# Examples of Mahler Measures as Multiple Polylogarithms

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## Mahler Measure

For $P \in \mathbb{C}[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, the (logarithmic) *Mahler measure* is:

$$\begin{aligned} m(P) &= \int_0^1 \ldots \int_0^1 \log |P(\mathrm{e}^{2\pi \mathrm{i}\theta_1}, \ldots, \mathrm{e}^{2\pi \mathrm{i}\theta_n})| \mathrm{d}\theta_1 \ldots \mathrm{d}\theta_n \\ &= \frac{1}{(2\pi \mathrm{i})^n} \int_{\mathbb{T}^n} \log |P(x_1, \ldots, x_n)| \frac{\mathrm{d}x_1}{x_1} \ldots \frac{\mathrm{d}x_n}{x_n} \end{aligned}$$

Jensen's formula $\longrightarrow$ simple expression in one-variable case.

Several-variable case?

**Some of Smyth's Examples with several variables**

The simplest example in two variables:

$$m(1+x+y) = \frac{3\sqrt{3}}{4\pi}\mathsf{L}(\chi_{-3},2) = \mathsf{L}'(\chi_{-3},-1)$$

$$\mathsf{L}(\chi_{-3},s) := \sum_{n=1}^{\infty} \frac{\chi_{-3}(n)}{n^s}$$

is the L-series in the character of conductor 3:

$$\chi_{-3}(n) = \begin{cases} 1 & \text{if} \quad n \equiv 1 \bmod 3 \\ -1 & \text{if} \quad n \equiv -1 \bmod 3 \\ 0 & \text{otherwise} \end{cases}$$

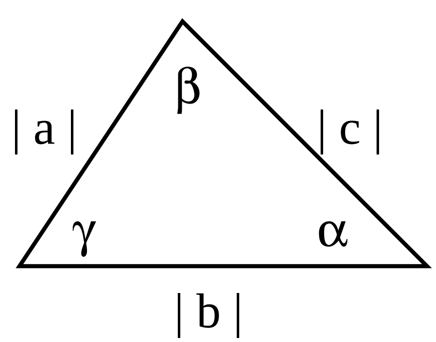
Another example in three variables

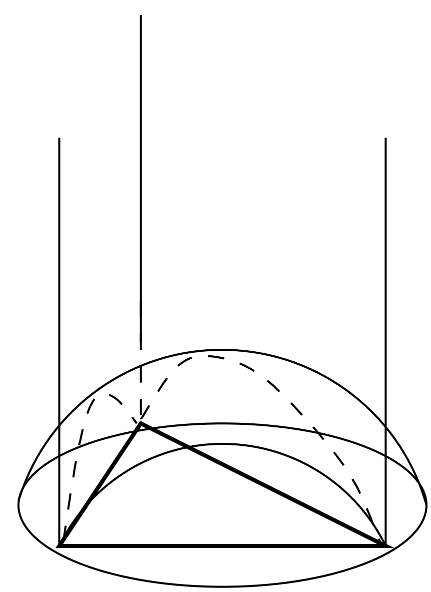
$$m(1+x+y+z) = \frac{7}{2\pi^2}\zeta(3)$$

## Cassaigne and Maillot's Example

$$\pi m(a + bx + cy) =$$

$$\begin{cases} D\left(\left|\frac{a}{b}\right| e^{i\gamma}\right) + \alpha \log|a| + \beta \log|b| + \gamma \log|c| & \triangle \\ \pi \log \max\{|a|, |b|, |c|\} & \text{not } \triangle \end{cases}$$



where

$$D(z) := \mathrm{Im}(\mathrm{Li}_2(z)) + \log|z| \arg(1-z) \qquad z \in \mathbb{C} \backslash [1, \infty)$$

## Results

| | |
|---|---|
| $\pi m((1+y)+\alpha(1-y)x)$ | $2\mathrm{L}(\chi_{-4},2)$ |
| $\pi^2 m((1+w)(1+y)+\alpha(1-w)(1-y)x)$ | $7\zeta(3)$ |
| $\pi^3 m((1+v)(1+w)(1+y)+\alpha(1-v)(1-w)(1-y)x)$ | $7\pi\zeta(3)+4\sum_{0\leq j<k}\frac{(-1)^j}{(2j+1)^2k^2}$ |
| $\pi^2 m((1+x)+\alpha(y+z))$ | $\frac{7}{2}\zeta(3)$ |
| $\pi^3 m((1+w)(1+x)+\alpha(1-w)(y+z))$ | $2\pi^2\mathrm{L}(\chi_{-4},2)+8\sum_{0\leq j<k}\frac{(-1)^{j+k+1}}{(2j+1)^3k}$ |
| $\pi^4 m((1+v)(1+w)(1+x)+\alpha(1-v)(1-w)(y+z))$ | $93\zeta(5)$ |
| $\pi^2 m((1+w)(1+y)+(1-w)(x-y))$ | $\frac{7}{2}\zeta(3)+\frac{\pi^2}{2}\log 2$ |

$$\chi_{-4}(n)=\begin{cases} 1 & \text{if} \quad n\equiv 1 \bmod 4 \\ -1 & \text{if} \quad n\equiv -1 \bmod 4 \\ 0 & \text{otherwise} \end{cases}$$

In the first column, $\alpha\in\mathbb{C}$. The values in the second column are the Mahler measures for the $\alpha=1$ case.

## Polylogarithms

*Multiple polylogarithms*:

$$\mathrm{Li}_{k_1,\ldots,k_m}(x_1,\ldots,x_m) := \sum_{0<n_1<n_2<\ldots<n_m} \frac{x_1^{n_1} x_2^{n_2} \ldots x_m^{n_m}}{n_1^{k_1} n_2^{k_2} \ldots n_m^{k_m}}$$

(convergent for $|x_i| < 1$)

*Hyperlogarithms*:

$$\mathrm{I}_{k_1,\ldots,k_m}(a_1 : \ldots : a_m : a_{m+1}) :=$$

$$\int_0^{a_{m+1}} \underbrace{\frac{\mathrm{d}t}{t-a_1} \circ \frac{\mathrm{d}t}{t} \circ \ldots \circ \frac{\mathrm{d}t}{t}}_{k_1} \circ \ldots \circ \underbrace{\frac{\mathrm{d}t}{t-a_m} \circ \frac{\mathrm{d}t}{t} \circ \ldots \circ \frac{\mathrm{d}t}{t}}_{k_m}$$

$k_i$ are integers, $a_i$ are complex numbers, and

$$\int_0^{b_{l+1}} \frac{\mathrm{d}t}{t-b_1} \circ \ldots \circ \frac{\mathrm{d}t}{t-b_l} =$$

$$\int_{0 \le t_1 \le \ldots \le t_l \le b_{l+1}} \frac{\mathrm{d}t_1}{t_1 - b_1} \ldots \frac{\mathrm{d}t_l}{t_l - b_l}$$

The value of the integral above depends on the homotopy class of the path connecting 0 and $a_{m+1}$ on $\mathbb{C} \setminus \{a_1, \ldots, a_m\}$.

**Proposition 1**

$$\mathrm{I}_{k_1,\ldots,k_m}(a_1 : \ldots : a_m : a_{m+1}) =$$

$$(-1)^m \mathrm{Li}_{k_1,\ldots,k_m}\left(\frac{a_2}{a_1}, \frac{a_3}{a_2}, \ldots, \frac{a_m}{a_{m-1}}, \frac{a_{m+1}}{a_m}\right)$$

$$\mathrm{Li}_{k_1,\ldots,k_m}(x_1, \ldots, x_m) =$$

$$(-1)^m \mathrm{I}_{k_1,\ldots,k_m}\left(\frac{1}{x_1 \ldots x_m} : \ldots : \frac{1}{x_m} : 1\right)$$

*(gives an analytic continuation to multiple polylogarithms)*

## Method

1. Let $P_\alpha \in \mathbb{C}[x_1, \dots, x_n]$ whose coefficients depend polynomially on $\alpha \in \mathbb{C}$.

   For example, $P_\alpha(x) = 1 + \alpha x$.

   $m(P) = \log^+ |\alpha|$.

2. Replace $\alpha$ by $\alpha\frac{1-y}{1+y}$. We obtain a polynomial $\tilde{P}_\alpha \in \mathbb{C}[x_1, \dots, x_n, y]$.

   In the example,
   $\tilde{P}_\alpha(x, y) = 1 + y + \alpha(1 - y)x$.

3. The Mahler measure of $\tilde{P}_\alpha$ is a certain integral of the Mahler measure of $P_\alpha$:

$$m(\tilde{P}_\alpha) = \frac{1}{2\pi \mathrm{i}} \int_{\mathbb{T}^1} m\left(P_{\alpha\frac{1-y}{1+y}}\right) \frac{\mathrm{d}y}{y}$$

4. If the Mahler measure depends just on $|\alpha|$, make $u = \left|\alpha \frac{1-y}{1+y}\right|$.

   First $y = \mathrm{e}^{\mathrm{i}\theta}$, then set $u = |\alpha| \tan\left(\frac{\theta}{2}\right)$.

$$
\begin{aligned}
m(\tilde{P}_\alpha) &= \frac{2}{\pi} \int_0^\infty m\left(P_u\right) \frac{|\alpha|\, \mathrm{d}u}{u^2 + |\alpha|^2} \\
&= \frac{\mathrm{i}}{\pi} \int_0^\infty m\left(P_u\right) \left( \frac{1}{u + \mathrm{i}|\alpha|} - \frac{1}{u - \mathrm{i}|\alpha|} \right) \mathrm{d}u
\end{aligned}
$$

In the example,

$$m(1+y+\alpha(1-y)x)$$

$$\begin{aligned}
&= \frac{\mathrm{i}}{\pi}\int_0^\infty \log^+ u \left(\frac{1}{u+\mathrm{i}|\alpha|} - \frac{1}{u-\mathrm{i}|\alpha|}\right) \mathrm{d}u \\
&= \frac{\mathrm{i}}{\pi}\int_0^1 \int_s^1 \frac{\mathrm{d}t}{t} \left(\frac{1}{s+\frac{\mathrm{i}}{|\alpha|}} - \frac{1}{s-\frac{\mathrm{i}}{|\alpha|}}\right) \mathrm{d}s \\
&= \frac{\mathrm{i}}{\pi}\left(\mathrm{I}_2\left(-\frac{\mathrm{i}}{|\alpha|} : 1\right) - \mathrm{I}_2\left(\frac{\mathrm{i}}{|\alpha|} : 1\right)\right) \\
&= -\frac{\mathrm{i}}{\pi}(\mathrm{Li}_2(\mathrm{i}|\alpha|) - \mathrm{Li}_2(-\mathrm{i}|\alpha|))
\end{aligned}$$

**How does $\zeta(5)$ show up?**

$$\pi^4 m((1+v)(1+w)(1+x)+(1-v)(1-w)(y+z))$$

$$\begin{aligned} &= 7\pi^2\zeta(3) + 8(\mathrm{Li}_{3,2}(1,1) - \mathrm{Li}_{3,2}(-1,1)) \\ &\quad +8(\mathrm{Li}_{3,2}(1,-1) - \mathrm{Li}_{3,2}(-1,-1)) \end{aligned}$$

$\mathrm{Li}_{3,2}(\pm 1, \pm 1)$ are alternating Euler sums.

Use the formula (Borwein, Bradley and Broadhurst)

$$\begin{aligned}\mathsf{Li}_{3,2}(x,y) &= -\frac{1}{2}\mathsf{Li}_5(x\,y)+\mathsf{Li}_3(x)\,\mathsf{Li}_2(y)+3\mathsf{Li}_5(x)\\&\quad+2\mathsf{Li}_5(y)-\mathsf{Li}_2(x\,y)(\mathsf{Li}_3(x)+2\mathsf{Li}_3(y))\end{aligned}$$

for $x, y = \pm 1$, together with

$$\mathsf{Li}_k(1) = \zeta(k) \quad \text{and} \quad \mathsf{Li}_k(-1) = \left(\frac{1}{2^{k-1}} - 1\right)\zeta(k)$$

We get

$$\mathrm{Li}_{3,2}(1,1)-\mathrm{Li}_{3,2}(-1,1)+\mathrm{Li}_{3,2}(1,-1)-\mathrm{Li}_{3,2}(-1,-1)$$

$$=-\frac{21}{4}\zeta(2)\zeta(3)+\frac{93}{8}\zeta(5)$$

We obtain the result by using that $\zeta(2)=\frac{\pi^2}{6}$