

Generalized multiplication tables

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	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

- There are 100 entries in the above table.
- Only 42 distinct integers appear among them.

Question (Erdős, 1955)

How big is $A(N) = |\{ab : a \leq N, b \leq N\}|$?

First answer

$$A(N) = o(N^2) \quad (N \rightarrow \infty).$$

For most n we have $\omega(n) \sim \log \log n$.

\Rightarrow For most pairs (a, b) with $a \leq N$ and $b \leq N$ we have $\omega(ab) \sim 2 \log \log N$.

\Rightarrow The density of $\{ab : a \leq N, b \leq N\}$ in $[1, N^2] \cap \mathbb{N}$ is 0.

Second answer

$$A(N) \ll \frac{N^2}{(\log N)^{Q(1/\log 2)} (\log \log N)^{1/2}},$$

where $Q(u) = u \log u - u + 1$.

To show this we need the following well-known result.

Theorem (Hardy-Ramanujan (1917))

There are absolute constants C_1 and C_2 such that for all $x \geq 2$ and all $r \in \mathbb{N}$ we have

$$\pi_r(x) := |\{n \leq x : \omega(n) = r\}| \leq \frac{C_1 x}{\log x} \frac{(\log \log x + C_2)^{r-1}}{(r-1)!}.$$

Fix $\lambda > 1$ and set $L = \lfloor \lambda \log \log N \rfloor$.

$$\begin{aligned}
 A^*(N) &:= |\{ab : a \leq N, b \leq N, (a, b) = 1\}| \\
 &\leq |\{n \leq N^2 : \omega(n) > L\}| \\
 &\quad + |\{(a, b) : a \leq N, b \leq N, \omega(a) + \omega(b) \leq L\}| \\
 &= \sum_{r > L} \pi_r(N^2) + \sum_{r+s \leq L} \pi_r(N) \pi_s(N) \\
 &\ll_{\lambda} (1 + (\log N)^{\lambda \log 2 - 1}) \frac{N^2}{(\log N)^{Q(\lambda)} (\log \log N)^{1/2}}
 \end{aligned}$$

So setting $\lambda = 1/\log 2$ yields

$$A^*(N) \ll \frac{N^2}{(\log N)^{Q(1/\log 2)} (\log \log N)^{1/2}}.$$

Thus setting $d = (a, b)$ for $a \leq N$ and $b \leq N$ yields

$$A(N) \leq \sum_{d \leq N} A^*(N/d) \ll \frac{N^2}{(\log N)^{Q(1/\log 2)} (\log \log N)^{1/2}}.$$

Theorem (Ford, 2004)

$$A(N) \asymp \frac{N^2}{(\log N)^{Q(1/\log 2)} (\log \log N)^{3/2}} \quad (N \geq 3).$$

A natural generalization of Erdős's question is to estimate the size of the $\underbrace{N \times \cdots \times N}_{k+1 \text{ times}}$ multiplication table, that is to bound

$$A_{k+1}(N) := |\{n_1 \cdots n_{k+1} : n_i \leq N (1 \leq i \leq k + 1)\}|.$$

Mimicking the argument we gave for $A(N)$ yields

$$A_{k+1}(N) \ll_k \frac{N^{k+1}}{(\log N)^{Q(k/\log(k+1))} (\log \log N)^{1/2}}.$$

In fact we have

Theorem (K, 2009)

$$A_{k+1}(N) \asymp_k \frac{N^{k+1}}{(\log N)^{Q(k/\log(k+1))} (\log \log N)^{3/2}} \quad (N \geq 3).$$

Next we broaden our scope further and ask for estimates on

$$A_{k+1}(N_1, \dots, N_{k+1}) := |\{n_1 \cdots n_{k+1} : n_i \leq N_i (1 \leq i \leq k+1)\}|.$$

When $k = 1$, Ford solved the problem completely.

Theorem (Ford (2004))

Let $3 \leq N_1 \leq N_2$. Then

$$A_2(N_1, N_2) \asymp \frac{N_1 N_2}{(\log N_1)^{Q(\frac{1}{\log 2})} (\log \log N_1)^{3/2}}.$$

When $k > 1$ we develop a heuristic argument which sheds some light to the behavior of $A_{k+1}(N_1, \dots, N_{k+1})$; it is a generalization of an argument given by Ford when $k = 1$.

Let $n \in \mathbb{N} \cap [N_1 \cdots N_{k+1}/4, N_1 \cdots N_{k+1}/2]$. For simplicity assume that n is square-free.

Goal: Understand when n is counted by $A_{k+1}(N_1, \dots, N_{k+1})$.

Consider the set

$$D_{k+1}(n; \mathbf{N}) = \{(\log n_1, \dots, \log n_k) : n_1 \cdots n_j \mid \prod_{p|n, p \leq N_j} p \ (1 \leq j \leq k)\}.$$

Main assumption: $D_{k+1}(n; \mathbf{N})$ is well-distributed in $[0, \log N_1] \times \cdots \times [0, \log N_k]$.

$$\begin{aligned}
\tau_{k+1}(n; \mathbf{N}) &= |\{n_1, \dots, n_{k+1} = n : n_j \leq N_j (1 \leq j \leq k+1)\}| \\
&= |\{n_1 \cdots n_k | n : n_j \leq N_j (1 \leq j \leq k), n_1 \cdots n_k \geq n/N_{k+1}\}| \\
&= \left| D_{k+1}(n; \mathbf{N}) \cap \left\{ \mathbf{x} \in \prod_{j=1}^k [0, \log N_j] : \sum_{j=1}^k x_j \geq \log(n/N_{k+1}) \right\} \right| \\
&\approx \frac{|D_{k+1}(n; \mathbf{N})|}{(\log N_1) \cdots (\log N_k)} = \prod_{j=1}^k \frac{(k-j+2)^{\omega(n; N_{j-1}, N_j)}}{\log N_j},
\end{aligned}$$

where $\omega(n; y, z) = |\{p | n : y < p \leq z\}|$. So if we let

$$\mathcal{H} = \left\{ (r_1, \dots, r_k) \in (\mathbb{N} \cup \{0\})^k : \sum_{i=1}^k r_i \log(k-i+2) \geq O(1) + \sum_{i=1}^k \log \log N_i \right\}$$

then $\tau_{k+1}(n; \mathbf{N}) \geq 1$ precisely when $(\omega(n; N_{j-1}, N_j))_{1 \leq j \leq k} \in \mathcal{H}$. So

$$A_{k+1}(N_1, \dots, N_{k+1}) \approx \sum_{\mathbf{r} \in \mathcal{H}} \left| \left\{ n \leq \prod_{i=1}^{k+1} N_i : \omega(n; N_{j-1}, N_j) = r_j (1 \leq j \leq k) \right\} \right|.$$

Consequently,

$$A_{k+1}(N_1, \dots, N_{k+1}) \approx \sum_{r \in \mathcal{H}} \frac{N_1 \cdots N_{k+1}}{\log N_k} \prod_{j=1}^k \frac{\ell_j^{r_j-1}}{(r_j-1)!},$$

where we have set $\ell_j = \log(3 \log N_j / \log N_{j-1})$ for $j \in \{1, \dots, k\}$. Using Lagrange multipliers and the saddle point method we find

$$A_{k+1}(N_1, \dots, N_{k+1}) \approx \frac{N_1 \cdots N_{k+1}}{\sqrt{\log \log N_k} \prod_{j=1}^k \left(\frac{\log N_j}{\log N_{j-1}} \right)^{Q((k-j+2)^\alpha)},}$$

where $\alpha = \alpha(k, \mathbf{N})$ is defined implicitly via the equation

$$\sum_{j=1}^k (k-j+2)^\alpha \log(k-j+2) \ell_j = \sum_{j=1}^k (k-j+1) \ell_j.$$

Theorem (K, 2010)

Let $3 = N_0 \leq N_1 \leq \dots \leq N_{k+1}$. Then

$$\frac{A_{k+1}(N_1, \dots, N_{k+1})}{N_1 \cdots N_{k+1}} \ll_k \frac{\min \left\{ 1, \frac{(\log \log 3N_{i_0-1})(\log 3 \frac{\log N_k}{\log N_{i_0}})}{\ell_{i_0}} \right\}}{\sqrt{\log \log N_k} \prod_{i=1}^k \left(\frac{\log N_i}{\log N_{i-1}} \right)^{Q((k-i+2)^\alpha)}},$$

where i_0 is such that $\ell_{i_0} = \max_{1 \leq i \leq k} \ell_i$.

Theorem (K, 2010)

Let $3 = N_0 \leq N_1 \leq \dots \leq N_{k+1}$. If

$$\alpha \geq 1 + \epsilon - \frac{1}{\log(k+1)} \log\left(\frac{(k+1)\log(k+1) - 2\log 2}{k-1}\right) \quad (*)$$

for some $\epsilon > 0$, then

$$\frac{A_{k+1}(N_1, \dots, N_{k+1})}{N_1 \cdots N_{k+1}} \asymp_{k, \epsilon} \frac{\min\left\{1, \frac{(\log \log 3N_{i_0-1})(\log 3 \frac{\log N_k}{\log N_{i_0}})}{\ell_{i_0}}\right\}}{\sqrt{\log \log N_k} \prod_{i=1}^k \left(\frac{\log N_i}{\log N_{i-1}}\right)^{Q((k-i+2)^\alpha)}}.$$

Remark

Condition () is optimal in the sense that for every fixed γ that satisfies*

$$\frac{1}{\log 2} \log\left(\frac{1}{\log 2}\right) < \gamma < 1 - \frac{1}{\log(k+1)} \log\left(\frac{(k+1)\log(k+1) - 2\log 2}{k-1}\right)$$

there is a choice of $N_1 \leq \dots \leq N_{k+1}$ such that $\alpha = \alpha(k; \mathbf{N}) = \gamma$ and for which the order of $A_{k+1}(N_1, \dots, N_{k+1})$ is smaller than the one stated in the previous slide.

When $k \in \{2, 3, 4, 5\}$ a computation shows that condition (*) is always satisfied. Consequently

Corollary

Let $k \in \{2, 3, 4, 5\}$ and $3 = N_0 \leq N_1 \leq \dots \leq N_{k+1}$. Then

$$\frac{A_{k+1}(N_1, \dots, N_{k+1})}{N_1 \cdots N_{k+1}} \asymp \frac{\min \left\{ 1, \frac{(\log \log 3N_{i_0-1})(\log 3 \frac{\log N_k}{\log N_{i_0}})}{\ell_{i_0}} \right\}}{\sqrt{\log \log N_k} \prod_{i=1}^k \left(\frac{\log N_i}{\log N_{i-1}} \right)^{Q((k-i+2)^\alpha)}}.$$

Thank you for your attention!