Generalized multiplication tables

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	1	2	3	4	5	6	7	8	9	10
1	1	2	3	4	5	6	7	8	9	10
2	2	4	6	8	10	12	14	16	18	20
3	3	6	9	12	15	18	21	24	27	30
4	4	8	12	16	20	24	28	32	36	40
5	5	10	15	20	25	30	35	40	45	50
6	6	12	18	24	30	36	42	48	54	60
7	7	14	21	28	35	42	49	56	63	70
8	8	16	24	32	40	48	56	64	72	80
9	9	18	27	36	45	54	63	72	81	90
10	10	20	30	40	50	60	70	80	90	100

- There are 100 entries in the above table.
- Only 42 distinct integers appear among them.

Question (Erdős, 1955)

How big is $A(N) = |\{ab : a \leq N, b \leq N\}|$?

First answer

$A(N) = o(N^2) \quad (N \to \infty)$.

For most n we have $\omega(n) \sim \log\log n$.

$\Rightarrow$ For most pairs (a, b) with $a \leq N$ and $b \leq N$ we have $\omega(ab) \sim 2\log\log N$.

$\Rightarrow$ The density of $\{ab : a \leq N, b \leq N\}$ in $[1, N^2] \cap \mathbb{N}$ is 0.

Second answer

$$A(N) \ll \frac{N^2}{(\log N)^{Q(1/\log 2)}(\log\log N)^{1/2}},$$

where $Q(u) = u\log u - u + 1$.

To show this we need the following well-known result.

Theorem (Hardy-Ramanujan (1917))

There are absolute constants C_1 and C_2 such that for all $x \geq 2$ and all $r \in \mathbb{N}$ we have

$$\pi_r(x) := |\{n \leq x : \omega(n) = r\}| \leq \frac{C_1 x}{\log x} \frac{(\log\log x + C_2)^{r-1}}{(r-1)!}.$$

Fix $\lambda > 1$ and set $L = \lfloor \lambda \log\log N \rfloor$.

$$\begin{aligned} A^*(N) &:= |\{ab : a \le N, b \le N, (a,b) = 1\}| \\ &\le |\{n \le N^2 : \omega(n) > L\}| \\ &\quad + |\{(a,b) : a \le N, b \le N, \omega(a) + \omega(b) \le L\}| \\ &= \sum_{r > L} \pi_r(N^2) + \sum_{r+s \le L} \pi_r(N)\pi_s(N) \\ &\ll_\lambda \left(1 + (\log N)^{\lambda \log 2 - 1}\right) \frac{N^2}{(\log N)^{Q(\lambda)} (\log\log N)^{1/2}} \end{aligned}$$

So setting $\lambda = 1/\log 2$ yields

$$A^*(N) \ll \frac{N^2}{(\log N)^{Q(1/\log 2)} (\log\log N)^{1/2}}.$$

Thus setting $d = (a, b)$ for $a \leq N$ and $b \leq N$ yields

$$A(N) \leq \sum_{d \leq N} A^*(N/d) \ll \frac{N^2}{(\log N)^{Q(1/\log 2)}(\log\log N)^{1/2}}.$$

Theorem (Ford,2004)

$$A(N) \asymp \frac{N^2}{(\log N)^{Q(1/\log 2)}(\log\log N)^{3/2}} \quad (N \geq 3).$$

A natural generalization of Erdős's question is to estimate the size of the $\underbrace{N \times \cdots \times N}_{k+1 \text{ times}}$ multiplication table, that is to bound

$$A_{k+1}(N) := |\{n_1 \cdots n_{k+1} : n_i \le N \ (1 \le i \le k+1)\}|.$$

Mimicking the argument we gave for $A(N)$ yields

$$A_{k+1}(N) \ll_k \frac{N^{k+1}}{(\log N)^{Q(k/\log(k+1))}(\log\log N)^{1/2}}.$$

In fact we have

Theorem (K, 2009)

$$A_{k+1}(N) \asymp_k \frac{N^{k+1}}{(\log N)^{Q(k/\log(k+1))}(\log\log N)^{3/2}} \quad (N \ge 3).$$

Next we broaden our scope further and ask for estimates on

$$A_{k+1}(N_1, \dots, N_{k+1}) := |\{n_1 \cdots n_{k+1} : n_i \le N_i \ (1 \le i \le k+1)\}|.$$

When $k = 1$, Ford solved the problem completely.

Theorem (Ford (2004))

Let $3 \le N_1 \le N_2$. *Then*

$$A_2(N_1, N_2) \asymp \frac{N_1 N_2}{(\log N_1)^{Q(\frac{1}{\log 2})} (\log\log N_1)^{3/2}}.$$

When $k > 1$ we develop a heuristic argument which sheds some light to the behavior of $A_{k+1}(N_1, \ldots, N_{k+1})$; it is a generalization of an argument given by Ford when $k = 1$.

Let $n \in \mathbb{N} \cap [N_1 \cdots N_{k+1}/4, N_1 \cdots N_{k+1}/2]$. For simplicity assume that n is square-free.

Goal: Understand when n is counted by $A_{k+1}(N_1, \ldots, N_{k+1})$.

Consider the set

$$D_{k+1}(n; \boldsymbol{N}) = \{(\log n_1, \ldots, \log n_k) : n_1 \cdots n_j | \prod_{p|n, p \le N_j} p \ (1 \le j \le k)\}.$$

Main assumption: $D_{k+1}(n; \boldsymbol{N})$ is well-distributed in $[0, \log N_1] \times \cdots \times [0, \log N_k]$.

$$\begin{aligned}
\tau_{k+1}(n;\boldsymbol{N}) &= |\{n_1,\dots,n_{k+1}=n : n_j\le N_j\ (1\le j\le k+1)\}| \\
&= |\{n_1\cdots n_k | n : n_j\le N_j\ (1\le j\le k),\ n_1\cdots n_k\ge n/N_{k+1}\}| \\
&= \left| D_{k+1}(n;\boldsymbol{N})\cap\left\{\boldsymbol{x}\in\prod_{j=1}^{k}[0,\log N_j] : \sum_{j=1}^{k}x_j\ge\log(n/N_{k+1})\right\}\right| \\
&\approx \frac{|D_{k+1}(n;\boldsymbol{N})|}{(\log N_1)\cdots(\log N_k)}=\prod_{j=1}^{k}\frac{(k-j+2)^{\omega(n;N_{j-1},N_j)}}{\log N_j},
\end{aligned}$$

where $\omega(n;y,z)=|\{p|n : y<p\le z\}|$. So if we let

$$\mathscr{H}=\left\{(r_1,\dots,r_k)\in(\mathbb{N}\cup\{0\})^k : \sum_{i=1}^{k}r_i\log(k-i+2)\ge O(1)+\sum_{i=1}^{k}\log\log N_i\right\}$$

then $\tau_{k+1}(n;\boldsymbol{N})\ge 1$ precisely when $(\omega(n;N_{j-1},N_j))_{1\le j\le k}\in\mathscr{H}$. So

$$A_{k+1}(N_1,\dots,N_{k+1})\approx\sum_{\boldsymbol{r}\in\mathscr{H}}\left|\left\{n\le\prod_{i=1}^{k+1}N_i : \omega(n;N_{j-1},N_j)=r_j\ (1\le j\le k)\right\}\right|.$$

Consequently,

$$A_{k+1}(N_1,\dots,N_{k+1}) \approx \sum_{\boldsymbol{r}\in\mathscr{H}} \frac{N_1\cdots N_{k+1}}{\log N_k} \prod_{j=1}^{k} \frac{\ell_j^{r_j-1}}{(r_j-1)!},$$

where we have set $\ell_j = \log(3\log N_j/\log N_{j-1})$ for $j\in\{1,\dots,k\}$. Using Lagrange multipliers and the saddle point method we find

$$A_{k+1}(N_1,\dots,N_{k+1}) \approx \frac{N_1\cdots N_{k+1}}{\sqrt{\log\log N_k}\prod_{j=1}^{k}\left(\frac{\log N_j}{\log N_{j-1}}\right)^{Q((k-j+2)^{\alpha})}},$$

where $\alpha=\alpha(k,\boldsymbol{N})$ is defined implicitly via the equation

$$\sum_{j=1}^{k}(k-j+2)^{\alpha}\log(k-j+2)\ell_j = \sum_{j=1}^{k}(k-j+1)\ell_j.$$

Theorem (K, 2010)

Let $3 = N_0 \le N_1 \le \cdots \le N_{k+1}$. *Then*

$$\frac{A_{k+1}(N_1,\ldots,N_{k+1})}{N_1\cdots N_{k+1}} \ll_k \frac{\min\left\{1, \frac{(\log\log 3N_{i_0-1})(\log 3\frac{\log N_k}{\log N_{i_0}})}{\ell_{i_0}}\right\}}{\sqrt{\log\log N_k}\prod_{i=1}^{k}\left(\frac{\log N_i}{\log N_{i-1}}\right)^{Q((k-i+2)^\alpha)}},$$

where i_0 *is such that* $\ell_{i_0} = \max_{1\le i\le k}\ell_i$.

Theorem (K, 2010)

Let $3 = N_0 \le N_1 \le \cdots \le N_{k+1}$. *If*

$$\alpha \ge 1 + \epsilon - \frac{1}{\log(k+1)} \log\Big(\frac{(k+1)\log(k+1) - 2\log 2}{k-1}\Big) \qquad (*)$$

for some $\epsilon > 0$, *then*

$$\frac{A_{k+1}(N_1, \ldots, N_{k+1})}{N_1 \cdots N_{k+1}} \asymp_{k,\epsilon} \frac{\min\Big\{1, \frac{(\log\log 3N_{i_0 - 1})(\log 3\frac{\log N_k}{\log N_{i_0}})}{\ell_{i_0}}\Big\}}{\sqrt{\log\log N_k} \prod_{i=1}^{k} \Big(\frac{\log N_i}{\log N_{i-1}}\Big)^{Q((k-i+2)^\alpha)}}.$$

Remark

Condition (*) is optimal in the sense that for every fixed γ that satisfies

$$\frac{1}{\log 2}\log\Big(\frac{1}{\log 2}\Big) < \gamma < 1-\frac{1}{\log(k+1)}\log\Big(\frac{(k+1)\log(k+1)-2\log 2}{k-1}\Big)$$

there is a choice of $N_1 \le \cdots \le N_{k+1}$ such that $\alpha = \alpha(k;\boldsymbol{N}) = \gamma$ and for which the order of $A_{k+1}(N_1,\ldots,N_{k+1})$ is smaller than the one stated in the previous slide.

When $k \in \{2,3,4,5\}$ a computation shows that condition (*) is always satisfied. Consequently

Corollary

Let $k \in \{2,3,4,5\}$ *and* $3 = N_0 \le N_1 \le \cdots \le N_{k+1}$ *. Then*

$$\frac{A_{k+1}(N_1,\dots,N_{k+1})}{N_1\cdots N_{k+1}} \asymp \frac{\min\left\{1, \frac{(\log\log 3N_{i_0-1})(\log 3\frac{\log N_k}{\log N_{i_0}})}{\ell_{i_0}}\right\}}{\sqrt{\log\log N_k}\prod_{i=1}^{k}\left(\frac{\log N_i}{\log N_{i-1}}\right)^{Q((k-i+2)^\alpha)}}.$$

Thank you for your attention!