

**Functional equations for Mahler measures
of genus-one curves**

(joint with Mat Rogers, University of British
Columbia)

Matilde N. Lalin

Clay Mathematics Institute,
University of British Columbia,
and Pacific Institute for the Mathematical
Sciences

`mlalin@math.ubc.ca`

`http://www.math.ubc.ca/~mlalin`

August, 2006

Mahler measure of one-variable polynomials

Pierce (1918): $P \in \mathbb{Z}[x]$ monic,

$$P(x) = \prod_i (x - \alpha_i)$$

$$\Delta_n = \prod_i (\alpha_i^n - 1)$$

$$P(x) = x - 2 \Rightarrow \Delta_n = 2^n - 1$$

Lehmer (1933):

$$\lim_{n \rightarrow \infty} \frac{|\alpha^{n+1} - 1|}{|\alpha^n - 1|} = \begin{cases} |\alpha| & \text{if } |\alpha| > 1 \\ 1 & \text{if } |\alpha| < 1 \end{cases}$$

For

$$P(x) = a \prod_i (x - \alpha_i)$$

$$M(P) = |a| \prod_i \max\{1, |\alpha_i|\}$$

$$m(P) = \log M(P) = \log |a| + \sum_i \log^+ |\alpha_i|$$

- Kronecker's Lemma: $P \in \mathbb{Z}[x]$, $P \neq 0$,

$$m(P) = 0 \Leftrightarrow P(x) = x^n \prod \phi_i(x)$$

- Lehmer's Question (1933): Does there exist $C > 0$ such that $P(x) \in \mathbb{Z}[x]$

$$m(P) = 0 \quad \text{or} \quad m(P) > C??$$

Is

$$\begin{aligned} m(x^{10} + x^9 - x^7 - x^6 - x^5 - x^4 - x^3 + x + 1) \\ = 0.162357612\dots \end{aligned}$$

the best possible?

$$\sqrt{\Delta_{379}} = 1,794,327,140,357$$

Mahler measure of multivariable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_n})| d\theta_1 \dots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n} \end{aligned}$$

Jensen's formula:

$$\int_0^1 \log |e^{2\pi i \theta} - \alpha| d\theta = \log^+ |\alpha|$$

recovers one-variable case.

- $m(P \cdot Q) = m(P) + m(Q)$
- $m(P) \geq 0$ if P has integral coefficients.
- $\alpha \in \bar{\mathbb{Q}}$, P_α is its minimal polynomial over \mathbb{Q} , then

$$m(P_\alpha) = [\mathbb{Q}(\alpha) : \mathbb{Q}] h(\alpha),$$

where h is the logarithmic Weil height.

- Boyd–Lawton: $P \in \mathbb{C}[x_1, \dots, x_n]$

$$\begin{aligned} \lim_{k_2 \rightarrow \infty} \dots \lim_{k_n \rightarrow \infty} m(P(x, x^{k_2}, \dots, x^{k_n})) \\ = m(P(x_1, \dots, x_n)) \end{aligned}$$

Examples

Smyth (1981)

$$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$$

Smyth(1981)

$$m(1 + x + y + z) = \frac{7}{2\pi^2} \zeta(3)$$

D'Andrea & L (2003)

$$m\left(z(1 - xy)^2 - (1 - x)(1 - y)\right) = \frac{4\sqrt{5}\zeta_{\mathbb{Q}(\sqrt{5})}(3)}{\zeta(3)\pi^2}$$

Boyd & L (2005)

$$m(x^2 + 1 + (x+1)y + (x-1)z) = \frac{L(\chi_{-4}, 2)}{\pi} + \frac{21}{8\pi^2} \zeta(3)$$

Examples in several variables

L (2003)

$$\begin{aligned} m \left(1 + \left(\frac{1-x_1}{1+x_1} \right) \left(\frac{1-x_2}{1+x_2} \right) \left(\frac{1-x_3}{1+x_3} \right) z \right) \\ = \frac{24}{\pi^3} L(\chi_{-4}, 4) + \frac{L(\chi_{-4}, 2)}{\pi} \end{aligned}$$

$$\begin{aligned} m \left(1 + \left(\frac{1-x_1}{1+x_1} \right) \cdots \left(\frac{1-x_4}{1+x_4} \right) z \right) \\ = \frac{62}{\pi^4} \zeta(5) + \frac{14}{3\pi^2} \zeta(3) \end{aligned}$$

$$m \left(1 + x + \left(\frac{1-x_1}{1+x_1} \right) (1+y)z \right) = \frac{24}{\pi^3} L(\chi_{-4}, 4)$$

$$m \left(1 + x + \left(\frac{1-x_1}{1+x_1} \right) \left(\frac{1-x_2}{1+x_2} \right) (1+y)z \right) = \frac{93}{\pi^4} \zeta(5)$$

The measures of a family of genus-one curves

Boyd, Deninger, Rodriguez-Villegas 1997-1998

$$m \left(x + \frac{1}{x} + y + \frac{1}{y} + k \right) \stackrel{?}{=} \frac{L'(E_k, 0)}{s_k} \quad k \in \mathbb{N} \neq 0, 4$$

E_k determined by $x + \frac{1}{x} + y + \frac{1}{y} = 0$.

L-functions \leftarrow Beilinson's conjectures

$k = 4\sqrt{2}$ (CM case)

$$m \left(x + \frac{1}{x} + y + \frac{1}{y} + 4\sqrt{2} \right) = L'(E_{4\sqrt{2}}, 0)$$

$k = 3\sqrt{2}$ (modular curve $X_0(24)$)

$$m \left(x + \frac{1}{x} + y + \frac{1}{y} + 3\sqrt{2} \right) = qL'(E_{3\sqrt{2}}, 0)$$

$$q \in \mathbb{Q}^*, \quad q \stackrel{?}{=} \frac{5}{2}$$

$$m(k) := m \left(x + \frac{1}{x} + y + \frac{1}{y} + k \right)$$

Theorem 1 (*Rodriguez-Villegas*)

$$\begin{aligned} m(k) &= \operatorname{Re} \left(\frac{16y\mu}{\pi^2} \sum'_{m,n} \frac{\chi_{-4}(m)}{(m+n4\mu)^2(m+n4\bar{\mu})} \right) \\ &= \operatorname{Re} \left(-\pi i \mu + 2 \sum_{n=1}^{\infty} \sum_{d|n} \chi_{-4}(d) d^2 \frac{q^n}{n} \right) \end{aligned}$$

where $j(E_k) = j\left(-\frac{1}{4\mu}\right)$

$$q = e^{2\pi i \mu} = q\left(\frac{16}{k^2}\right) = \exp \left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, 1 - \frac{16}{k^2}\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, \frac{16}{k^2}\right)} \right)$$

and y_μ is the imaginary part of μ .

Theorem 2 For $h \in \mathbb{R}^*$,

$$m(4h^2) + m\left(\frac{4}{h^2}\right) = 2m\left(2\left(h + \frac{1}{h}\right)\right).$$

(also due to Kurokawa & Ochiai)

For $h \in \mathbb{R}^*$, $|h| < 1$,

$$m\left(2\left(h + \frac{1}{h}\right)\right) + m\left(2\left(ih + \frac{1}{ih}\right)\right) = m\left(\frac{4}{h^2}\right).$$

Corollary 3

$$m(8) = 4m(2) = \frac{8}{5}m(3\sqrt{2})$$

The elliptic regulator

F field. Matsumoto Theorem:

$$K_2(F) = \langle \{a, b\}, a, b \in F \rangle / \langle \text{bilinear}, \{a, 1 - a\} \rangle$$

$K_2(E) \otimes \mathbb{Q}$ subgroup of $K_2(\mathbb{Q}(E)) \otimes \mathbb{Q}$ determined by kernels of tame symbols.

$$x, y \in \mathbb{C}(E)$$

$$\eta(x, y) := \log |x| d \arg y - \log |y| d \arg x$$

1-form on $E(\mathbb{C}) \setminus S$ for any loop $\gamma \in E(\mathbb{C}) \setminus S$

$$(\gamma, \eta(x, y)) = \frac{1}{2\pi} \int_{\gamma} \eta(x, y)$$

The regulator map (Beilinson, Bloch):

$$r : K_2(E) \otimes \mathbb{Q} \rightarrow H^1(E, \mathbb{R})$$

$$\{x, y\} \rightarrow \left\{ \gamma \rightarrow \int_{\gamma} \eta(x, y) \right\}$$

for $\gamma \in H_1(E, \mathbb{Z})$.

$H^1(E, \mathbb{R})$ dual of $H_1(E, \mathbb{Z})$.

Follows from $\eta(x, 1 - x) = dD(x)$,

$$D(x) = \text{Im}(\text{Li}_2(x)) + \arg(1 - x) \log |x|$$

is the Bloch-Wigner dilogarithm

We may think of $\gamma \in H_1(E, \mathbb{Z})^-$.

$E(\mathbb{C}) \cong \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z} \cong \mathbb{C}^*/q^{\mathbb{Z}}$ where $z \bmod \Lambda = \mathbb{Z} + \tau\mathbb{Z}$ is identified with $e^{2i\pi z}$.

Bloch regulator function given by a Kronecker-Eisenstein series

$$R_{\tau}(e^{2\pi i\alpha}) = \frac{y_{\tau}^2}{\pi} \sum'_{m,n \in \mathbb{Z}} \frac{e^{2\pi i(bn-am)}}{(m\tau + n)^2(m\bar{\tau} + n)}$$

if $\alpha = a + b\tau$ and y_{τ} is the imaginary part of τ .

Elliptic dilogarithm

$$D_{\tau}(z) := \sum_{n \in \mathbb{Z}} D(zq^n)$$

Regulator function given by

$$R_{\tau} = D_{\tau} - iJ_{\tau}$$

$$\mathbb{Z}[E(\mathbb{C})]^- = \mathbb{Z}[E(\mathbb{C})]/\sim \quad [-P] \sim -[P].$$

R_τ is an odd function,

$$\mathbb{Z}[E(\mathbb{C})]^- \rightarrow \mathbb{C}.$$

$$(x) = \sum m_i(a_i), \quad (y) = \sum n_j(b_j).$$

$$\mathbb{C}(E)^* \otimes \mathbb{C}(E)^* \rightarrow \mathbb{Z}[E(\mathbb{C})]^-$$

$$(x) \diamond (y) = \sum m_i n_j (a_i - b_j).$$

Theorem 4 (Beilinson) X/\mathbb{C} smooth algebraic curve, x, y non-constant functions in $\mathbb{C}(X)$, $\omega \in \Omega^1$

$$\int_{E(\mathbb{C})} \eta(x, y) \wedge \bar{\omega} = R_{\tau}((x) \diamond (y))$$

Corollary 5 (after an idea of Deninger) x, y are non-constant functions in $\mathbb{C}(E)$ with trivial tame symbols

$$-r\{x, y\} = - \int_{\gamma} \eta(x, y) = \text{Im} \left(\frac{\Omega}{y_{\tau} \Omega_0} R_{\tau}((x) \diamond (y)) \right)$$

where Ω_0 is the real period and $\Omega = \int_{\gamma} \omega$.

The relation with Mahler measures

Deninger

$$m(k) \sim_{\mathbb{Z}} \frac{1}{2\pi} r(\{x, y\})(\gamma)$$

In the example,

$$yP_k(x, y) = (y - y_{(1)}(x))(y - y_{(2)}(x)),$$

$$m(k) = \frac{1}{2\pi i} \int_{\mathbb{T}^1} (\log^+ |y_{(1)}(x)| + \log^+ |y_{(2)}(x)|) \frac{dx}{x}.$$

By Jensen's formula respect to y .

$$m(k) = \frac{1}{2\pi i} \int_{\mathbb{T}^1} \log |y| \frac{dx}{x} = -\frac{1}{2\pi} \int_{\mathbb{T}^1} \eta(x, y),$$

$$\mathbb{T}^1 \in H_1(E, \mathbb{Z}).$$

Modularity for the regulator

Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbb{Z})$ and let $\tau' = \frac{\alpha\tau + \beta}{\gamma\tau + \delta}$, such that

$$\begin{pmatrix} b' \\ a' \end{pmatrix} = \begin{pmatrix} \delta & -\gamma \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix}$$

Then:

$$R_{\tau'} \left(e^{2\pi i(a' + b'\tau')} \right) = \frac{1}{\gamma\bar{\tau} + \delta} R_{\tau} \left(e^{2\pi i(a + b\tau)} \right).$$

Functional equations for the regulator

From

$$J(z) = p \sum_{x^p=z} J(x)$$

Let p prime,

$$(1 + \chi_{-4}(p)p^2) J_{4\tau}(e^{2\pi i\tau}) = \sum_{j=0}^{p-1} p J_{\frac{4(\tau+j)}{p}} \left(e^{\frac{2\pi i(\tau+j)}{p}} \right) \\ + \chi_{-4}(p) J_{4p\tau}(e^{2\pi ip\tau})$$

In particular, $p = 2$,

$$J_{4\tau}(e^{2\pi i\tau}) = 2J_{2\tau}(e^{\pi i\tau}) + 2J_{2(\tau+1)}(e^{\pi i(\tau+1)})$$

Also:

$$J_{\frac{2\tau+1}{2}}(e^{\pi i\tau}) = J_{2\tau}(e^{\pi i\tau}) - J_{2\tau}(-e^{\pi i\tau})$$

Idea of Proof

$$x + \frac{1}{x} + y + \frac{1}{y} + k = 0$$

Weierstrass form:

$$x = \frac{kX - 2Y}{2X(X - 1)} \quad y = \frac{kX + 2Y}{2X(X - 1)}.$$

$$Y^2 = X \left(X^2 + \left(\frac{k^2}{4} - 2 \right) X + 1 \right).$$

$P = \left(1, \frac{k}{2} \right)$, torsion point of order 4.

$$(x) \diamond (y) = 4(P) - 4(-P) = 8(P).$$

$$P \equiv -\frac{1}{4} \pmod{\mathbb{Z} + \tau\mathbb{Z}} \quad k \in \mathbb{R}$$

$$\tau = iy_\tau \quad k \in \mathbb{R}, |k| > 4,$$

$$\tau = \frac{1}{2} + iy_\tau \quad k \in \mathbb{R}, |k| < 4$$

Understand cycle $[|x| = 1] \in H_1(E, \mathbb{Z})$

$$\Omega = \tau\Omega_0 \quad k \in \mathbb{R}$$

$$m(k) = \frac{4}{\pi} \operatorname{Im} \left(\frac{\tau}{y_\tau} R_\tau(-i) \right), \quad k \in \mathbb{R}$$

Take $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(\mathbb{Z})$.

$$m(k) = -\frac{4|\tau|^2}{\pi y_\tau} J_{-\frac{1}{\tau}} \left(e^{-\frac{2\pi i}{4\tau}} \right)$$

If we let $\mu = -\frac{1}{4\tau}$, then

$$\begin{aligned} m(k) &= -\frac{1}{\pi y_\mu} J_{4\mu} \left(e^{2\pi i \mu} \right) \\ &= \operatorname{Re} \left(\frac{16y_\mu}{\pi^2} \sum'_{m,n} \frac{\chi_{-4}(m)}{(m+n4\mu)^2(m+n4\bar{\mu})} \right) \end{aligned}$$

•

$$J_{4\mu} \left(e^{2\pi i \mu} \right) = 2J_{2\mu} \left(e^{\pi i \mu} \right) + 2J_{2(\mu+1)} \left(e^{\frac{2\pi i(\mu+1)}{2}} \right)$$

set $\tau = -\frac{1}{2\mu}$, for $|h| < 1$ so $\mu \in i\mathbb{R}$

$$D_{\frac{\tau}{2}}(-i) = D_{\tau}(-i) + \frac{1}{y_{2(\mu+1)}} J_{2(\mu+1)} \left(e^{\frac{2\pi i(\mu+1)}{2}} \right)$$

this is the first equality.

•

$$J_{\frac{2\mu+1}{2}} \left(e^{\frac{2\pi i \mu}{2}} \right) = J_{2\mu} \left(e^{\pi i \mu} \right) - J_{2\mu} \left(-e^{\pi i \mu} \right)$$

Set $\tau = -\frac{1}{2\mu}$ and use $\begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}$.

$$D_{\frac{\tau-1}{2}}(-i) = D_{\tau}(-i) - \frac{1}{y_{2(\mu+1)}} J_{2(\mu+1)} \left(e^{\frac{2\pi i(\mu+1)}{2}} \right)$$

Putting things together,

$$2D_{\tau}(-i) = D_{\frac{\tau}{2}}(-i) + D_{\frac{\tau-1}{2}}(-i)$$

this is the second equality.

It turns out that

$$m(k) = \operatorname{Re} \left(-\pi i \mu - \pi i \int_{i\infty}^{\mu} (e(z) - 1) dz \right)$$

where

$$e(\mu) = 1 - 4 \sum_{n=1}^{\infty} \sum_{d|n} \chi_{-4}(d) d^2 q^n$$

is an Eisenstein series. Hence the equations can be also deduced from identities of Hecke operators.

Parameter k .

$$q = q\left(\frac{16}{k^2}\right) = \exp\left(-\pi \frac{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, 1 - \frac{16}{k^2}\right)}{{}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1, \frac{16}{k^2}\right)}\right)$$

Second degree modular equation, $|h| < 1$, $h \in \mathbb{R}$,

$$q^2\left(\left(\frac{2h}{1+h^2}\right)^2\right) = q(h^4).$$

$h \rightarrow ih$

$$-q\left(\left(\frac{2h}{1+h^2}\right)^2\right) = q\left(\left(\frac{2ih}{1-h^2}\right)^2\right).$$

Then the equation with J becomes

$$m\left(q\left(\left(\frac{2h}{1+h^2}\right)^2\right)\right) + m\left(q\left(\left(\frac{2ih}{1-h^2}\right)^2\right)\right) = m\left(q(h^4)\right).$$

$$m\left(2\left(h + \frac{1}{h}\right)\right) + m\left(2\left(ih + \frac{1}{ih}\right)\right) = m\left(\frac{4}{h^2}\right).$$

The identity with $h = \frac{1}{\sqrt{2}}$

$$m(2) + m(8) = 2m(3\sqrt{2})$$

$$m(3\sqrt{2}) + m(i\sqrt{2}) = m(8)$$

$$f = \frac{\sqrt{2}Y - X}{2} \text{ in } \mathbb{C}(E_{3\sqrt{2}}).$$

$$(f) \diamond (1-f) = 6(P) - 10(P+Q) \Rightarrow 6(P) \sim 10(P+Q).$$

$Q = \left(-\frac{1}{h^2}, 0\right)$ has order 2.

$$\phi : E_{3\sqrt{2}} \rightarrow E_{i\sqrt{2}} \quad (X, Y) \rightarrow (-X, iY)$$

$$r_{i\sqrt{2}}(\{x, y\}) = r_{3\sqrt{2}}(\{x \circ \phi, y \circ \phi\})$$

But

$$(x \circ \phi) \diamond (y \circ \phi) = 8(P + Q)$$

$$(x) \diamond (y) = 8(P)$$

$$6r_{3\sqrt{2}}(\{x, y\}) = 10r_{i\sqrt{2}}(\{x, y\})$$

and

$$3m(3\sqrt{2}) = 5m(i\sqrt{2}).$$

Consequently,

$$m(8) = \frac{8}{5}m(3\sqrt{2})$$

$$m(2) = \frac{2}{5}m(3\sqrt{2})$$

Other families

- Hesse family

$$h(a^3) = m \left(x^3 + y^3 + 1 - \frac{3xy}{a} \right)$$

(studied by Rodriguez-Villegas)

$$h(u^3) = \sum_{j=0}^2 h \left(1 - \left(\frac{1 - \xi_3^j u}{1 + 2\xi_3^j u} \right)^3 \right) \quad |u| \text{ small}$$

- More complicated equations for examples studied by Stienstra:

$$m \left((x + 1)(y + 1)(x + y) - \frac{xy}{t} \right)$$

and Zagier and Stienstra:

$$m \left((x + y + 1)(x + 1)(y + 1) - \frac{xy}{t} \right)$$