

# Examples of Mahler Measures as Multiple Polylogarithms

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## Mahler Measure

For  $P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ , the (logarithmic) *Mahler measure* is:

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_n})| d\theta_1 \dots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n} \end{aligned}$$

Jensen's formula  $\longrightarrow$  simple expression in one-variable case.

Several-variable case?

## Some of Smyth's Examples with several variables

The simplest example in two variables:

$$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = L'(\chi_{-3}, -1)$$

$$L(\chi_{-3}, s) := \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s}$$

is the L-series in the character of conductor 3:

$$\chi_{-3}(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{3} \\ -1 & \text{if } n \equiv -1 \pmod{3} \\ 0 & \text{otherwise} \end{cases}$$

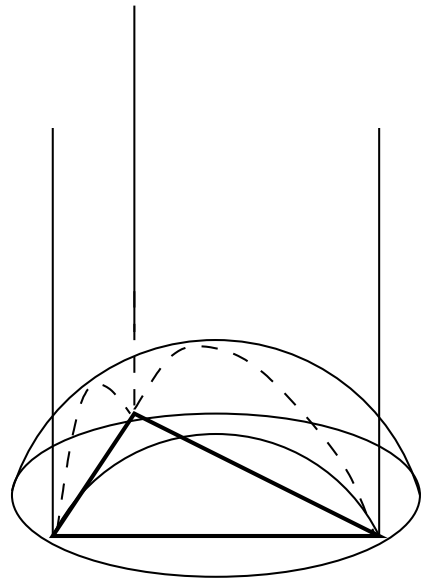
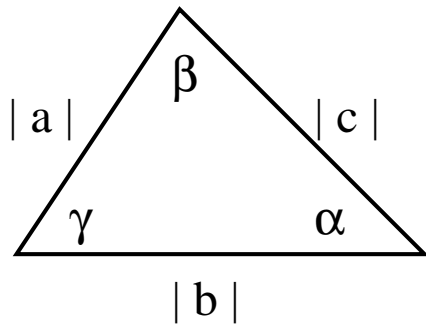
Another example in three variables

$$m(1 + x + y + z) = \frac{7}{2\pi^2} \zeta(3)$$

## Cassaigne and Maillot's Example

$$\pi m(a + bx + cy) =$$

$$\begin{cases} D\left(\left|\frac{a}{b}\right| e^{i\gamma}\right) + \alpha \log |a| + \beta \log |b| + \gamma \log |c| & \Delta \\ \pi \log \max\{|a|, |b|, |c|\} & \text{not } \Delta \end{cases}$$



where

$$D(z) := \text{Im}(\text{Li}_2(z)) + \log |z| \arg(1-z) \quad z \in \mathbb{C} \setminus [1, \infty)$$

## Results

|   |  |
|---|--|
| $\pi m((1 + y) + \alpha(1 - y)x)$                               | $2L(\chi_{-4}, 2)$   |
| $\pi^2 m((1 + w)(1 + y) + \alpha(1 - w)(1 - y)x)$               | $7\zeta(3)$  |
| $\pi^3 m((1 + v)(1 + w)(1 + y) + \alpha(1 - v)(1 - w)(1 - y)x)$ | $7\pi\zeta(3) + 4 \sum_{0 \leq j < k} \frac{(-1)^j}{(2j+1)^2 k^2}$               |
| $\pi^2 m((1 + x) + \alpha(y + z))$                              | $\frac{7}{2}\zeta(3)$  |
| $\pi^3 m((1 + w)(1 + x) + \alpha(1 - w)(y + z))$                | $2\pi^2 L(\chi_{-4}, 2) + 8 \sum_{0 \leq j < k} \frac{(-1)^{j+k+1}}{(2j+1)^3 k}$ |
| $\pi^4 m((1 + v)(1 + w)(1 + x) + \alpha(1 - v)(1 - w)(y + z))$  | $93\zeta(5)$   |
| $\pi^2 m((1 + w)(1 + y) + (1 - w)(x - y))$                      | $\frac{7}{2}\zeta(3) + \frac{\pi^2}{2} \log 2$                                   |

$$\chi_{-4}(n) = \begin{cases} 1 & \text{if } n \equiv 1 \pmod{4} \\ -1 & \text{if } n \equiv -1 \pmod{4} \\ 0 & \text{otherwise} \end{cases}$$

In the first column,  $\alpha \in \mathbb{C}$ . The values in the second column are the Mahler measures for the  $\alpha = 1$  case.

## Polylogarithms

Multiple polylogarithms:

$$\text{Li}_{k_1, \dots, k_m}(x_1, \dots, x_m) := \sum_{0 < n_1 < n_2 < \dots < n_m} \frac{x_1^{n_1} x_2^{n_2} \dots x_m^{n_m}}{n_1^{k_1} n_2^{k_2} \dots n_m^{k_m}}$$

(convergent for  $|x_i| < 1$ )

Hyperlogarithms:

$$\mathbb{I}_{k_1, \dots, k_m}(a_1 : \dots : a_m : a_{m+1}) := \int_0^{a_{m+1}} \underbrace{\frac{dt}{t-a_1} \circ \frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{k_1} \circ \dots \circ \underbrace{\frac{dt}{t-a_m} \circ \frac{dt}{t} \circ \dots \circ \frac{dt}{t}}_{k_m}$$

$k_i$  are integers,  $a_i$  are complex numbers, and

$$\int_0^{b_{l+1}} \frac{dt}{t-b_1} \circ \dots \circ \frac{dt}{t-b_l} = \int_{0 \leq t_1 \leq \dots \leq t_l \leq b_{l+1}} \frac{dt_1}{t_1 - b_1} \cdots \frac{dt_l}{t_l - b_l}$$

The value of the integral above depends on the homotopy class of the path connecting 0 and  $a_{m+1}$  on  $\mathbb{C} \setminus \{a_1, \dots, a_m\}$ .

### Proposition 1

$$\begin{aligned} & \mathbf{I}_{k_1, \dots, k_m}(a_1 : \dots : a_m : a_{m+1}) = \\ & (-1)^m \mathbf{Li}_{k_1, \dots, k_m} \left( \frac{a_2}{a_1}, \frac{a_3}{a_2}, \dots, \frac{a_m}{a_{m-1}}, \frac{a_{m+1}}{a_m} \right) \end{aligned}$$

$$\begin{aligned} & \mathbf{Li}_{k_1, \dots, k_m}(x_1, \dots, x_m) = \\ & (-1)^m \mathbf{I}_{k_1, \dots, k_m} \left( \frac{1}{x_1 \dots x_m} : \dots : \frac{1}{x_m} : 1 \right) \end{aligned}$$

*(gives an analytic continuation to multiple polylogarithms)*

## Method

1. Let  $P_\alpha \in \mathbb{C}[x_1, \dots, x_n]$  whose coefficients depend polynomially on  $\alpha \in \mathbb{C}$ .

For example,  $P_\alpha(x) = 1 + \alpha x$ .

$$m(P) = \log^+ |\alpha|.$$

2. Replace  $\alpha$  by  $\alpha \frac{1-y}{1+y}$ . We obtain a polynomial  $\tilde{P}_\alpha \in \mathbb{C}[x_1, \dots, x_n, y]$ .

In the example,

$$\tilde{P}_\alpha(x, y) = 1 + y + \alpha(1 - y)x.$$

3. The Mahler measure of  $\tilde{P}_\alpha$  is a certain integral of the Mahler measure of  $P_\alpha$ :

$$m(\tilde{P}_\alpha) = \frac{1}{2\pi i} \int_{\mathbb{T}^1} m \left( P_{\alpha \frac{1-y}{1+y}} \right) \frac{dy}{y}$$



4. If the Mahler measure depends just on  $|\alpha|$ , make  $u = \left| \alpha \frac{1-y}{1+y} \right|$ .

First  $y = e^{i\theta}$ , then set  $u = |\alpha| \tan\left(\frac{\theta}{2}\right)$ .

$$\begin{aligned} m(\tilde{P}_\alpha) &= \frac{2}{\pi} \int_0^\infty m(P_u) \frac{|\alpha| du}{u^2 + |\alpha|^2} \\ &= \frac{i}{\pi} \int_0^\infty m(P_u) \left( \frac{1}{u + i|\alpha|} - \frac{1}{u - i|\alpha|} \right) du \end{aligned}$$

In the example,

$$\begin{aligned} & m(1 + y + \alpha(1 - y)x) \\ &= \frac{i}{\pi} \int_0^\infty \log^+ u \left( \frac{1}{u + i|\alpha|} - \frac{1}{u - i|\alpha|} \right) du \\ &= \frac{i}{\pi} \int_0^1 \int_s^1 \frac{dt}{t} \left( \frac{1}{s + \frac{i}{|\alpha|}} - \frac{1}{s - \frac{i}{|\alpha|}} \right) ds \\ &= \frac{i}{\pi} \left( \mathbf{I}_2 \left( -\frac{i}{|\alpha|} : 1 \right) - \mathbf{I}_2 \left( \frac{i}{|\alpha|} : 1 \right) \right) \\ &= -\frac{i}{\pi} (\text{Li}_2(i|\alpha|) - \text{Li}_2(-i|\alpha|)) \end{aligned}$$

**How does  $\zeta(5)$  show up?**

$$\begin{aligned} & \pi^4 m((1+v)(1+w)(1+x) + (1-v)(1-w)(y+z)) \\ &= 7\pi^2 \zeta(3) + 8(\text{Li}_{3,2}(1, 1) - \text{Li}_{3,2}(-1, 1)) \\ & \quad + 8(\text{Li}_{3,2}(1, -1) - \text{Li}_{3,2}(-1, -1)) \end{aligned}$$

$\text{Li}_{3,2}(\pm 1, \pm 1)$  are alternating Euler sums.

Use the formula (Borwein, Bradley and Broadhurst)

$$\text{Li}_{3,2}(x, y) = -\frac{1}{2}\text{Li}_5(xy) + \text{Li}_3(x)\text{Li}_2(y) + 3\text{Li}_5(x) \\ + 2\text{Li}_5(y) - \text{Li}_2(xy)(\text{Li}_3(x) + 2\text{Li}_3(y))$$

for  $x, y = \pm 1$ , together with

$$\text{Li}_k(1) = \zeta(k) \quad \text{and} \quad \text{Li}_k(-1) = \left(\frac{1}{2^{k-1}} - 1\right) \zeta(k)$$

We get

$$\begin{aligned} & \operatorname{Li}_{3,2}(1, 1) - \operatorname{Li}_{3,2}(-1, 1) + \operatorname{Li}_{3,2}(1, -1) - \operatorname{Li}_{3,2}(-1, -1) \\ &= -\frac{21}{4}\zeta(2)\zeta(3) + \frac{93}{8}\zeta(5) \end{aligned}$$

We obtain the result by using that  $\zeta(2) = \frac{\pi^2}{6}$