

# Mahler measure under variations of the base group

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# Mahler measure of several variable polynomials

$P \in \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ , the (logarithmic) *Mahler measure* is :

$$\begin{aligned} m(P) &= \int_0^1 \dots \int_0^1 \log |P(e^{2\pi i \theta_1}, \dots, e^{2\pi i \theta_n})| d\theta_1 \dots d\theta_n \\ &= \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log |P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \dots \frac{dx_n}{x_n} \end{aligned}$$

By Jensen's formula,

$$m\left(a \prod (x - \alpha_j)\right) = \log |a| + \sum \log \max\{1, |\alpha_j|\}.$$



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# Examples in several variables

- Smyth (1981)

$$m(1 + x + y) = \frac{3\sqrt{3}}{4\pi} L(\chi_{-3}, 2) = \frac{\text{Vol}(\text{Fig8})}{2\pi}$$

- Boyd, Deninger, Rodriguez-Villegas (1997)

$$m\left(x + \frac{1}{x} + y + \frac{1}{y} - 1\right) \stackrel{?}{=} L'(E_1, 0)$$

$E_1$  elliptic curve, projective closure of  $x + \frac{1}{x} + y + \frac{1}{y} - 1 = 0$ .  
(50 decimal places)



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# The general technique

Rodriguez-Villegas (1997)

$$P_\lambda(x, y) = 1 - \lambda P(x, y) \quad P(x, y) = x + \frac{1}{x} + y + \frac{1}{y}$$

Reciprocal

$$m(P, \lambda) := m(P_\lambda)$$

$$m(P, \lambda) = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log |1 - \lambda P(x, y)| \frac{dx}{x} \frac{dy}{y}.$$



## Note

$$|\lambda P(x, y)| < 1, \quad \lambda \text{ small}, \quad x, y \in \mathbb{T}^2$$

$$\begin{aligned} \tilde{m}(P, \lambda) &= \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \log(1 - \lambda P(x, y)) \frac{dx}{x} \frac{dy}{y} \\ &= - \sum_{n=1}^{\infty} \frac{\lambda^n}{n} \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} P(x, y)^n \frac{dx}{x} \frac{dy}{y} = - \sum_{n=1}^{\infty} \frac{a_n \lambda^n}{n} \\ &\quad a_n := [P(x, y)^n]_0 \end{aligned}$$



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Let

$$u(P, \lambda) = \frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \frac{1}{1 - \lambda P(x, y)} \frac{dx}{x} \frac{dy}{y} = \sum_{n=0}^{\infty} a_n \lambda^n$$

$$\frac{d\tilde{m}(P, \lambda)}{d\lambda} = -\frac{1}{(2\pi i)^2} \int_{\mathbb{T}^2} \frac{P(x, y)}{1 - \lambda P(x, y)} \frac{dx}{x} \frac{dy}{y}$$



In the case  $P = x + \frac{1}{x} + y + \frac{1}{y}$ ,

$$a_n = \begin{cases} \binom{2m}{m}^2 & n = 2m \\ 0 & \textit{otherwise} \end{cases}$$



## Definition

$\Gamma$  finitely generated group with generators  $x_1, \dots, x_l$

$$Q = Q(x_1, \dots, x_l) = \sum_{g \in \Gamma} c_g g \in \mathbb{C}\Gamma,$$

$$Q^* = \sum_{g \in \Gamma} \overline{c_g} g^{-1} \in \mathbb{C}\Gamma \text{ reciprocal.}$$

$P = P(x_1, \dots, x_l) \in \mathbb{C}\Gamma$ ,  $P = P^*$ ,  $|\lambda|^{-1} >$  length of  $P$ ,

$$m_\Gamma(P, \lambda) = - \sum_{n=1}^{\infty} \frac{a_n \lambda^n}{n},$$

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We also write

$$u_{\Gamma}(P, \lambda) = \sum_{n=0}^{\infty} a_n \lambda^n$$

for the generating function of the  $a_n$ .

$$Q(x_1, \dots, x_l) \in \mathbb{C}\Gamma$$

$$QQ^* = \frac{1}{\lambda} (1 - (1 - \lambda QQ^*))$$

for  $\lambda$  real and positive and  $1/\lambda$  larger than the length of  $QQ^*$ .

$$m_{\Gamma}(Q) = -\frac{\log \lambda}{2} - \sum_{n=1}^{\infty} \frac{b_n}{2n}, \quad b_n = [(1 - \lambda QQ^*)^n]_0.$$



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# Lück's combinatorial $L^2$ -torsion.

$K$  knot

$$\Gamma = \pi_1(S^3 \setminus K) = \langle x_1, \dots, x_g \mid r_1, \dots, r_{g-1} \rangle$$

Let

$$F = \begin{pmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_g} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_{g-1}}{\partial x_1} & \cdots & \frac{\partial r_{g-1}}{\partial x_g} \end{pmatrix} \in M^{(g-1) \times g}(\mathbb{C}\Gamma)$$

Fox matrix.

$$D(u + v) = D(u) + D(v)$$

$$D(u \cdot v) = D(u)\epsilon(v) + uD(v)$$

Delete a column  $F \rightsquigarrow A \in M^{(g-1) \times (g-1)}(\mathbb{C}\Gamma)$ .



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## Theorem (Lück, 2002)

Suppose  $K$  is a hyperbolic knot. Then, for  $\lambda$  sufficiently large

$$\frac{1}{3\pi} \text{Vol}(K) = -(g-1) \ln \lambda - \sum_{n=1}^{\infty} \frac{1}{n} \text{tr}_{\mathbb{C}\Gamma} ((1 - \lambda AA^*)^n).$$



# Cayley Graphs

$\Gamma$  of order  $m$

$$\alpha : \Gamma \rightarrow \mathbb{C} \quad \alpha(g) = \overline{\alpha(g^{-1})} \quad \forall g \in \Gamma$$

Weighted Cayley graph:

- Vertices  $g_1, \dots, g_m$ .
- (directed) Edge between  $g_i$  and  $g_j$  has weight  $\alpha(g_i^{-1}g_j)$ .

Weighted adjacency matrix

$$A(\Gamma, \alpha) = \{\alpha(g_i^{-1}g_j)\}_{i,j}$$



# The Mahler measure over finite groups

$\Gamma$

$$P \in \mathbb{C}\Gamma$$

reciprocal

Assume monomials generate  $\Gamma$ .

## Theorem

For  $\Gamma$  finite

$$m_{\Gamma}(P, \lambda) = \frac{1}{|\Gamma|} \log \det(I - \lambda A),$$

$A$  is the adjacency matrix of the Cayley graph (with weights) and  $\frac{1}{\lambda} > \rho(A)$ .

Analytic continuation for  $m_{\Gamma}(P, \lambda)$  to  $\mathbb{C} \setminus \text{Spec}(A)$ .



# Finite Abelian Groups

$$\Gamma = \mathbb{Z}/m_1\mathbb{Z} \times \cdots \times \mathbb{Z}/m_l\mathbb{Z}$$

## Corollary

$$m_\Gamma(P, \lambda) = \frac{1}{|\Gamma|} \log \left( \prod_{j_1, \dots, j_l} (1 - \lambda P(\xi_{m_1}^{j_1}, \dots, \xi_{m_l}^{j_l})) \right)$$

where  $\xi_k$  is a primitive root of unity.

Babai (1979): Spectra of Cayley graph is related to irreducible characters of  $\Gamma$ .



# Approximations

## Proposition

For small  $\lambda$ ,



$$\lim_{m_1, \dots, m_l \rightarrow \infty} m_{\mathbb{Z}/m_1\mathbb{Z} \times \dots \times \mathbb{Z}/m_l\mathbb{Z}}(P, \lambda) = m_{\mathbb{Z}^l}(P, \lambda).$$

Where the limit is with  $m_1, \dots, m_l$  going to infinity independently.

- For  $\Gamma = D_\infty$ ,  $\Gamma_m = D_m$ ,

$$\lim_{m \rightarrow \infty} m_{D_m}(P, \lambda) = m_{D_\infty}(P, \lambda).$$

- For  $\Gamma = PSL_2(\mathbb{Z}) = \langle x, y \mid x^2, y^3 \rangle$ ,  $\Gamma_m = \langle x, y \mid x^2, y^3, (xy)^m \rangle$ ,

$$\lim_{m \rightarrow \infty} m_{\Gamma_m}(P, \lambda) = m_{PSL_2(\mathbb{Z})}(P, \lambda).$$

# $x + x^{-1} + y + y^{-1}$ revisited

Now  $P = x + x^{-1} + y + y^{-1}$ .

$$u_{\mathbb{Z} \times \mathbb{Z}}(P, \lambda) = \sum_{n=0}^{\infty} \binom{2n}{n}^2 \lambda^{2n} = {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; 16\lambda^2\right)$$

$$u_{\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}}(P, \lambda) = \sum_{n=0}^{\infty} \binom{4n}{2n} \lambda^{2n}$$

$$u_{\mathbb{Z} * \mathbb{Z}}(P, \lambda) = \frac{3}{1 + 2\sqrt{1 - 12\lambda^2}}$$



Generating function of the circuits of a  $d$ -regular tree (Bartholdi, 1999).

$$g_d(\lambda) = \frac{2(d-1)}{d-2 + d\sqrt{1-4(d-1)\lambda^2}}.$$

$$x_1 + x_1^{-1} + \cdots + x_l + x_l^{-1}$$

$$(1 + x_1 + \cdots + x_{l-1})(1 + x_1^{-1} + \cdots + x_{l-1}^{-1})$$



# Recurrence relations $x + x^{-1} + y + y^{-1}$

Coefficients satisfy recurrence relations

$$\mathbb{Z} \times \mathbb{Z} : \quad n^2 a_{2n} - 4(2n - 1)^2 a_{2n-2} = 0$$

$$\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} : \quad n(2n - 1)a_{2n} - 2(4n - 1)(4n - 3)a_{2n-2} = 0$$

$$\mathbb{Z} * \mathbb{Z} : \quad na_{2n} - 2(14n - 9)a_{2n-2} + 96(2n - 3)a_{2n-4} = 0$$





- $\mathbb{Z}^l$

Rodriguez - Villegas:  $u(\lambda)$  periods of a holomorphic differential in the curve defined by  $1 = \lambda P(x, y)$ . By Griffiths (1969)

$$A(\lambda)u'' + B(\lambda)u' + C(\lambda)u = 0,$$

Picard-Fuchs differential equation ( $A, B, C$  polynomials).

$\Rightarrow$  Recurrence of the coefficients.

- This recurrence result extends to the case of  $\Gamma$  finitely generated abelian group.

- $\mathbb{F}_l$

By Haiman (1993):  $u(\lambda)$  is algebraic.

Algebraic functions in non-commuting variables.



$$P = x + x^{-1} + y + y^{-1}$$

$$\Gamma = \langle x, y \mid x^2y = yx^2, y^2x = xy^2 \rangle$$

Domb (1960)

$$a_{2n} = \sum_{k=0}^n \binom{n}{k} \binom{2k}{k} \binom{2n-2k}{n-k}$$

Same as ordinary Mahler measure for

$$1 - \lambda (x + x^{-1} + z (y + y^{-1})) (x + x^{-1} + z^{-1} (y + y^{-1}))$$



$$n^3 a_{2n} - 2(2n - 1)(5n^2 - 5n + 2)a_{2n-2} + 6(n - 1)^3 a_{2n-4} = 0$$

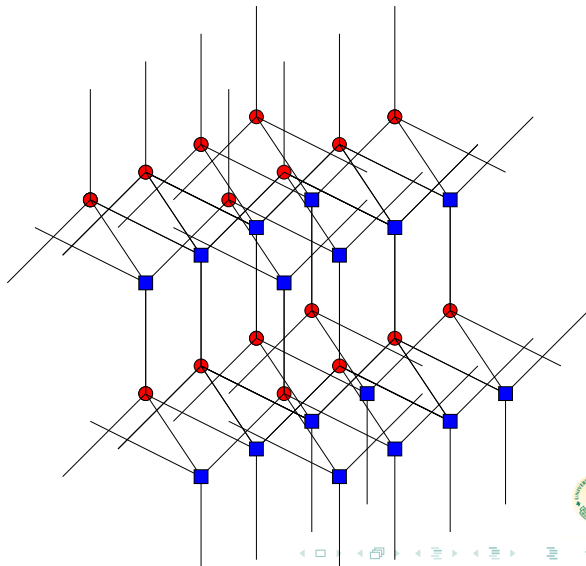
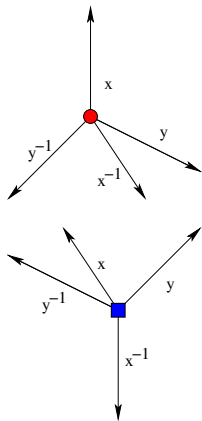
Rogers (2007)

$$1 - \lambda (4 + (x + x^{-1})(y + y^{-1}) + (y + y^{-1})(z + z^{-1}) + (z + z^{-1})(x + x^{-1}))$$

$${}_3F_2 \left( \frac{1}{3}, \frac{1}{2}, \frac{2}{3}; 1, 1; -\frac{108\lambda}{(1 - 16\lambda)^3} \right) = (1 - 16\lambda) \sum_{n=0}^{\infty} a_{2n} \lambda^n$$



# The diamond lattice



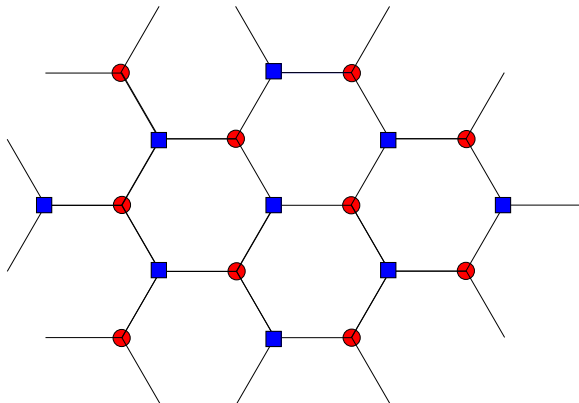
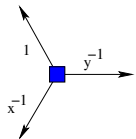
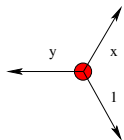
$$Q = (1 + x + y)(1 + x^{-1} + y^{-1})$$

$$[Q^n]_0 = a_n$$

$$n^2 a_n - (10n^2 - 10n + 3)a_{n-1} + 9(n-1)^2 a_{n-2} = 0,$$



# Honeycomb lattice $(1 + x + y)(1 + x^{-1} + y^{-1})$



$$P = x + x^{-1} + y + y^{-1} + xy^{-1} + x^{-1}y$$

$$[P^n]_0 = b_n$$

$$n^2 b_n - n(n-1)b_{n-1} - 24(n-1)^2 b_{n-2} - 36(n-2)(n-1)b_{n-3} = 0.$$

$$Q = 3 + P$$

$$b_n = \sum_{j=0}^n \binom{n}{j} (-3)^{n-j} a_j$$



# Triangular lattice $x + x^{-1} + y + y^{-1} + xy^{-1} + x^{-1}y$

