

Bernoulli Numbers

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I will mostly follow [2].

Definition and some identities

Definition 1 *Bernoulli numbers are defined as $B_0 = 1$ and recursively as*

$$(m+1)B_m = -\sum_{k=0}^{m-1} \binom{m+1}{k} B_k,$$

so we find $B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, B_4 = -\frac{1}{30}, B_5 = 0, B_6 = \frac{1}{42}, \dots, B_{12} = -\frac{691}{2730}, \dots$

Lemma 2

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!}.$$

PROOF. Write $\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} a_m \frac{t^m}{m!}$ and multiply by $e^t - 1$,

$$t = \sum_{n=1}^{\infty} \frac{t^n}{n!} \sum_{m=0}^{\infty} a_m \frac{t^m}{m!},$$

equate coefficients for t^{m+1} gives $a_0 = 1$ and

$$\sum_{k=0}^m \binom{m+1}{k} a_k = 0.$$

□

Theorem 3 (*J. Bernoulli*) *Let m be a positive integer and define*

$$S_m(n) = 1^m + \dots + (n-1)^m,$$

then

$$(m+1)S_m(n) = \sum_{k=0}^m \binom{m+1}{k} B_k n^{m+1-k}.$$

PROOF. In $e^{kt} = \sum_{m=0}^{\infty} k^m \frac{t^m}{m!}$ substitute $k = 0, 1, \dots, n-1$ and add,

$$\begin{aligned} \sum_{m=0}^{\infty} S_m(n) \frac{t^m}{m!} &= 1 + e^t + \dots + e^{(n-1)t} = \frac{e^{nt} - 1}{t} \frac{t}{e^t - 1} \\ &= \sum_{k=1}^{\infty} n^k \frac{t^{k-1}}{k!} \sum_{j=0}^{\infty} B_j \frac{t^j}{j!} \end{aligned}$$

now equate the coefficients of t^m and multiply by $(m+1)!$. □

Definition 4

$$B_m(x) = \sum_{k=0}^m \binom{m}{k} B_k x^{m-k}$$

are called *Bernoulli polynomials*.

So $B_0(x) = 1$, $B_1(x) = x - \frac{1}{2}$, etc.

Then Theorem 3 may be stated as

$$S_m(n) = \frac{1}{m+1} (B_{m+1}(n) - B_{m+1}).$$

Lemma 5

$$\frac{te^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}.$$

PROOF. First note that

$$B'_m(x) = \sum_{k=0}^{m-1} \binom{m}{k} (m-k) B_k x^{m-1-k} = m B_{m-1}(x).$$

Also

$$\int_0^1 B_m(x) dx = \frac{1}{m+1} \sum_{k=0}^m \binom{m+1}{k} B_k = 0, \quad m \geq 1.$$

Now let $F(x, t) = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}$, differentiating,

$$\frac{\partial}{\partial x} F(x, t) = \sum_{m=1}^{\infty} B_{m-1}(x) \frac{t^m}{(m-1)!} = t F(x, t).$$

Now we solve using separation of variables, $F(x, t) = T(t)e^{xt}$, then

$$\int_0^1 F(x, t) dx = \int_0^1 T(t)e^{xt} dx = T(t) \frac{e^t - 1}{t}$$

but

$$\int_0^1 F(x, t) dx = \sum_{m=0}^{\infty} \frac{t^m}{m!} \int_0^1 B_m(x) dx = 1$$

and this proves the statement (Castellanos, [1]). \square

Proposition 6 1. $B_m(x+1) - B_m(x) = \sum_{k=0}^m \binom{m}{k} B_{m-k}(x) = mx^{m-1}$ (Roman [5]).

2. $B_m(1-x) = (-1)^m B_m(x)$.

3. $B_m = \sum_{k=0}^m \frac{1}{k+1} \sum_{r=0}^k (-1)^r \binom{k}{r} r^m$ (Rademacher [4]).

4. $\sum_{k=0}^m (-1)^{k+m} \binom{m}{k} B_m(k) = m!$ (Ruiz [6]).

5. $B_m(kx) = k^{q-1} \sum_{j=0}^{k-1} B_m\left(x + \frac{j}{k}\right)$.

Euler MacLaurin sum formula (Rademacher, [4]).

Let $f(x)$ smooth. Since $B_1'(x) = 1$,

$$\begin{aligned} \int_0^1 f(x)dx &= B_1(x)f(x)\Big|_0^1 - \int_0^1 B_1(x)f'(x)dx \\ &= \dots = \sum_{m=1}^q (-1)^{m-1} \frac{B_m(x)}{m!} f^{(m-1)}(x)\Big|_0^1 + (-1)^q \int_0^1 \frac{B_q(x)}{q!} f^{(q)}(x)dx \end{aligned}$$

Evaluating in $x = 1$,

$$f(1) = \int_0^1 f(x)dx + \sum_{m=1}^q (-1)^m \frac{B_m}{m!} (f^{(m-1)}(1) - f^{(m-1)}(0)) + (-1)^{q-1} \int_0^1 \frac{B_q(x)}{q!} f^{(q)}(x)dx.$$

Changing $f(x)$ by $f(n-1+x)$ and adding, we obtain the formula

$$\sum_{n=a+1}^b f(n) = \int_a^b f(x)dx + \sum_{m=1}^q (-1)^m \frac{B_m}{m!} (f^{(m-1)}(b) - f^{(m-1)}(a)) + R_q$$

where

$$R_q = \frac{(-1)^{q-1}}{q!} \int_a^b B_q(x - [x]) f^{(q)}(x)dx$$

An integral and some identities

Proposition 7 *We have:*

$$\int_0^\infty \frac{x \log^k x dx}{(x^2 + a^2)(x^2 + b^2)} = \left(\frac{\pi}{2}\right)^{k+1} \frac{P_k\left(\frac{2 \log a}{\pi}\right) - P_k\left(\frac{2 \log b}{\pi}\right)}{a^2 - b^2}.$$

where

$$P_k(x) = \frac{2i^{k+1}}{k+1} \left(B_{k+1}\left(\frac{x}{i}\right) - 2^k B_{k+1}\left(\frac{x}{2i}\right) \right) + \frac{(2^{k+1} - 2)i^{k+1}}{k+1} B_{k+1}$$

PROOF. (Idea) We first prove that

$$\int_0^\infty \frac{x^\alpha dx}{(x^2 + a^2)(x^2 + b^2)} = \frac{\pi(a^{\alpha-1} - b^{\alpha-1})}{2 \cos \frac{\pi\alpha}{2} (b^2 - a^2)} \quad \text{for } 0 < \alpha < 1, \quad a, b \neq 0$$

by first computing

$$\int_0^\infty \frac{x^\alpha dx}{x^2 + a^2} = \frac{\pi a^{\alpha-1}}{2 \cos \frac{\pi\alpha}{2}}.$$

We note that the polynomials P_k may be defined recursively as

$$P_k(x) = \frac{x^{k+1}}{k+1} + \frac{1}{k+1} \sum_{j>1(\text{odd})}^{k+1} (-1)^{\frac{j+1}{2}} \binom{k+1}{j} P_{k+1-j}(x).$$

The idea, suggested by Rodriguez-Villegas, is to obtain the value of the integral in the statement by differentiating k times the integral of with α and then evaluating at $\alpha = 1$. Let

$$f(\alpha) = \frac{\pi(a^{\alpha-1} - b^{\alpha-1})}{2 \cos \frac{\pi\alpha}{2} (b^2 - a^2)}$$

which is the value of the integral with α . In other words, we have

$$f^{(k)}(1) = \int_0^\infty \frac{x \log^k x dx}{(x^2 + a^2)(x^2 + b^2)}.$$

By developing in power series around $\alpha = 1$, we obtain

$$f(\alpha) \cos \frac{\pi\alpha}{2} = \frac{\pi}{2(b^2 - a^2)} \sum_{n=0}^{\infty} \frac{\log^n a - \log^n b}{n!} (\alpha - 1)^n.$$

By differentiating k times,

$$\sum_{j=0}^k \binom{k}{j} f^{(k-j)}(\alpha) \left(\cos \frac{\pi\alpha}{2}\right)^{(j)} = \frac{\pi}{2(b^2 - a^2)} \sum_{n=0}^{\infty} \frac{\log^{n+k} a - \log^{n+k} b}{n!} (\alpha - 1)^n.$$

We evaluate in $\alpha = 1$,

$$\sum_{j=0(\text{odd})}^k (-1)^{\frac{j+1}{2}} \binom{k}{j} f^{(k-j)}(1) \left(\frac{\pi}{2}\right)^j = \frac{\pi(\log^k a - \log^k b)}{2(b^2 - a^2)}.$$

As a consequence, we obtain

$$f^{(k)}(1) = \frac{1}{k+1} \sum_{j>1(\text{odd})}^{k+1} (-1)^{\frac{j+1}{2}} \binom{k+1}{j} f^{(k+1-j)}(1) \left(\frac{\pi}{2}\right)^{j-1} + \frac{\log^{k+1} a - \log^{k+1} b}{(k+1)(a^2 - b^2)}.$$

When $k = 0$,

$$f^{(0)}(1) = f(1) = \frac{\log a - \log b}{a^2 - b^2} = \frac{\pi}{2} \frac{P_0\left(\frac{2 \log a}{\pi}\right) - P_0\left(\frac{2 \log b}{\pi}\right)}{a^2 - b^2}.$$

The general result follows by induction on k and the definition of P_k . \square

Theorem 8 *We have the following identities:*

- For $1 \leq l \leq n$:

$$\begin{aligned} & s_{n-l}(1^2, \dots, (2n-1)^2) \\ &= n \sum_{s=0}^{n-l} s_{n-l-s}(2^2, \dots, (2n-2)^2) \frac{1}{l+s} B_{2s} \binom{2(l+s)}{2s} (2^{2s} - 2) (-1)^{s+1}. \end{aligned}$$

- For $1 \leq n$:

$$\left(\frac{(2n)!}{2^n n!}\right)^2 = 2n \sum_{s=1}^n s_{n-s}(2^2, \dots, (2n-2)^2) \frac{1}{s} B_{2s} (2^{2s} - 1) (-1)^{s+1}.$$

- For $0 \leq l \leq n$:

$$(2l+1)s_{n-l}(2^2, \dots, (2n)^2)$$

$$= (2n+1) \sum_{s=0}^{n-l} s_{n-l-s}(1^2, \dots, (2n-1)^2) B_{2s} \binom{2(l+s)}{2s} (2^{2s}-2)(-1)^{s+1}.$$

- For $1 \leq n$:

$$\sum_{s=1}^n s_{n-s}(2^2, \dots, (2n-2)^2)(-1)^{s+1} \frac{2^{2s}(2^{2s}-1)}{s} B_{2s} = 2(2n-1)!$$

where

$$s_l(a_1, \dots, a_k) = \begin{cases} 1 & \text{if } l=0 \\ \sum_{i_1 < \dots < i_l} a_{i_1} \dots a_{i_l} & \text{if } 0 < l \leq k \\ 0 & \text{if } k < l \end{cases}$$

are the elementary symmetric polynomials, i.e.,

$$\prod_{i=1}^k (x+a_i) = \sum_{l=0}^k s_l(a_1, \dots, a_k) x^{k-l}$$

Some big classic results

Theorem 9 (Euler)

$$2\zeta(2m) = (-1)^{m+1} \frac{(2\pi)^{2m}}{(2m)!} B_{2m}.$$

PROOF. We will need

$$\cot x = \frac{1}{x} - 2 \sum_{n=1}^{\infty} \frac{x}{n^2\pi^2 - x^2}.$$

This identity may be deduced by applying the logarithmic derivative to

$$\sin x = x \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right).$$

Then

$$x \cot x = 1 - 2 \sum_{n=1}^{\infty} \frac{x^2}{n^2\pi^2} \sum_{k=0}^{\infty} \left(\frac{x}{n\pi}\right)^{2k} = 1 - 2 \sum_{m=1}^{\infty} \zeta(2m) \frac{x^{2m}}{\pi^{2m}}$$

On the other hand,

$$x \cot x = x \frac{\cos x}{\sin x} = x \frac{i(e^{ix} + e^{-ix})}{e^{ix} - e^{-ix}} = ix + \frac{2ix}{e^{2ix} - 1} = 1 + \sum_{n=2}^{\infty} B_n \frac{(2ix)^n}{n!}$$

and compare coefficients of x^{2m} .

For instance, $\zeta(2) = \frac{\pi^2}{6}$, $\zeta(4) = \frac{\pi^4}{90}$, etc.

Corollary 10 1. $(-1)^{m+1} B_{2m} > 0$.

2. $\left| \frac{B_{2m}}{2m} \right| \rightarrow \infty$ or $B_{2m} \sim (-1)^{m+1} \frac{2(2m)!}{(2\pi)^{2m}}$ as $m \rightarrow \infty$.

PROOF. The first assertion is consequence of the fact that $\zeta(2m)$ is positive. The second is consequence of the fact that $\zeta(2m) > 1$ implies

$$|B_{2m}| > \frac{2(2m)!}{(2\pi)^{2m}}.$$

□

Theorem 11 (Claussen, von Staudt) For $m \geq 1$

$$B_{2m} \equiv - \sum_{(p-1)|2m, p \text{ prime}} \frac{1}{p} \pmod{1}$$

We will need the following

Definition 12 For every rational number r and p prime write $r = p^k \frac{a}{b}$ where a, b are integers such that $p \nmid ab$. Then $\text{ord}_p(r) = k$. We say that r is p -integral if $\text{ord}_p(r) \geq 0$.

Lemma 13 Let p be a prime number and k a positive integer, then

1. $\frac{p^k}{k+1}$ is p -integral.
2. $\frac{p^k}{k+1} \equiv 0 \pmod{p}$ if $k \geq 2$.
3. $\frac{p^{k-2}}{k+1}$ is p -integral if $k \geq 3$ and $p \geq 5$.

PROOF. By induction, $k+1 \leq p^k$. Let $k+1 = p^a q$. Then $\frac{p^k}{k+1} = \frac{p^{k-a}}{q} \geq 1$ implies $k \geq a$. For the second case use that $k+1 < p^k$ for $k \geq 2$. The third case is consequence of $k+1 < p^{k-2}$ for $k \geq 3$ and $p \geq 5$. □

Proposition 14 Let p be a prime and m a positive integer. Then pB_m is p -integral. Also, if m is even $pB_m \equiv S_m(p) \pmod{p}$

PROOF. For the first statement we will use induction. It is clear for $m = 1$. Now note that for $m \geq k$ we have

$$\binom{m+1}{k} = \frac{m+1}{m-k+1} \binom{m}{k}$$

Then Theorem 3 becomes

$$S_m(n) = \sum_{k=0}^m \binom{m}{k} B_k \frac{n^{m+1-k}}{m+1-k} = \sum_{k=0}^m \binom{m}{k} B_{m-k} \frac{n^{k+1}}{k+1} \quad (1)$$

Now set $n = p$ and since $S_m(p)$ is integer, it suffices to prove that

$$\binom{m}{k} p B_{m-k} \frac{p^k}{k+1}$$

is p -integral for $k = 1, \dots, m$, but that is true by induction and Lemma 13.

For the congruence it suffices to see that

$$\text{ord}_p \left(\binom{m}{k} \left(pB_{m-k} \frac{p^k}{k+1} \right) \right) \geq 1$$

for $k \geq 1$. By Lemma 13 this is true for $k \geq 2$. The case with $k = 1$ corresponds to $\frac{m}{2}(pB_{m-1})p$ and it is true because m is even and the only nontrivial case is with $m = 2$.

Lemma 15 *Let p be a prime. If $p - 1 \nmid m$, then $S_m(p) \equiv 0 \pmod{p}$. If $p - 1 \mid m$ then $S_m(p) \equiv -1 \pmod{p}$*

PROOF. First suppose that $p - 1 \nmid m$. Let g be a primitive root modulo p . Then

$$S_m(p) = 1^m + \dots + (p-1)^m \equiv 1^m + g^m + \dots + g^{(p-2)m} \pmod{p}$$

and

$$(g^m - 1)S_m(p) \equiv g^{m(p-1)} - 1 \equiv 0 \pmod{p}$$

the result follows. Now suppose that $p - 1 \mid m$, then

$$S_m(p) \equiv 1 + 1 + \dots + 1 \equiv p - 1 \pmod{p}$$

□

PROOF. (Theorem 11) Assume m is even. By Proposition 14, pB_m is p -integral and $\equiv S_m(p) \pmod{p}$. By Lemma 15, B_m is a p -integer if $p - 1 \nmid m$ and $pB_m \equiv -1 \pmod{p}$ if $p - 1 \mid m$. Then

$$B_m + \sum_{p-1 \mid m} \frac{1}{p}$$

is a p -integer for all primes p , then it must be integral. □

More Congruences

Corollary 16 *If $p - 1 \nmid 2m$, then B_{2m} is p -integral. If $p - 1 \mid 2m$ then $pB_{2m} + 1$ is p -integral and*

$$\text{ord}_p(pB_{2m} + 1) = \text{ord}_p \left(p \left(B_{2m} + \frac{1}{p} \right) \right) \geq 1$$

so $pB_{2m} \equiv -1 \pmod{p}$. Also 6 always divides the denominator of B_{2m} .

From now on write $B_m = \frac{U_m}{V_m}$ as a fraction in lowest terms with $V_m > 0$.

Proposition 17 *For m even and > 1 ,*

$$V_m S_m(n) \equiv U_m n \pmod{n^2}$$

PROOF. We will use equation (1), for $k \geq 1$ write

$$\binom{m}{k} \left(B_{m-k} \frac{n^{k-1}}{k+1} \right) n^2 = A_k^m n^2.$$

If we show that for $p|n$, $p \neq 2, 3$, then $\text{ord}_p(A_k^m) \geq 0$ and if $p|n$, $p = 2$ or 3 , $\text{ord}_p(A_k^m) \geq -1$, then (A_k^m, n) must divide 6 and the same is true for the greater common divisor between the sum of A_k^m and n . Then we may write

$$S_m(n) = B_m n + \frac{A n^2}{lB}$$

with $(B, n) = 1$ and $l|6$. Multiplying by BV_m and using the fact that $6|V_m$ (by Corollary 16) the result is proved.

Use Corollary 16 to see that $\text{ord}_p(B_{m-k}) \geq -1$. Assume $p|n$ and $p \neq 2, 3$. The cases $k = 1, 2$ are simple. If $k \geq 3$,

$$\text{ord}_p\left(B_{m-k} \frac{n^{k-1}}{k+1}\right) \geq -1 + (k-1)\text{ord}_p n - \text{ord}_p(k+1) \geq k-2 - \text{ord}_p(k+1) \geq 0$$

by Lemma 15.

Now let $p = 2$. If $k = 1$, then $B_{m-1} = 0$ for $m > 2$ and $A_1^2 = 2B_1 \frac{1}{2} = -\frac{1}{2}$. For $k > 1$ note that $B_{m-k} = 0$ unless k is even or $k = m-1$. k even implies $\text{ord}_2(k+1) = 0$ and $k = m-1$, $A_{m-1}^m = -\frac{1}{2}n^{m-2}$ which has order greater or equal to -1 .

When $p = 3$, $\text{ord}_3(A_2^m) \geq -1$ and $\text{ord}_3(A_3^m) \geq 1$. For $k \geq 4$, one shows that $\text{ord}_3\left(\frac{3^{k-2}}{k+1}\right) \geq 0$. \square

Corollary 18 *Let m be even and p prime with $p-1 \nmid m$. Then*

$$S_m(p) \equiv B_m p \pmod{p^2}.$$

PROOF. By Theorem 11, $p \nmid V_m$. Now put $n = p$ in the above Proposition and divide by V_m which is permissible since $p \nmid V_m$. \square

Proposition 19 (*Voronoi's congruence*) *Let m even and > 1 . Suppose that a and n are positive coprime integers. Then*

$$(a^m - 1)U_m \equiv ma^{m-1}V_m \sum_{j=1}^{n-1} j^{m-1} \left[\frac{ja}{n} \right] \pmod{n}.$$

PROOF. Write $ja = q_j n + r_j$ with $0 \leq r_j < n$. Then

$$j^m a^m \equiv r_j^m + m q_j n r_j^{m-1} \pmod{n^2}.$$

But $r_j \equiv ja \pmod{n}$, then

$$j^m a^m \equiv r_j^m + ma^{m-1} q_j n j^{m-1} \pmod{n^2}.$$

Summing for $j = 1, \dots, n-1$,

$$S_m(n)a^m \equiv S_m(n) + ma^{m-1}n \sum_{j=1}^{n-1} j^{m-1} \left[\frac{ja}{n} \right] \pmod{n^2}.$$

Now multiply by V_m and use Proposition 17. \square

Proposition 20 *If $p - 1 \nmid m$, then $\frac{B_m}{m}$ is p -integral.*

PROOF. By Theorem 11, B_m is a p -integer. Let $m = p^t m_0$ with $p \nmid m_0$. In Voronoi congruence put $n = p^t$. Then $(a^m - 1)U_m \equiv 0 \pmod{p^t}$. Now let a be a primitive root modulo p . Since $p - 1 \nmid m$, then $p \nmid a^m - 1$. Then $U_m \equiv 0 \pmod{p^t}$. Then $\frac{B_m}{m} = \frac{U_m}{mV_m}$ is p -integer. \square

Theorem 21 (*Kummer congruences*) *Suppose $m \geq 2$ is even, p prime, and $p - 1 \nmid m$. Let $C_m = \frac{(1-p^{m-1})B_m}{m}$. If $m' \equiv m \pmod{\phi(p^e)}$, then $C_{m'} \equiv C_m \pmod{p^e}$.*

PROOF. We will see the case $e = 1$. Let $t = \text{ord}_p(m)$. By Proposition 20, $p^t \mid U_m$. In Voronoi's congruence, set $n = p^{e+t}$. Since p^t divides both U_m and m , and $\frac{mV_m}{p^t}$ is prime to p , we obtain,

$$\frac{(a^m - 1)B_m}{m} \equiv a^{m-1} \sum_{j=1}^{p^{e+t}-1} j^{m-1} \left[\frac{ja}{p^{e+t}} \right] \pmod{p^e}.$$

The right-hand side is unchanged if we replace m by $m' \equiv m \pmod{p-1}$. Then

$$\frac{(a^{m'} - 1)B_{m'}}{m'} \equiv \frac{(a^m - 1)B_m}{m} \pmod{p}.$$

Choose a to be a primitive root modulo p . Since $p - 1 \nmid m$ we have $a^{m'} - 1 \equiv a^m - 1 \not\equiv 0 \pmod{p}$. Then

$$\frac{B_{m'}}{m'} \equiv \frac{B_m}{m} \pmod{p}.$$

\square

Definition 22 *An odd prime number p is said to be regular if p does not divide the numerator of any of the numbers B_2, B_4, \dots, B_{p-3} . The prime 3 is regular. Equivalently, p is regular if it does not divide the class number of $\mathbb{Q}(\xi_p)$*

The first irregular primes are 37 and 59.

Theorem 23 (*Kummer*) *Let p be a regular prime. Then $x^p + y^p = z^p$ has no solution in positive integers.*

Theorem 24 (*Jensen*) *The set of irregular primes is infinite.*

PROOF. Let $\{p_1, \dots, p_s\}$ be the set of irregular primes. Let $k \geq 2$ be even and $n = k(p_1 - 1) \dots (p_s - 1)$. Choose k large such that $\left| \frac{B_n}{n} \right| > 1$ and p prime such that $\text{ord}_p \left(\frac{B_n}{n} \right) > 0$. Then $p - 1 \nmid n$ and so $p \neq p_i$. We will prove that p is also irregular.

Let $n \equiv m \pmod{p-1}$ where $0 < m < p - 1$. Then m is even and $2 \leq m \leq p - 3$. By the Kummer congruence,

$$\frac{B_n}{n} \equiv \frac{B_m}{m} \pmod{p}.$$

Since $\text{ord}_p \left(\frac{B_n}{n} \right) > 0$ and $\text{ord}_p \left(\frac{B_m}{m} - \frac{B_n}{n} \right) > 0$, then

$$\text{ord}_p \left(\frac{B_m}{m} \right) = \text{ord}_p B_m > 0$$

and p is irregular. \square

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