} p\tau}\right)$$

In particular, $p = 2$,

$$J_{4\tau}\left(\mathrm{e}^{2\pi \mathrm{i}\tau}\right) = 2J_{2\tau}\left(\mathrm{e}^{\pi \mathrm{i}\tau}\right) + 2J_{2(\tau+1)}\left(\mathrm{e}^{\pi \mathrm{i}(\tau+1)}\right)$$

Also:

$$J_{\frac{2\tau+1}{2}}\left(\mathrm{e}^{\pi \mathrm{i}\tau}\right) = J_{2\tau}\left(\mathrm{e}^{\pi \mathrm{i}\tau}\right) - J_{2\tau}\left(-\mathrm{e}^{\pi \mathrm{i}\tau}\right)$$

Idea of Proof

$$x + \frac{1}{x} + y + \frac{1}{y} + k = 0$$

Weierstrass form:

$$x = \frac{kX - 2Y}{2X(X-1)} \qquad y = \frac{kX + 2Y}{2X(X-1)}.$$

$$Y^2 = X\left(X^2 + \left(\frac{k^2}{4} - 2\right)X + 1\right).$$

$P = \left(1, \frac{k}{2}\right)$, torsion point of order 4.

$$(x) \diamond (y) = 4(P) - 4(-P) = 8(P).$$

$$P \equiv -\frac{1}{4} \mod \mathbb{Z} + \tau\mathbb{Z} \qquad k \in \mathbb{R}$$

$$\tau = \mathrm{i}y_\tau \qquad k \in \mathbb{R}, |k| > 4,$$

$$\tau = \frac{1}{2} + \mathrm{i}y_\tau \qquad k \in \mathbb{R}, |k| < 4$$

Understand cycle $[|x| = 1] \in H_1(E, \mathbb{Z})$

$$\Omega = \tau\Omega_0 \quad k \in \mathbb{R}$$

$$m(k) = \frac{4}{\pi} \operatorname{Im}\left(\frac{\tau}{y_\tau} R_\tau(-\mathrm{i})\right), \quad k \in \mathbb{R}$$

Take $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$.

$$m(k) = -\frac{4|\tau|^2}{\pi y_\tau} J_{-\frac{1}{\tau}}\left(\mathrm{e}^{-\frac{2\pi \mathrm{i}}{4\tau}}\right)$$

If we let $\mu = -\frac{1}{4\tau}$, then

$$m(k) = -\frac{1}{\pi y_\mu} J_{4\mu}\left(\mathrm{e}^{2\pi \mathrm{i}\mu}\right)$$

$$= \operatorname{Re}\left(\frac{16 y_\mu}{\pi^2} \sum_{m,n}{}' \frac{\chi_{-4}(m)}{(m + n4\mu)^2 (m + n4\bar{\mu})}\right)$$

- $$J_{4\mu}\left(\mathrm{e}^{2\pi \mathrm{i}\mu}\right) = 2J_{2\mu}\left(\mathrm{e}^{\pi \mathrm{i}\mu}\right) + 2J_{2(\mu+1)}\left(\mathrm{e}^{\frac{2\pi \mathrm{i}(\mu+1)}{2}}\right)$$

  set $\tau = -\frac{1}{2\mu}$, for $|h| < 1$ so $\mu \in \mathrm{i}\mathbb{R}$

  $$D_{\frac{\tau}{2}}(-\mathrm{i}) = D_\tau(-\mathrm{i}) + \frac{1}{y_{2(\mu+1)}} J_{2(\mu+1)}\left(\mathrm{e}^{\frac{2\pi \mathrm{i}(\mu+1)}{2}}\right)$$

  this is the first equality.

- $$J_{\frac{2\mu+1}{2}}\left(\mathrm{e}^{\frac{2\pi \mathrm{i}\mu}{2}}\right) = J_{2\mu}\left(\mathrm{e}^{\pi \mathrm{i}\mu}\right) - J_{2\mu}\left(-\mathrm{e}^{\pi \mathrm{i}\mu}\right)$$

  Set $\tau = -\frac{1}{2\mu}$ and use $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$.

  $$D_{\frac{\tau-1}{2}}(-\mathrm{i}) = D_\tau(-\mathrm{i}) - \frac{1}{y_{2(\mu+1)}} J_{2(\mu+1)}\left(\mathrm{e}^{\frac{2\pi \mathrm{i}(\mu+1)}{2}}\right)$$

Putting things together,

$$2D_\tau(-\mathrm{i}) = D_{\frac{\tau}{2}}(-\mathrm{i}) + D_{\frac{\tau-1}{2}}(-\mathrm{i})$$

this is the second equality.

It turns out that

$$m(k) = \mathrm{Re}\left(-\pi\mathrm{i}\mu - \pi\mathrm{i}\int_{\mathrm{i}\infty}^{\mu}(e(z)-1)\mathrm{d}z\right)$$

where

$$e(\mu) = 1 - 4\sum_{n=1}^{\infty}\sum_{d\ n}\chi_{-4}(d)d^2q^n$$

is an Eisenstein series. Hence the equations can be also deduced from identities of Hecke operators.

Parameter k.

$$q = q\left(\frac{16}{k^2}\right) = \exp\left(-\pi\frac{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,1-\frac{16}{k^2}\right)}{{}_2F_1\left(\frac{1}{2},\frac{1}{2};1,\frac{16}{k^2}\right)}\right)$$

Second degree modular equation, $|h| < 1$, $h \in \mathbb{R}$,

$$q^2\left(\left(\frac{2h}{1+h^2}\right)^2\right) = q\left(h^4\right).$$

$h \to \mathrm{i}h$

$$-q\left(\left(\frac{2h}{1+h^2}\right)^2\right) = q\left(\left(\frac{2\mathrm{i}h}{1-h^2}\right)^2\right).$$

Then the equation with J becomes

$$m\left(q\left(\left(\frac{2h}{1+h^2}\right)^2\right)\right)+m\left(q\left(\left(\frac{2\mathrm{i}h}{1-h^2}\right)^2\right)\right)=m\left(q\left(h^4\right)\right).$$

$$m\left(2\left(h+\frac{1}{h}\right)\right)+m\left(2\left(\mathrm{i}h+\frac{1}{\mathrm{i}h}\right)\right)=m\left(\frac{4}{h^2}\right).$$

The identity with $h = \frac{1}{\sqrt{2}}$

$$m(2) + m(8) = 2m\left(3\sqrt{2}\right)$$

$$m\left(3\sqrt{2}\right) + m\left(\mathrm{i}\sqrt{2}\right) = m(8)$$

$f = \frac{\sqrt{2}Y - X}{2}$ in $\mathbb{C}(E_{3\sqrt{2}})$.

$(f)\diamond(1-f) = 6(P) - 10(P+Q) \Rightarrow 6(P) \sim 10(P+Q)$.

$Q = \left(-\frac{1}{h^2}, 0\right)$ has order 2.

$$\phi : E_{3\sqrt{2}} \to E_{\mathrm{i}\sqrt{2}} \qquad (X,Y) \to (-X, \mathrm{i}Y)$$

$$r_{\mathrm{i}\sqrt{2}}(\{x,y\}) = r_{3\sqrt{2}}(\{x \circ \phi, y \circ \phi\})$$

But

$$(x \circ \phi) \diamond (y \circ \phi) = 8(P + Q)$$

$$(x) \diamond (y) = 8(P)$$

$$6 r_{3\sqrt{2}}(\{x, y\}) = 10 r_{\mathrm{i}\sqrt{2}}(\{x, y\})$$

and

$$3m(3\sqrt{2}) = 5m(\mathrm{i}\sqrt{2}).$$

Consequently,

$$m(8) = \frac{8}{5} m(3\sqrt{2})$$

$$m(2) = \frac{2}{5} m(3\sqrt{2})$$

Other families

- Hesse family

$$h(a^3) = m\left(x^3 + y^3 + 1 - \frac{3xy}{a}\right)$$

(studied by Rodriguez-Villegas)

$$h(u^3) = \sum_{j=0}^{2} h\left(1 - \left(\frac{1 - \xi_3^j u}{1 + 2\xi_3^j u}\right)^3\right) \qquad |u| \text{ small}$$

- More complicated equations for examples studied by Stienstra:

$$m\left((x+1)(y+1)(x+y) - \frac{xy}{t}\right)$$

and Zagier and Stienstra:

$$m\left((x+y+1)(x+1)(y+1) - \frac{xy}{t}\right)$$