AN ALMOST SHARP QUANTITATIVE VERSION OF THE DUFFIN–SCHAEFFER CONJECTURE

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ABSTRACT. We prove a quantitative version of the Duffin–Schaeffer conjecture with an almost sharp error term. Precisely, let $\psi : \mathbb{N} \to [0, 1/2]$ be a function such that the series $\sum_{q=1}^{\infty} \varphi(q)\psi(q)/q$ diverges. In addition, given $\alpha \in \mathbb{R}$ and $Q \geqslant 1$, let $N(\alpha; Q)$ be the number of coprime pairs $(a, q) \in \mathbb{Z} \times \mathbb{N}$ with $q \leqslant Q$ and $|\alpha - a/q| < \psi(q)/q$. Lastly, let $\Psi(Q) = \sum_{q \leqslant Q} 2\varphi(q)\psi(q)/q$, which is the expected value of $N(\alpha; Q)$ when α is uniformly chosen from $[0, 1]$. We prove that $N(\alpha; Q) = \Psi(Q) + O_{\alpha,\varepsilon}(\Psi(Q)^{1/2+\varepsilon})$ for almost all α (in the Lebesgue sense) and for every fixed $\varepsilon > 0$. This improves upon results of Koukoulopoulos–Maynard and of Aistleitner–Borda–Hauke.

1. INTRODUCTION

A classical result of Khintchine [21] states that if $\psi : \mathbb{N} \to [0, 1/2]$ is a function such that $q\psi(q)$ is non-increasing and $\sum_q \psi(q) = \infty$, then for almost all $\alpha \in \mathbb{R}$ (that is, for α in a set of full Lebesgue measure) there are infinitely many pairs $(a, q) \in \mathbb{Z} \times \mathbb{N}$ such that

$$\left|\alpha - \frac{a}{q}\right| < \frac{\psi(q)}{q}. \tag{1.1}$$

Conversely, if $\sum_q \psi(q) < \infty$ then the Borel–Cantelli lemma implies that for almost all $\alpha \in \mathbb{R}$ there are only finitely many pairs $(a, q) \in \mathbb{Z} \times \mathbb{N}$ satisfying (1.1). Hence, Khintchine's theorem is optimal up to the hypothesis that $q\psi(q)$ is non-increasing. In fact, in modern treatments of Khintchine's theorem (e.g., see [18]) this condition is relaxed to simply require that ψ is non-increasing.

Erdős [12] and Schmidt [29] proved a quantitative version of Khintchine's theorem: if ψ is non-increasing and we let

$$\Psi_1(Q) := \sum_{q \leqslant Q} \psi(q),$$

then we have that [18, Theorem 4.1]

$$\#\Big\{(a, q) \in \mathbb{Z}^2 : \ q \leqslant Q, \ (1.1) \text{ holds}\Big\} = \Psi_1(Q) + O_\varepsilon\Big(\Psi_1(Q)^{1/2}(\log \Psi_1(Q))^{2+\varepsilon}\Big) \tag{1.2}$$

for almost all $\alpha \in \mathbb{R}$, and for all Q sufficiently large in terms of α and ψ. As a matter of fact, if we impose some even stronger conditions on ψ, we know from work of Fuchs [15] that the quantity on the left-hand side of (1.2) satisfies a Central Limit Theorem. In particular, the error term on the right-hand side of (1.2) is nearly sharp.

Duffin and Schaeffer [10] undertook a study to understand to what extent Khintchine's theorem holds without the hypothesis that ψ is non-increasing. They realized that the fact that the fractions a/q are not reduced can lead to degenerate situations, with the series $\sum_{q=1}^{\infty} \psi(q)$ diverging but with

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the inequality (1.1) having only finitely many solutions for almost all α. This led them to formulate the following conjecture: if $\psi : \mathbb{N} \to \mathbb{R}_{\geqslant 0}$ is a function such that $\sum_{q=1}^{\infty} \psi(q)\varphi(q)/q = \infty$ (with φ denoting the Euler totient function), then for almost all $\alpha \in \mathbb{R}$ the inequality (1.1) is satisfied by infinitely many pairs $(a, q) \in \mathbb{Z} \times \mathbb{N}$ of *coprime* integers. Conversely, the first Borel–Cantelli lemma readily implies that if $\sum_{q=1}^{\infty} \psi(q)\varphi(q)/q < \infty$, then for almost all $\alpha \in \mathbb{R}$ there are only finitely many coprime pairs $(a, q) \in \mathbb{Z} \times \mathbb{N}$ satisfying (1.1). This would therefore give a full 0-1 law with a simple criterion to determine whether the measure of approximable α is full or null.

The Duffin–Schaeffer conjecture was resolved in full by the first two named authors [24] after important partial results by many authors [1, 2, 4, 6, 7, 8, 9, 11, 13, 14, 16, 20, 27, 30]. See also [23] for an exposition on the history of the conjecture.

The work of [24] left open the quantitative question of the number of approximations, corresponding to (1.2). Given a function $\psi : \mathbb{N} \to [0, 1/2]$, let

$$\Psi(Q) := 2 \sum_{q \leqslant Q} \frac{\varphi(q)\psi(q)}{q}. \tag{1.3}$$

Aistleitner, Borda and Hauke [3] proved that if $\lim_{Q\to\infty} \Psi(Q) = \infty$, then for almost all α and all $C > 0$, we have

$$\#\Big\{(a, q) \in \mathbb{Z} \times \mathbb{N} :\ q \leqslant Q,\ (1.1) \text{ holds},\ \gcd(a, q) = 1\Big\} = \Psi(Q) + O_C\Big(\frac{\Psi(Q)}{(\log \Psi(Q))^C}\Big), \tag{1.4}$$

whenever Q was sufficiently large in terms of α and ψ. Therefore the expected asymptotic holds for almost all $\alpha \in \mathbb{R}$, in keeping with (1.2). The main goal of our paper is to establish a refinement of (1.4), which obtains a near-sharp error term similar to the one in (1.2).

Theorem 1. *Let $\psi : \mathbb{N} \to [0, 1/2]$. Assume that $\Psi(Q) = 2\sum_{q\leqslant Q} \psi(q)\varphi(q)/q \to \infty$ as $Q \to \infty$. Then there exists a set $\mathcal{B}$ of Lebesgue measure 0 such that, for all $\alpha \in \mathbb{R} \setminus \mathcal{B}$ and $\varepsilon > 0$,*

$$\#\Big\{(a, q) \in \mathbb{Z} \times \mathbb{N} :\ q \leqslant Q,\ \Big|\alpha - \frac{a}{q}\Big| < \frac{\psi(q)}{q},\ \gcd(a, q) = 1\Big\} = \Psi(Q) + O_\varepsilon\Big(\Psi(Q)^{\frac{1}{2}+\varepsilon}\Big)$$

for all Q sufficiently large in terms of α and ψ.

LeVeque [26] showed that when ψ satisfies certain more stringent conditions, the quantity on the left-hand side of (1.4) satisfies a Central Limit Theorem analogous to Fuchs' result [15] for (1.2). In particular, the error term in Theorem 1 cannot be $O(\Psi(Q)^{1/2})$, and so Theorem 1 is sharp up to a factor of $\Psi(Q)^{o(1)}$. We have made no attempt to quantify more precisely this $\Psi(Q)^{o(1)}$ factor; as written the argument here would likely give a quantification $\Psi(Q)^{1/2+O((\log\log \Psi(Q))^{-c})}$ for some $c > 0$, but with a bit more care (particularly in Proposition 8.3) this could surely be improved.

Remark. LeVeque originally claimed a result analogous to Fuch's theorem [25]. He corrected his claim in [26], noticing that his proof is instead suited for the left-hand side of (1.4).

In Section 16, we give a different and simple proof of the optimality of the exponent $1/2$ in Theorem 1.

Notation. We shall use the letter λ to denote the Lebesgue measure on $\mathbb{R}$.

Sets will be typically denoted by capital calligraphic letters such as $\mathcal{A}, \mathcal{V}$ and $\mathcal{E}$. A triple $G = (\mathcal{V}, \mathcal{W}, \mathcal{E})$ denotes a bipartite graph with vertex sets $\mathcal{V}$ and $\mathcal{W}$ and edge set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{W}$.

Given a set or an event $\mathcal{E}$, we let $\mathbb{1}_{\mathcal{E}}$ denote its indicator function.

The letter p will always denote a prime number. We also write $p^k \| n$ to mean that p^k is the exact power of p dividing the integer n.

2. Proof outline

Throughout the rest of this paper, we fix a function $\psi : \mathbb{N} \to [0,1/2]$ such that $\Psi(Q) \to \infty$ (where Ψ is as defined in (1.3)) as $Q \to \infty$. In addition, we set

$$N(\alpha;Q) = \#\Big\{(a,q) \in \mathbb{Z}\times\mathbb{N} :\, q \leqslant Q,\ \Big|\alpha - \frac{a}{q}\Big| < \frac{\psi(q)}{q},\ \gcd(a,q)=1\Big\}.$$

With this notation, the conclusion of Theorem 1 is that

$$N(\alpha;Q) = \Psi(Q) + O_\varepsilon\Big(\Psi(Q)^{\frac{1}{2}+\varepsilon}\Big)$$

for every fixed $\varepsilon > 0$, for almost all $\alpha \in \mathbb{R}$ and for all Q sufficiently large in terms of α. As a matter of fact, the function $\alpha \to N(\alpha;Q)$ is 1-periodic, so that it suffices to prove the above estimate for almost all $\alpha \in [0,1]$.

Now, let

$$\begin{aligned}\mathcal{A}_q &:= \Big\{\alpha \in [0,1] : \Big|\alpha - \frac{a}{q}\Big| < \frac{\psi(q)}{q} \text{ for some } a \in \mathbb{Z} \text{ coprime with } q\Big\}\\ &= [0,1] \cap \bigsqcup_{\substack{0\leqslant a\leqslant q\\ \gcd(a,q)=1}} \Big(\frac{a}{q} - \frac{\psi(q)}{q}, \frac{a}{q} + \frac{\psi(q)}{q}\Big),\end{aligned}$$

where $\sqcup$ denotes a disjoint union. The fact that the above union is disjoint follows readily from our assumption that $\psi(q) \in [0,1/2]$. For the same reason, we restricted the union to $a \in [0,q]$, because otherwise the interval $(\frac{a-\psi(q)}{q}, \frac{a+\psi(q)}{q})$ does not intersect $[0,1]$. We thus have

$$N(\alpha;Q) = \sum_{q\leqslant Q} \mathbb{1}_{\mathcal{A}_q}(\alpha) \quad \text{for all } \alpha \in [0,1], \tag{2.1}$$

as well as

$$\lambda(\mathcal{A}_q) = \frac{2\varphi(q)\psi(q)}{q}.$$

As a consequence,

$$\int_0^1 N(\alpha;Q)\,\mathrm{d}\alpha = \sum_{q\leqslant Q} \lambda(\mathcal{A}_q) = \Psi(Q). \tag{2.2}$$

We shall deduce Theorem 1 from the following bound on the variance of the random variable $[0,1] \ni \alpha \to N(\alpha;Q)$.

Theorem 2. *Assume the above notation. Then, for every fixed $\varepsilon > 0$, we have*

$$\int_0^1 \big(N(\alpha;Q) - \Psi(Q)\big)^2 \mathrm{d}\alpha \leqslant \Psi(Q) + O_\varepsilon\big(\Psi(Q)^{1+\varepsilon}\big)\,;$$

the implied constant depends at most on ε.

As with all previous works, the key to Theorem 2 is to show that $\lambda(\mathcal{A}_q \cap \mathcal{A}_r) \lessapprox \lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r)$ for 'most' pairs q, r, which can be thought of as a quantitative 'quasi-independence on average' of the events $\mathcal{A}_q$. Typically, one first attacks this via a counting argument, which shows

$$\frac{\lambda(\mathcal{A}_q\cap\mathcal{A}_r)}{\lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r)}\leqslant\frac{\sum_{\gcd(c,n)=1} w\Big(\frac{c}{C}\Big)\sum_{d|\gcd(\ell,c)} f(d)}{\Big(\frac{\varphi(n)}{n}\Big)\Big(\int_0^\infty w\Big(\frac{t}{C}\Big)dt\Big)\Big(\sum_{d|\ell}\frac{f(d)}{d}\Big)},$$

for some integers n,ℓ which divide qr, length of summation $C=\max(r\psi(q),q\psi(r))/\gcd(q,r)$, continuous compactly supported weight function w and multiplicative function f. Thus, to show the right hand side is close to 1 we wish to show that the numerator, a weighted sum of a multiplicative function, is close to its expected size. The function f and weight w only cause minor technical inconveniences, but more substantial complications are caused by the restriction $\gcd(c,n)=1$. Thus morally the key issue is to get a good upper bound for the number of $c\leqslant C$ which are coprime to n. We want an upper bound which is as close to the expected $\varphi(n)C/n$ as possible, but with good uniformity in the size of n and C. The work of Pollington and Vaughan [27] uses a standard upper bound sieve, which essentially gives

$$\sum_{\substack{c\leqslant C\\ \gcd(c,n)=1}} 1\ll\frac{\varphi(n)}{n}C\prod_{\substack{p|n\\ p\geqslant C}}\Big(1+\frac{1}{p}\Big). \tag{2.3}$$

The loss factor of the product over $p\geqslant C$ is typically close to 1, and the arguments of [24] use this to show $\lambda(\mathcal{A}_q\cap\mathcal{A}_r)\ll\lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r)$ for 'most' q,r, which is sufficient to resolve the Duffin-Schaeffer conjecture, but not to give better quantitative estimates.

In [3], a fundamental-lemma style sieve was used, giving for any choice of the parameter u

$$\sum_{\substack{c\leqslant C\\ \gcd(c,n)=1}} 1\leqslant\frac{\varphi(n)}{n}C(1+O(u^{-u}))\prod_{\substack{p|n\\ p\geqslant C^{1/u}}}\Big(1+\frac{1}{p}\Big). \tag{2.4}$$

This improves upon the Pollington-Vaughan bound because it no longer loses a constant factor if u is moderately large and the final product is close to 1. By choosing u appropriately and using this bound in the arguments of [24], they were able to show $\lambda(\mathcal{A}_q\cap\mathcal{A}_r)\leqslant(1+o(1))\lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r)$ for most q, r, which then gave the result (1.4). Unfortunately the quantification of the $o(1)$ is severely limited by the limitations on the possible choices of the parameter u to ensure that the $O(u^{-u})$ error term is small but the product over primes $p>C^{1/u}$ is still typically close to 1, and it appears that there is no version of this argument which could yield a power-saving bound in (1.4).

In our work, we instead use a Legendre sieve to get an upper bound when q and r have suitably generic prime factorisations. Let n^* denote the t-smooth part of n. Then we have

$$\sum_{\substack{c\leqslant C\\ \gcd(c,n)=1}} 1\leqslant\sum_{\substack{c\leqslant C\\ \gcd(c,n^*)=1}} 1=C\frac{\varphi(n^*)}{n^*}+O\Big(2^{\#\{p|n^*\}}\Big).$$

When t is chosen appropriately, this gives a good estimate provided q and r have somewhat typical factorisations. In particular, this gives a good quantitative estimate for suitably 'generic' q, r. We then separately handle 'non-generic' pairs q, r using (2.3) and adapting the methods of [24] to exploit the unusual factorisations. The crucial point is that for those 'non-generic' pairs, we have an extra decay (roughly speaking, either exponential decay or polynomial decay) in the summation. Thus the crude estimate $\lambda(\mathcal{A}_q\cap\mathcal{A}_r)\ll\lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r)$ is sufficient for our purpose.

An adoption of the above strategy would enable one to get a result similar to Theorem 1, but with the exponent $1/2$ in the error term replaced by some number in $(1/2, 1)$ close to 1. To get the full quantitative strength of Theorem 1 we need to modify some of the technical definitions in [24] (most notably, the definition of 'quality' of a GCD graph) and verify that the arguments of [24] still hold with these quantitatively stronger definitions.

Remark. Hauke–Saez–Walker [19] have recently refined the argument of [3] to obtain an estimate $\Psi(Q) + O(\Psi(Q)\exp(-(\log\Psi(Q))^{1/2+o(1)}))$ for the left hand side of (1.4), which appears to be close to the strongest possible quantification obtainable from using only the overlap estimate (2.4). Their strategy (building on the earlier work of Green–Walker [17]) also contains some additional ideas similar to the modification of the definition of quality mentioned above.

3. Structure of the paper

The first part of the paper consists of Sections 4-7 whose goal is to reduce the proof of Theorem 1 to three bilinear estimates (Propositions 7.1-7.3). Specifically, in Section 4, we show how Theorem 2 implies Theorem 1. In Section 5 we establish a technical estimate about the measure of the intersection of two events $\mathcal{A}_q$ and $\mathcal{A}_r$; this is key for our improvement over the Aistleitner–Borda–Hauke estimate (1.4). In Section 6, we present two auxiliary statistical lemmas about the multiplicative 'anatomy' of a random integer; these serve as the underlying cause for the additional decay in two of the bilinear estimates of Section 7 (Propositions 7.2 and 7.3). Lastly, in Section 7, we state the three key bilinear estimates (Propositions 7.1-7.3) and deduce Theorem 2 from them.

The second part of the paper consists of Sections 8-15. In this part, we prove Propositions 7.1-7.3 using the language GCD graphs developed in [24]. In Section 8, we present the definition of a GCD graph and of other related notions, and we present three iterative Propositions (Propositions 8.1-8.3) and three structural lemmas (Lemmas 8.4-8.6) about GCD graphs. In Section 9, we show how the results of Section 8 imply the three bilinear estimates of Section 7 (Propositions 7.1-7.3). In Section 10, we prove Lemma 8.6. In Section 11, we collect some preliminary properties of GCD graph. Section 12 is dedicated to the proof of the anatomical Lemmas 8.4 and 8.5, Section 13 to the proof of Proposition 8.1. Finally, in Sections 14 and 15, we prove Propositions 8.3 and 8.2, respectively.

4. Proof of Theorem 1 assuming Theorem 2

This claimed deduction follows by a straightforward adaption of Lemma 1.5 in [18]. Let $Q_0 = 0$ and, for each $j \in \mathbb{N}$, let

$$Q_j := \max\{Q \geqslant 0 : \Psi(Q) < j\}.$$

We have that $\Psi(Q_j) \in [j-1, j)$ for all $j \geqslant 1$. Hence, if $Q \in [Q_j, Q_{j+1})$, then $\Psi(Q) = j + O(1)$, and we also have that

$$N(\alpha; Q_j) \leqslant N(\alpha; Q) \leqslant N(\alpha; Q_{j+1}).$$

This reduces Theorem 1 to proving that

$$N(\alpha; Q_j) = \Psi(Q_j) + O_\varepsilon\Big(\Psi(Q_j)^{\frac{1}{2}+\varepsilon}\Big) \tag{4.1}$$

for almost all $\alpha \in [0, 1]$ and all j sufficiently large in terms of α.

For each $r \in \mathbb{N}$, let $\mathcal{E}_r$ be the set of $\alpha \in [0, 1]$ such that

$$\big|N(\alpha; Q_j) - \Psi(Q_j)\big| > 2^{(\frac{1}{2}+\varepsilon)r} \quad \text{for some } j \in [2^r, 2^{r+1}),$$

and let $\mathcal{B} = \limsup_{r\to\infty} \mathcal{E}_r$ be the set of $\alpha \in [0,1]$ which lie in infinitely many $\mathcal{E}_r$. Since we also have that $\Psi(Q_j) \asymp 2^r$ whenever $j \in [2^{r-1}, 2^r)$, we have (4.1) for all $\alpha \in [0,1]\setminus\mathcal{B}$ and j sufficiently large in terms of α. Thus it suffices to show that $\lambda(\mathcal{B}) = 0$. We will prove that

$$\lambda(\mathcal{E}_r) \ll_\varepsilon 1/r^2. \tag{4.2}$$

This relation implies $\sum_{r=1}^\infty \lambda(\mathcal{E}_r) < \infty$, so the first Borel–Cantelli lemma [18, Lemma 1.2] then shows that $\lambda(\mathcal{B}) = 0$, giving Theorem 1.

Now, let us fix $r \in \mathbb{N}$ and demonstrate (4.2). Each integer $j \in [2^r, 2^{r+1})$ can be written in binary form as $j = \sum_{t=0}^r b_t 2^t$ with $b_t \in \{0,1\}$ for $t < r$ and $b_r = 1$. For $s = 0, 1, \dots, r+1$, let us define $j_s = \sum_{t\geqslant s} b_t 2^t$ with the convention that $j_{r+1} = 0$. We then have that for any $j \in [2^r, 2^{r+1})$

$$\begin{aligned}\sum_{q\leqslant Q_j} \big(\mathbb{1}_{\mathcal{A}_q}(\alpha) - \lambda(\mathcal{A}_q)\big) &= \sum_{s=0}^r \sum_{Q_{j_{s+1}}<q\leqslant Q_{j_s}} \big(\mathbb{1}_{\mathcal{A}_q}(\alpha) - \lambda(\mathcal{A}_q)\big) \\ &\leqslant (r+1)^{1/2}\Bigg(\sum_{s=0}^r \Big|\sum_{Q_{j_{s+1}}<q\leqslant Q_{j_s}} \big(\mathbb{1}_{\mathcal{A}_q}(\alpha) - \lambda(\mathcal{A}_q)\big)\Big|^2\Bigg)^{1/2}\end{aligned}$$

by the Cauchy–Schwarz inequality. Evidently, we may restrict the summation to those numbers s for which $j_s > j_{s+1}$. For such s we have that $j_s = j_{s+1} + 2^s$ as well as $j_{s+1} = 2^{s+1}i$ with $i < 2^{r-s}$. Hence, if we set

$$\Delta(i,s,\alpha) := \sum_{Q_{i2^{s+1}}<q\leqslant Q_{2^s+i2^{s+1}}} \big(\mathbb{1}_{\mathcal{A}_q}(\alpha) - \lambda(\mathcal{A}_q)\big),$$

then we conclude that

$$\Big|\sum_{q\leqslant Q_j} \big(\mathbb{1}_{\mathcal{A}_q}(\alpha) - \lambda(\mathcal{A}_q)\big)\Big|^2 \leqslant (r+1)\sum_{0\leqslant s\leqslant r}\sum_{0\leqslant i<2^{r-s}} \big|\Delta(i,s,\alpha)\big|^2$$

for each $j \in [2^r, 2^{r+1})$. We now note that the right-hand side above doesn't depend on j. Hence, if $\alpha \in \mathcal{E}_r$, then we must have

$$\sum_{0\leqslant s\leqslant r}\sum_{0\leqslant i<2^{r-s}} \big|\Delta(i,s,\alpha)\big|^2 > \frac{2^{(1+2\varepsilon)r}}{r+1}.$$

Using Markov's inequality, we arrive at the bound

$$\lambda(\mathcal{E}_r) \leqslant \frac{r+1}{2^{(1+2\varepsilon)r}}\sum_{0\leqslant s\leqslant r}\sum_{0\leqslant i<2^{r-s}} \int_0^1 \big|\Delta(i,s,\alpha)\big|^2\,\mathrm{d}\alpha. \tag{4.3}$$

Fix for the moment s and i as above, and let $Q' = Q_{i2^{s+1}}$, $Q = Q_{2^s+i2^{s+1}}$ and $\psi'(q) = \mathbb{1}_{q>Q'}\psi(q)$. We see that

$$\sum_{q\leqslant Q} \frac{\psi'(q)\varphi(q)}{q} = \sum_{Q'<q\leqslant Q} \frac{\varphi(q)\psi(q)}{q} = \Psi(Q) - \Psi(Q') = 2^s + O(1),$$

since $\Psi(Q_j) = j + O(1)$ for all $j \in \mathbb{Z}_{\geqslant 0}$. To estimate the right-hand side of (4.3), we apply Theorem 2 with the above choice of Q and $\psi'(q)$ in place of ψ. As a consequence,

$$\lambda(\mathcal{E}_r) \ll_\varepsilon \frac{r+1}{2^{(1+2\varepsilon)r}}\sum_{0\leqslant s\leqslant r}\sum_{0\leqslant i<2^{r-s}} 2^{s(1+\varepsilon)} \ll_\varepsilon \frac{r+1}{2^{\varepsilon r}} \ll_\varepsilon \frac{1}{r^2}.$$

This proves (4.2), thus completing the deduction of Theorem 1 from Theorem 2.

5. Overlap estimates

Throughout the rest of the paper, we let

$$D = D(q,r) = \frac{\max\left(r\psi(q), q\psi(r)\right)}{\gcd(q,r)}. \tag{5.1}$$

for any pair of natural numbers (q,r). In addition, if $t \geqslant 1$ is a real number, then we let

$$L_t(q,r) := \sum_{\substack{p \mid qr/\gcd(q,r)^2 \\ p>t}} \frac{1}{p} \qquad \text{and} \qquad \omega_t(q,r) := \#\left\{p \,\middle|\, \frac{qr}{\gcd(q,r)^2} : p \leqslant t\right\}. \tag{5.2}$$

It does not seem possible to improve the overlap estimate in [3, Lemma 5] for arbitrary pair of integers q and r, so we improve the overlap estimate for generic pairs of integers.

Lemma 5.1 (Overlap estimate for generic pairs of integers). *Let $q \neq r$ be two natural numbers and let $D = D(q,r)$. For any real number $t \geqslant 1$, we have*

$$\lambda(\mathcal{A}_q \cap \mathcal{A}_r) \leqslant \mathbb{1}_{D\geqslant 1/2} \cdot \lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r)e^{2L_t(q,r)}\left(1 + O\left(\frac{2^{\omega_t(q,r)}\log(4D)}{D}\right)\right).$$

Proof. If $D < 1/2$, we know from [27, Section 3] that $\mathcal{A}_q \cap \mathcal{A}_r = \emptyset$. So let us assume that $D \geqslant 1/2$. Let $\nu_p(\cdot)$ denote the p-adic valuation function, let

$$\ell = \prod_{p\colon\, \nu_p(q)=\nu_p(r)} p^{\nu_p(q)}, \quad m = \prod_{p\colon\, \nu_p(q)\neq\nu_p(r)} p^{\min\{\nu_p(q),\nu_p(r)\}}, \quad n = \prod_{p\colon\, \nu_p(q)\neq\nu_p(r)} p^{\max\{\nu_p(q),\nu_p(r)\}},$$

and let

$$w(y) = \begin{cases} 2\delta & \text{if } 0 \leqslant y \leqslant \Delta - \delta, \\ \Delta + \delta - y & \text{if } \Delta - \delta < y \leqslant \Delta + \delta, \\ 0 & \text{otherwise,} \end{cases}$$

with

$$\delta = \min\left\{\frac{\psi(q)}{q}, \frac{\psi(r)}{r}\right\} \quad \text{and} \quad \Delta = \max\left\{\frac{\psi(q)}{q}, \frac{\psi(r)}{r}\right\}.$$

Finally, let $s = \mathbb{1}_{2|\ell}$. Then, arguing as in the proof of Lemma 5 of [3], we have

$$\lambda(\mathcal{A}_q \cap \mathcal{A}_r) = 2^{s+1}\varphi(m)\frac{\varphi(\ell)^2}{\ell}\prod_{\substack{p|\ell \\ p>2}}\left(1 - \frac{1}{(p-1)^2}\right)\sum_{\substack{c\geqslant 1 \\ \gcd(c,n)=1}} w\left(\frac{2^s c}{\ell n}\right)\prod_{\substack{p|\gcd(\ell,c) \\ p>2}}\left(1 + \frac{1}{p-2}\right).$$

Since w is supported on $[0, \Delta + \delta] \subseteq [0, 2\Delta]$, we must have $c \leqslant 2\Delta\ell n = 2D$. In particular, we may restrict the last product to primes $p \leqslant 2D$. In addition, we may replace the condition $\gcd(c,n) = 1$ by the condition $\gcd(c,n^*) = 1$, where n^* denotes the t-smooth part of n, at the cost of replacing the exact expression for $\lambda(\mathcal{A}_q \cap \mathcal{A}_r)$ by an upper bound. We detect the condition that $\gcd(c,n^*) = 1$ using Möbius inversion.

In conclusion, the above discussion implies that

$$\lambda(\mathcal{A}_q \cap \mathcal{A}_r) \leqslant 2^{s+1}\varphi(m)\frac{\varphi(\ell)^2}{\ell}\prod_{\substack{p|\ell \\ p>2}}\left(1 - \frac{1}{(p-1)^2}\right)\sum_{a|n^*}\mu(a)\sum_{\substack{c\geqslant 1 \\ a|c}} w\left(\frac{2^s c}{\ell n}\right)\prod_{\substack{p|\gcd(\ell,c) \\ 2<p\leqslant 2D}}\left(1 + \frac{1}{p-2}\right)$$

with μ denoting the Möbius function throughout this proof. If we let $c = ab$, then $\gcd(\ell, c) = \gcd(\ell, b)$ because $a|n^*$, and n^* and ℓ are co-prime. Moreover, note that

$$\prod_{\substack{p|k \\ 2<p\leqslant 2D}} \Big(1+\frac{1}{p-2}\Big) = \sum_{j|k} f(j),$$

where f is the unique multiplicative function that is supported on square-free integers and satisfies $f(p) = \mathbb{1}_{2<p\leqslant 2D}/(p-2)$. Hence,

$$\lambda(\mathcal{A}_q\cap\mathcal{A}_r) \leqslant 2^{s+1}\varphi(m)\frac{\varphi(\ell)^2}{\ell}\prod_{\substack{p|\ell \\ p>2}}\Big(1-\frac{1}{(p-1)^2}\Big)\sum_{a|n^*}\mu(a)\sum_{j|\ell}f(j)\sum_{i\geqslant 1} w\Big(i\cdot\frac{2^s aj}{\ell n}\Big),$$

where we set $b = ij$. Uniformly for $\varrho > 0$, the Euler–McLaurin formula[1] [22, Theorem 1.10] implies

$$\sum_{i\geqslant 1} w(\varrho i) = \int_0^\infty w(\varrho t)\,\mathrm{d}t + \int_0^\infty \varrho w'(\varrho t)\{t\}\,\mathrm{d}t = 2\delta\Delta\varrho^{-1} + O(\delta). \tag{5.3}$$

Therefore,

$$\lambda(\mathcal{A}_q\cap\mathcal{A}_r) \leqslant 4\delta\Delta\ell n\varphi(m)\frac{\varphi(\ell)^2}{\ell}\prod_{\substack{p|\ell \\ p>2}}\Big(1-\frac{1}{(p-1)^2}\Big)\sum_{a|n^*}\frac{\mu(a)}{a}\sum_{j|\ell}\frac{f(j)}{j}$$
$$+ O\bigg(\delta\varphi(m)\frac{\varphi(\ell)^2}{\ell}\prod_{\substack{p|\ell \\ p>2}}\Big(1-\frac{1}{(p-1)^2}\Big)\sum_{a|n^*}\mu^2(a)\sum_{j|\ell}f(j)\bigg).$$

We have

$$\sum_{a|n^*}\frac{\mu(a)}{a} = \frac{\varphi(n^*)}{n^*}, \quad \sum_{a|n^*}\mu^2(a) = 2^{\#\{p|n^*\}} = 2^{\omega_t(q,r)},$$

as well as

$$\sum_{j|\ell}\frac{f(j)}{j} \leqslant \prod_{\substack{p|\ell \\ p>2}}\Big(1-\frac{1}{(p-1)^2}\Big)^{-1}, \quad \sum_{j|\ell} f(j) = \prod_{\substack{p|\ell \\ 2<p\leqslant 2D}}\Big(1+\frac{1}{p-2}\Big) \ll \log(4D).$$

Since we also have that $D = \Delta\ell n$ and $4\delta\Delta\varphi(\ell)^2\varphi(m)\varphi(n) = \lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r)$, we conclude that

$$\lambda(\mathcal{A}_q\cap\mathcal{A}_r) \leqslant \lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r)\cdot\frac{\varphi(n^*)/n^*}{\varphi(n)/n}\cdot\bigg(1+O\bigg(\frac{2^{\omega_t(q,r)}\log(4D)}{D}\bigg)\bigg).$$

In order to complete the proof, note that

$$\frac{\varphi(n^*)/n^*}{\varphi(n)/n} = \prod_{p|n,\ p>t}\Big(1+\frac{1}{p-1}\Big) \leqslant e^{2L_t(q,r)},$$

where we used the inequality $1 + 1/(y-1) \leqslant e^{2/y}$ for $y \geqslant 2$. □

For non-generic pairs of integers, we use the following estimate.

[1] In [22], the Euler–McLaurin formula is stated for continuously differentiable functions, but it can be easily generalized to continuous functions that are piecewise differentiable. This requires Theorem 7.35 in [5].

Lemma 5.2 (Pollington–Vaughan [27]). *If q, r and D are as in Lemma 5.1, then we have*

$$\lambda(\mathcal{A}_q\cap\mathcal{A}_r)\ll \mathbb{1}_{D\geqslant 1/2}\cdot\lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r)e^{L_D(q,r)}.$$

6. Auxiliary anatomical lemmas

Lemma 6.1 (Bounds on multiplicative functions). *Let $k\in\mathbb{N}$ and write τ_k for the k-th divisor function. If f is a multiplicative function such that $0\leqslant f\leqslant\tau_k$, then*

$$\sum_{n\leqslant x} f(n)\ll_k x\cdot\exp\Big\{\sum_{p\leqslant x}\frac{f(p)-1}{p}\Big\}.$$

Proof. This is [22, Theorem 14.2, p. 145]. □

Lemma 6.2 (Few numbers with many prime factors). *Fix $C\geqslant 1$. For $x,t,s\geqslant 1$, we have the uniform estimate*

$$\#\Big\{n\leqslant x:\sum_{\substack{p|n\\ p>t}}\frac{1}{p}\geqslant\frac{1}{s}\Big\}\ll_C xe^{-Ct/s}.$$

Proof. The quantity in question is

$$\leqslant e^{-Ct/s}\sum_{n\leqslant x}f(n)$$

with $f(n)=\prod_{p|n}e^{\mathbb{1}_{p>t}Ct/p}$. In particular, $f(p)=1+O_C(\mathbb{1}_{p>t}t/p)$. Hence we may use Lemma 6.1 to complete the proof. □

Lemma 6.3 (Few numbers with many prime factors). *Fix $C\geqslant 1$ and $\kappa>0$. For $x,t,s\geqslant 1$, we have the uniform estimate*

$$\#\Big\{n\leqslant x:\#\{p|n:p\leqslant t\}\geqslant\kappa\log t\Big\}\ll_{C,\kappa} x\cdot t^{-C}.$$

Proof. The quantity in question is

$$\leqslant t^{-C-1}\sum_{n\leqslant x}f(n)$$

with $f(n)=\prod_{p|n}e^{\mathbb{1}_{p\leqslant t}\frac{C+1}{\kappa}}$. Using Lemma 6.1 completes the proof. □

7. Reduction to three propositions

In this section, we reduce the proof of Theorem 2 (and hence Theorem 1) to the following three propositions. In their statements, recall the definitions of $D(q,r)$, $L_t(q,r)$ and $\omega_t(q,r)$ given in the beginning of Section 5.

Proposition 7.1. *Fix $\varepsilon>0$. Let $\psi:\mathbb{N}\to\mathbb{R}_{\geqslant 0}$ and $y\geqslant 1$. We then have the uniform estimate*

$$\sum\sum_{\substack{(q,r)\in[1,Q]^2\\ D(q,r)\leqslant y}}\frac{\psi(q)\varphi(q)}{q}\cdot\frac{\psi(r)\varphi(r)}{r}\ll_\varepsilon y^{1-\varepsilon}\Psi(Q)^{1+\varepsilon}.$$

Proposition 7.2. *Fix $\varepsilon>0$ and $C\geqslant 1$. Let $\psi:\mathbb{N}\to\mathbb{R}_{\geqslant 0}$ and $y,t,s\geqslant 1$. We then have the uniform estimate*

$$\sum\sum_{\substack{(q,r)\in[1,Q]^2\\ D(q,r)\leqslant y,\ L_t(q,r)\geqslant 1/s}}\frac{\psi(q)\varphi(q)}{q}\cdot\frac{\psi(r)\varphi(r)}{r}\ll_{\varepsilon,C} e^{-Ct/s}y^{1-\varepsilon}\Psi(Q)^{1+\varepsilon}.$$

Proposition 7.3. *Fix $\varepsilon,\kappa>0$ and $C\geqslant 1$. Let $\psi:\mathbb{N}\to\mathbb{R}_{\geqslant 0}$ and $y,t\geqslant 1$. We then have the uniform estimate*

$$\sum\sum_{\substack{(q,r)\in[1,Q]^2\\ D(q,r)\leqslant y,\ \omega_t(q,r)\geqslant\kappa\log t}} \frac{\psi(q)\varphi(q)}{q}\cdot\frac{\psi(r)\varphi(r)}{r} \ll_{\varepsilon,\kappa,C} t^{-C}y^{1-\varepsilon}\Psi(Q)^{1+\varepsilon}.$$

Deduction of Theorem 2 from Propositions 7.1-7.3. Using (2.1) and (2.2), we find that

$$0\leqslant\int_0^1\big(N(\alpha;Q)-\Psi(Q)\big)^2\,\mathrm{d}\alpha=\int_0^1\Big(\sum_{q\leqslant Q}\mathbb{1}_{\mathcal{A}_q}\Big)^2\mathrm{d}\alpha-\Psi(Q)^2$$
$$=\sum\sum_{q,r\leqslant Q}\lambda(\mathcal{A}_q\cap\mathcal{A}_r)-\Psi(Q)^2.$$

So in order to prove Theorem 2, it suffices to establish the upper bound

$$\sum\sum_{q,r\leqslant Q}\lambda(\mathcal{A}_q\cap\mathcal{A}_r)\leqslant\Psi(Q)^2+\Psi(Q)+O_\varepsilon\big(\Psi(Q)^{1+\varepsilon}\big).$$

The terms with $q=r$ contribute

$$\sum_{q\leqslant Q}\lambda(\mathcal{A}_q)=\Psi(Q).$$

It thus remains to show that

$$\sum\sum_{\substack{q,r\leqslant Q\\ q\neq r}}\lambda(\mathcal{A}_q\cap\mathcal{A}_r)\leqslant\Psi(Q)^2+O_\varepsilon\big(\Psi(Q)^{1+\varepsilon}\big). \tag{7.1}$$

From the first part of Lemma 5.1, we may assume that $D(q,r)\geqslant 1/2$. To this end, let

$$\mathcal{E}=\{(q,r)\in[1,Q]^2:q\neq r,\ D(q,r)\geqslant 1/2\},$$

so that the sum in (7.1) is over all pairs $(q,r)\in\mathcal{E}$. Writing $D=D(q,r)$, we split $\mathcal{E}$ into the following three subsets:

$$\mathcal{E}^{(1)}=\Big\{(q,r)\in\mathcal{E}:L_{D^2}(q,r)\leqslant\frac{1}{D},\ \omega_{D^2}(q,r)\leqslant\frac{\varepsilon}{4}\log(2D)\Big\},$$
$$\mathcal{E}^{(2)}=\Big\{(q,r)\in\mathcal{E}:L_{D^2}(q,r)>\frac{1}{D}\Big\},$$
$$\mathcal{E}^{(3)}=\Big\{(q,r)\in\mathcal{E}:L_{D^2}(q,r)\leqslant\frac{1}{D},\ \omega_{D^2}(q,r)>\frac{\varepsilon}{4}\log(2D)\Big\}.$$

We claim the following estimates:

$$\sum\sum_{(q,r)\in\mathcal{E}^{(1)}}\lambda(\mathcal{A}_q\cap\mathcal{A}_r)\leqslant\Psi(Q)^2+O_\varepsilon\big(\Psi(Q)^{1+\varepsilon}\big)\,; \tag{7.2}$$

$$\sum\sum_{(q,r)\in\mathcal{E}^{(2)}}\lambda(\mathcal{A}_q\cap\mathcal{A}_r)\ll_\varepsilon\Psi(Q)^{1+\varepsilon}\,; \tag{7.3}$$

$$\sum\sum_{(q,r)\in\mathcal{E}^{(3)}}\lambda(\mathcal{A}_q\cap\mathcal{A}_r)\ll_\varepsilon\Psi(Q)^{1+\varepsilon}. \tag{7.4}$$

These bounds suffice to complete the proof, since they readily imply

$$\sum_{\substack{q,r\leqslant Q\\ q\neq r}}\sum \lambda(\mathcal{A}_q\cap\mathcal{A}_r)=\sum_{i=1}^{3}\sum_{(q,r)\in\mathcal{E}^{(i)}}\sum \lambda(\mathcal{A}_q\cap\mathcal{A}_r)\leqslant \Psi(Q)^2+O_\varepsilon\Big(\Psi(Q)^{1+\varepsilon}\Big).$$

This gives (7.1), and so completes the proof of Theorem 2.

Let us now prove (7.2). Here the summation is over pairs $(q,r)\in\mathcal{E}^{(1)}$. We then use Lemma 5.1 with $t=D^2$ to find that

$$\lambda(\mathcal{A}_q\cap\mathcal{A}_r)\leqslant \lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r)\big(1+O\big(D^{-1+\varepsilon/2}\big)\big).$$

Therefore

$$\sum_{(q,r)\in\mathcal{E}^{(1)}}\sum \lambda(\mathcal{A}_q\cap\mathcal{A}_r)\leqslant M+O_\varepsilon(R),$$

where

$$M:=\sum_{(q,r)\in\mathcal{E}^{(1)}}\sum \lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r)\quad\text{and}\quad R:=\sum_{(q,r)\in\mathcal{E}^{(1)}}\sum D(q,r)^{\varepsilon/2-1}\lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r).$$

For the main term M we simply notice that it is $\leqslant\Psi(Q)^2$. Thus we focus on the $O_\varepsilon(R)$ term. Let I be the smallest non-negative integer such that $2^I\geqslant\Psi(Q)$. We then split R according to the size of $D(q,r)$ as

$$R=\sum_{i=0}^{I}R_i+R'$$

where R_i is the part of the sum with $D(q,r)\in[2^{i-1},2^i)$, and R' is the part with $D(q,r)\geqslant 2^I\geqslant\Psi(Q)$. Now, we trivially have that

$$R'\leqslant\Psi(Q)^{\varepsilon/2-1}\sum_{(q,r)\in\mathcal{E}^{(1)}}\sum \lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r)\leqslant\Psi(Q)^{1+\varepsilon/2}.$$

For each $i\in\{0,1,\dots,I\}$, Proposition 7.1 implies that

$$R_i\ll_\varepsilon (2^i)^{\varepsilon/2-1}\cdot(2^i)^{1-\varepsilon}\Psi(Q)^{1+\varepsilon}=2^{-i\varepsilon/2}\Psi(Q)^{1+\varepsilon}.$$

We thus conclude that $R\ll_\varepsilon\Psi(Q)^{1+\varepsilon}$, whence (7.2) follows.

Let us now prove (7.3). For $(q,r)\in\mathcal{E}^{(2)}$ we use Lemma 5.2 to find that

$$\sum_{(q,r)\in\mathcal{E}^{(2)}}\sum \lambda(\mathcal{A}_q\cap\mathcal{A}_r)\ll\sum_{(q,r)\in\mathcal{E}^{(2)}}\sum \lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r)e^{L_D(q,r)}.$$

Note that if $D\in[2^{i-1},2^i)$ and $(q,r)\in\mathcal{E}^{(2)}$, then $L_{4^{i-1}}(q,r)>1/2^i$. We may thus write

$$\mathcal{E}^{(2)}\subseteq\bigcup_{i=0}^{\infty}\bigcup_{j=i}^{\infty}\mathcal{E}^{(2)}_{i,j},$$

where $\mathcal{E}^{(2)}_{i,i}$ is the set of pairs $(q,r)\in\mathcal{E}^{(2)}$ with $D\in[2^{i-1},2^i)$ and $L_{4^{i-1}}(q,r)\leqslant 1$, and $\mathcal{E}^{(2)}_{i,j}$ with $j>i$ is the set of pairs $(q,r)\in\mathcal{E}^{(2)}$ with $D\in[2^{i-1},2^i)$ and $L_{4^{j-1}}(q,r)\leqslant 1<L_{4^{j-2}}(q,r)$. In particular, if $(q,r)\in\mathcal{E}^{(2)}_{i,j}$, then

$$L_D(q,r)\leqslant L_{2^{i-1}}(q,r)\leqslant\sum_{2^{i-1}<p\leqslant 4^{j-1}}\frac{1}{p}+L_{4^{j-1}}(q,r)\leqslant\log\frac{j+1}{i+1}+O(1),$$

and thus

$$\sum\sum_{(q,r)\in\mathcal{E}^{(2)}} \lambda(\mathcal{A}_q\cap\mathcal{A}_r) \ll \sum\sum_{j\geqslant i\geqslant 0} \frac{j+1}{i+1} \sum\sum_{(q,r)\in\mathcal{E}^{(2)}_{i,j}} \lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r). \tag{7.5}$$

For each $i\geqslant 0$, Proposition 7.2 applied with $y=2^{i+1}$, $t=\max\{1,4^{i-1}\}$, $s=2^i$ and $C=8$ implies that

$$\sum\sum_{(q,r)\in\mathcal{E}^{(2)}_{i,i}} \lambda(\mathcal{A}_q\cap\mathcal{A}_r) \ll_\varepsilon e^{-2^i}\Psi(Q)^{1+\varepsilon}.$$

In addition, if $j>i\geqslant 0$, then Proposition 7.2 applied with $y=2^i$, $t=\max\{1,4^{j-2}\}$, $s=1$ and $C=17$ implies that

$$\sum\sum_{(q,r)\in\mathcal{E}^{(2)}_{i,j}} \lambda(\mathcal{A}_q\cap\mathcal{A}_r) \ll_\varepsilon e^{-4^j}\Psi(Q)^{1+\varepsilon}.$$

Inserting the two above bounds into (7.5) completes the proof of (7.3).

It remains to prove (7.4). We note that for pairs $(q,r)\in\mathcal{E}^{(3)}$ we have

$$L_D(q,r)\leqslant \sum_{D<p\leqslant D^2}\frac{1}{p}+L_{D^2}(q,r)\ll 1.$$

Therefore, using Lemma 5.2, we find that

$$\sum\sum_{(q,r)\in\mathcal{E}^{(3)}} \lambda(\mathcal{A}_q\cap\mathcal{A}_r) \ll \sum\sum_{(q,r)\in\mathcal{E}^{(3)}} \lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r).$$

Note that if $D\in[2^{i-1},2^i)$ and $\omega_{D^2}(q,r)>0.25\varepsilon\log(2D)$, then $\omega_{4^i}(q,r)>0.1\varepsilon\log(4^i)$. Therefore,

$$\sum\sum_{(q,r)\in\mathcal{E}^{(3)}} \lambda(\mathcal{A}_q\cap\mathcal{A}_r) \ll \sum_{i\geqslant 0}\sum\sum_{(q,r)\in\mathcal{E}^{(3)}_i} \lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r), \tag{7.6}$$

where $\mathcal{E}^{(3)}_i$ is the set of pairs $(q,r)\in\mathcal{E}^{(3)}$ with $D\in[2^{i-1},2^i)$ and $\omega_{4^i}(q,r)>0.1\varepsilon\log(4^i)$. Proposition 7.3 applied with $y=2^i$, $t=4^i$, $\kappa=\varepsilon/10$, and $C=2$ implies that

$$\sum\sum_{(q,r)\in\mathcal{E}^{(3)}_i} \lambda(\mathcal{A}_q\cap\mathcal{A}_r) \ll_\varepsilon 2^{-i}\Psi(Q)^{1+\varepsilon}$$

for all $i\geqslant 0$. Inserting this bound into (7.6) completes the proof of (7.4), and thus of Theorem 2. ∎

8. GCD GRAPHS

In order to prove Propositions 7.1-7.3, we shall use the method of GCD graphs developed in [24] with some important modifications. The definition of a GCD graph is almost the same as in [24]:

Definition 1 (GCD graph)**.** Let G be a septuple $(\mu,\mathcal{V},\mathcal{W},\mathcal{E},\mathcal{P},f,g)$ such that:

(a) μ is a measure on $\mathbb{N}$ such that $\mu(n)<\infty$ for all $n\in\mathbb{N}$; we extend to $\mathbb{N}^2$ by letting

$$\mu(\mathcal{N}):=\sum_{(n_1,n_2)\in\mathcal{N}}\mu(n_1)\mu(n_2)\quad\text{for}\quad \mathcal{N}\subseteq\mathbb{N}^2;$$

(b) $\mathcal{V}$ and $\mathcal{W}$ are finite sets of positive integers;

(c) $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{W}$, that is to say $(\mathcal{V}, \mathcal{W}, \mathcal{E})$ is a bipartite graph;

(d) $\mathcal{P}$ is a set of primes;

(e) f and g are functions from $\mathcal{P}$ to $\mathbb{Z}_{\geqslant 0}$ such that for all $p \in \mathcal{P}$ we have:
   (i) $p^{f(p)} | v$ for all $v \in \mathcal{V}$, and $p^{g(p)} | w$ for all $w \in \mathcal{W}$;
   (ii) if $(v, w) \in \mathcal{E}$, then $p^{\min\{f(p),g(p)\}} \| \gcd(v, w)$;

We then call G a (bipartite) *GCD graph* with *sets of vertices* $(\mathcal{V}, \mathcal{W})$, *set of edges* $\mathcal{E}$ and *multiplicative data* $(\mathcal{P}, f, g)$. We will also refer to $\mathcal{P}$ as the *set of primes* of G. If $\mathcal{P} = \emptyset$, we say that G has *trivial* set of primes and we view $f = f_\emptyset$ and $g = g_\emptyset$ as two copies of the empty function from $\emptyset$ to $\mathbb{Z}_{\geqslant 0}$.

Remark. In [24], the definition of a GCD graph included one additional condition: if $f(p) \neq g(p)$, then we required that $p^{f(p)} \| v$ for all $v \in \mathcal{V}$, and $p^{g(p)} \| w$ for all $w \in \mathcal{W}$. Even though the GCD graphs we will consider do satisfy this condition, we no longer need to include it in the definition due to a small strengthening of Proposition 8.1, which allows us to specify the p-adic valuation of all vertices in G' and for all primes $p \in \mathcal{P}' \setminus \mathcal{P}$.

To each GCD graph G, we associate a *quality* $q(G)$. This will be defined differently to [24], and it will depend on some parameter $\tau \in (0, \frac{1}{100})$, which we can take to be arbitrarily small.

Definition 2. For a GCD graph $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$, we define:

(a) The *edge density*
$$\delta(G) = \frac{\mu(\mathcal{E})}{\mu(\mathcal{V})\mu(\mathcal{W})},$$
provided that $\mu(\mathcal{V})\mu(\mathcal{W}) \neq 0$. If $\mu(\mathcal{V})\mu(\mathcal{W}) = 0$, we define $\delta(G)$ to be 0.

(b) The *quality*
$$q(G) := \delta^{2+\tau} \mu(\mathcal{V})\mu(\mathcal{W}) \prod_{p \in \mathcal{P}} p^{|f(p)-g(p)|} \left(1 - \frac{\mathbb{1}_{f(p)=g(p)\geqslant 1}}{p}\right)^{-2} \left(1 - \frac{1}{p^{1+\frac{\tau}{4}}}\right)^{-3}.$$

Definition 3 (GCD subgraph)**.** Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ and $G' = (\mu', \mathcal{V}', \mathcal{W}', \mathcal{E}', \mathcal{P}', f', g')$ be two GCD graphs. We say that G' is a ***GCD subgraph*** of G if:
$$\mu' = \mu, \quad \mathcal{V}' \subseteq \mathcal{V}, \quad \mathcal{W}' \subseteq \mathcal{W}, \quad \mathcal{E}' \subseteq \mathcal{E}, \quad \mathcal{P}' \supseteq \mathcal{P}, \quad f'\big|_{\mathcal{P}} = f, \quad g'\big|_{\mathcal{P}} = g.$$
We write $G' \preceq G$ if G' is a GCD subgraph of G. Lastly, we say that G' is a *non-trivial GCD subgraph* of G if $\mu(\mathcal{E}') > 0$, that is to say G' is non-trivial as a GCD graph.

Definition 4 (maximal GCD graph)**.** Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph. We say that G is *maximal* or, more precisely, that G is $(\mathcal{P}, f, g)$*-maximal* if for every $G' \preceq G$ with the same multiplicative data $(\mathcal{P}, f, g)$ we have that $q(G') \leqslant q(G)$.

Remark. For any given GCD graph $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$, since both the sets $\mathcal{V}$ and $\mathcal{W}$ are finite, there exists a GCD subgraph G' of G with the same multiplicative data $(\mathcal{P}, f, g)$, such that G' is $(\mathcal{P}, f, g)$-maximal. To see this, just list all subgraphs of G with the same multiplicative data, and pick out one with the largest possible quality. We will use this fact several times in the later sections without mentioning it again.

We recall from [24] the special vertex sets $\mathcal{V}_{p^k}$ and GCD graphs G_{p^k,p^ℓ} formed by restricted elements by their prime-power divisibilty.

Definition 5. Let p be a prime number, and let $k,\ell\in\mathbb{Z}_{\geqslant 0}$.

(a) If $\mathcal{V}$ is a set of integers, we set

$$\mathcal{V}_{p^k}=\{v\in\mathcal{V}:p^k\|v\}.$$

Here we have the understanding that $\mathcal{V}_{p^0}$ denotes the set of $v\in\mathcal{V}$ that are coprime to p.

(b) Let $G=(\mu,\mathcal{V},\mathcal{W},\mathcal{E},\mathcal{P},f,g)$ be a GCD graph such that $p\notin\mathcal{P}$. We then define the septuple

$$G_{p^k,p^\ell}=(\mu,\mathcal{V}_{p^k},\mathcal{W}_{p^\ell},\mathcal{E}\cap(\mathcal{V}_{p^k}\times\mathcal{W}_{p^\ell}),\mathcal{P}\cup\{p\},f_{p^k},g_{p^\ell})$$

where the functions f_{p^k}, g_{p^ℓ} are defined on $\mathcal{P}\cup\{p\}$ by the relations $f_{p^k}|_{\mathcal{P}}=f$, $g_{p^\ell}|_{\mathcal{P}}=g$,

$$f_{p^k}(p)=k \quad\text{and}\quad g_{p^\ell}(p)=\ell.$$

We further fix a constant $M\geqslant 2$ and we introduce the auxiliary quantities $C_1,\dots,C_7$ as follows:

$$\begin{aligned}&C_1=\frac{10^4}{\tau}, \qquad C_2=10MC_1^3, \qquad C_3=10^3C_1^3, \qquad C_4=10^{10}M^2C_2^2,\\ &C_5=\max\{C_3,(50\log C_4)^3\}, \qquad C_6=\max\{C_4,10^4MC_2,C_2^{\frac{10}{\tau}}\}, \qquad C_7=C_5^{C_6}.\end{aligned} \tag{8.1}$$

Definition 6. Let $G=(\mu,\mathcal{V},\mathcal{W},\mathcal{E},\mathcal{P},f,g)$ be a GCD graph. We let $\mathcal{R}(G)$ be given by

$$\mathcal{R}(G):=\{p\notin\mathcal{P}:\exists(v,w)\in\mathcal{E}\text{ such that }p|\gcd(v,w)\}.$$

That is to say $\mathcal{R}(G)$ is the set of primes occurring in a GCD which we haven't yet accounted for. We split this into two further subsets:

- The set $\mathcal{R}^\sharp(G)$ consisting of all primes $p\in\mathcal{R}(G)$ for which there exists an integer $k\geqslant 0$ such that
$$\frac{\mu(\mathcal{V}_{p^k})}{\mu(\mathcal{V})}\geqslant 1-\frac{C_2}{p}\quad\text{and}\quad\frac{\mu(\mathcal{W}_{p^k})}{\mu(\mathcal{W})}\geqslant 1-\frac{C_2}{p},$$
and
$$q(G_{p^a,p^b})<M\cdot q(G)\quad\text{for all }(a,b)\in\mathbb{Z}_{\geqslant 0}^2\text{ with }a\neq b.$$
- The set $\mathcal{R}^\flat(G):=\mathcal{R}(G)\setminus\mathcal{R}^\sharp(G)$.

In the next section, we reduce the proof of Propositions 7.1-7.3 (and hence Theorem 2) to the existence of GCD subgraphs with nice properties as described in the following three propositions and three lemmas.

Proposition 8.1 (Iteration when $\mathcal{R}^\flat(G)\neq\emptyset$)**.** *Let $G=(\mu,\mathcal{V},\mathcal{W},\mathcal{E},\mathcal{P},f,g)$ be a GCD graph with edge density $\delta>0$ such that*

$$\mathcal{R}(G)\subseteq\{p>C_6\}\quad\textit{and}\quad\mathcal{R}^\flat(G)\neq\emptyset.$$

Then there is a GCD subgraph G' of G with edge density $\delta'>0$ and multiplicative data $(\mathcal{P}',f',g')$ such that:

(a) G' is $(\mathcal{P}',f',g')$-maximal;

(b) $\mathcal{P}\subsetneq\mathcal{P}'\subseteq\mathcal{P}\cup\mathcal{R}(G)$;

(c) $\mathcal{R}(G')\subsetneq\mathcal{R}(G)$;

(d) $q(G') \geqslant M^N q(G)$ *with* $N = \#\{p \in \mathcal{P}' \setminus \mathcal{P} : f'(p) \neq g'(p)\}$*;*

(e) $p^{f'(p)} \| v$ *and* $p^{g'(p)} \| w$ *for all* $p \in \mathcal{P}' \setminus \mathcal{P}$*,* $(v, w) \in \mathcal{V} \times \mathcal{W}$*.*

Proposition 8.2 (Iteration when $\mathcal{R}^\flat(G) = \emptyset$)**.** *Let* $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ *be a GCD graph with edge density* $\delta > 0$ *such that*

$$\mathcal{R}(G) \subseteq \{p > C_6\}, \quad \mathcal{R}^\flat(G) = \emptyset, \quad \mathcal{R}^\sharp(G) \neq \emptyset.$$

Then there is a GCD subgraph $G' = (\mu, \mathcal{V}', \mathcal{W}', \mathcal{E}', \mathcal{P}', f', g')$ *of* G *such that*

$$\mathcal{P} \subsetneq \mathcal{P}' \subseteq \mathcal{P} \cup \mathcal{R}(G), \quad \mathcal{R}(G') \subsetneq \mathcal{R}(G), \quad q(G') \geqslant q(G).$$

Proposition 8.3 (Bounded quality loss for small primes)**.** *Let* $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ *be a GCD graph with edge density* $\delta > 0$*. Then there is a GCD subgraph* $G' = (\mu, \mathcal{V}', \mathcal{W}', \mathcal{E}', \mathcal{P}', f', g')$ *of* G *with edge density* $\delta' > 0$ *such that*

$$\mathcal{P}' \subseteq \mathcal{P} \cup \big(\mathcal{R}(G) \cap \{p \leqslant C_6\}\big), \quad \mathcal{R}(G') \subseteq \{p > C_6\}, \quad q(G') \geqslant q(G)/C_7.$$

Lemma 8.4 (Removing the effect of $\mathcal{R}(G)$ from $L_t(v, w)$)**.** *Let* $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ *be a maximal GCD graph with edge density* $\delta > 0$ *and let* $s \geqslant 1$ *and* $t \geqslant C_6 s$*. Assume further that*

$$\mathcal{R}^\flat(G) = \emptyset \quad \text{and} \quad \mathcal{E} \subseteq \big\{(v, w) \in \mathcal{V} \times \mathcal{W} : L_t(v, w) \geqslant 1/s\big\}.$$

Then there exists a GCD subgraph $G' = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}', \mathcal{P}, f, g)$ *of* G *such that*

$$q(G') \geqslant \frac{q(G)}{2} > 0 \quad \text{and} \quad \mathcal{E}' \subseteq \Bigg\{(v, w) \in \mathcal{V} \times \mathcal{W} : \sum_{\substack{p | vw/\gcd(v,w)^2 \\ p > t,\ p \notin \mathcal{R}(G)}} \frac{1}{p} \geqslant \frac{1}{2s}\Bigg\}.$$

Lemma 8.5 (Removing the effect of $\mathcal{R}(G)$ from $\omega_t(v, w)$)**.** *Let* $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ *be a maximal GCD graph with edge density* $\delta > 0$ *and let* $t \geqslant \exp\exp(C_6)$ *and* $U \geqslant 2C_6 \log\log t$*. Assume further that*

$$\mathcal{R}^\flat(G) = \emptyset \quad \text{and} \quad \mathcal{E} \subseteq \big\{(v, w) \in \mathcal{V} \times \mathcal{W} : \omega_t(v, w) \geqslant U\big\}.$$

Then there exists a GCD subgraph $G' = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}', \mathcal{P}, f, g)$ *of* G *such that*

$$q(G') \geqslant \frac{q(G)}{2} > 0 \quad \text{and} \quad \mathcal{E}' \subseteq \Bigg\{(v, w) \in \mathcal{V} \times \mathcal{W} : \sum_{\substack{p | vw/\gcd(v,w)^2 \\ C_6 < p \leqslant t,\ p \notin \mathcal{R}(G)}} 1 \geqslant \frac{U}{2}\Bigg\}.$$

Lemma 8.6 (Subgraph with high-degree vertices)**.** *Let* $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ *be a GCD graph with edge density* $\delta > 0$*. Then there is a GCD subgraph* $G' = (\mu, \mathcal{V}', \mathcal{W}', \mathcal{E}', \mathcal{P}, f, g)$ *of* G *with edge density* $\delta' > 0$ *such that:*

(a) $q(G') \geqslant q(G)$*;*

(b) For all $v \in \mathcal{V}'$ *and for all* $w \in \mathcal{W}'$*, we have*

$$\mu(\Gamma_{G'}(v)) \geqslant \frac{1+\tau}{2+\tau} \cdot \delta' \mu(\mathcal{W}') \quad \text{and} \quad \mu(\Gamma_{G'}(w)) \geqslant \frac{1+\tau}{2+\tau} \cdot \delta' \mu(\mathcal{V}').$$

9. Proof of Propositions 7.1-7.3 assuming the results of Section 8

Proof of Proposition 7.1. Let $\mathcal{S} = \mathbb{Z} \cap [1, Q]$, $\mu(q) = \varphi(q)\psi(q)/q$ and $\mathcal{E} = \{(q, r) \in \mathcal{S}^2 : D(q, r) \leqslant y\}$, and consider the GCD graph $G = (\mu, \mathcal{S}, \mathcal{S}, \mathcal{E}, \emptyset, f_\emptyset, g_\emptyset)$ with trivial set of primes. Let $\delta = \delta(G)$ be its edge density. It suffices to show that

$$q(G) \ll_\tau y^2. \tag{9.1}$$

Indeed, we have that

$$q(G) = \delta^{2+\tau}\mu(\mathcal{S})^2 = \mu(\mathcal{E})^{2+\tau}\mu(\mathcal{S})^{-2-2\tau}$$

and $\mu(\mathcal{S}) = \frac{1}{2}\Psi(Q)$. Hence, if (9.1) holds, then

$$\mu(\mathcal{E}) \ll_\tau y^{\frac{2}{2+\tau}}\Psi(Q)^{\frac{2+2\tau}{2+\tau}}.$$

This proves Proposition 7.1 with $\varepsilon = \tau/(2+\tau)$. Since $\tau > 0$ can be taken to be arbitrarily small, this completes the proof of Proposition 7.1.

Let us now prove (9.1). If $\delta = 0$, relation (9.1) holds trivially, so let us assume that $\delta > 0$. We use the results of Section 8 with $M = 2$. We first use Proposition 8.3, and we then iterate Propositions 8.1 and 8.2. Lastly, we apply Lemma 8.6. Thus, we find that there exists a GCD subgraph $G' = (\mu, \mathcal{V}', \mathcal{W}', \mathcal{E}', \mathcal{P}', f', g')$ of G of density $\delta' > 0.$ such that:

(a) $\mathcal{R}(G') = \emptyset$;

(b) For all $v \in \mathcal{V}'$, we have $\mu(\Gamma_{G'}(v)) \geqslant \frac{1+\tau}{2+\tau}\delta'\mu(\mathcal{W}') \geqslant 0.5\delta'\mu(\mathcal{W}')$;

(c) For all $w \in \mathcal{W}'$, we have $\mu(\Gamma_{G'}(w)) \geqslant \frac{1+\tau}{2+\tau}\delta'\mu(\mathcal{V}') \geqslant 0.5\delta'\mu(\mathcal{V}')$;

(d) $q(G') \gg_\tau q(G)$.

Hence, it suffices to prove that

$$q(G') \ll_\tau y^2. \tag{9.2}$$

To do this, we largely follow the proof of Proposition 6.3 in [24], with some important simplifications because we do not need to keep track of the 'anatomical' condition $L_t(v, w) \geqslant 10$ as we did there. For the ease of the reader, we repeat all the details instead of referring to [24].

Set

$$a := \prod_{p\in\mathcal{P}'} p^{f'(p)} \quad \text{and} \quad b := \prod_{p\in\mathcal{P}'} p^{g'(p)}. \tag{9.3}$$

The definition of a GCD graph implies that

$$a|v \quad \text{for all } v \in \mathcal{V}', \qquad b|w \quad \text{for all } w \in \mathcal{W}'. \tag{9.4}$$

Moreover, since $\mathcal{R}(G') = \emptyset$, and $p^{\min\{f'(p),g'(p)\}} \| \gcd(v, w)$ for all $(v, w) \in \mathcal{E}'$, we have that

$$\gcd(v, w) = \gcd(a, b) \quad \text{for all} \quad (v, w) \in \mathcal{E}'. \tag{9.5}$$

Now, note that

$$\prod_{p\in\mathcal{P}'} p^{|f'(p)-g'(p)|} = \prod_{p\in\mathcal{P}'} p^{\max\{f'(p),g'(p)\}-\min\{f'(p),g'(p)\}} = \frac{\mathrm{lcm}(a, b)}{\gcd(a, b)} = \frac{ab}{\gcd(a, b)^2},$$

as well as

$$\prod_{p\in\mathcal{P}'} \frac{1}{(1 - \mathbb{1}_{f'(p)=g'(p)\geqslant 1}/p)^2(1 - 1/p^{1+\frac{\tau}{4}})^3} \ll_\tau \prod_{p\in\mathcal{P}'} \frac{1}{(1 - \mathbb{1}_{f'(p)=g'(p)\geqslant 1}/p)^2} \leqslant \frac{ab}{\varphi(a)\varphi(b)}.$$

Consequently, from the definition of $q(\cdot)$, we find

$$
\begin{aligned}
q(G') &= (\delta')^{2+\tau}\mu(\mathcal{V}')\mu(\mathcal{W}')\prod_{p\in\mathcal{P}'}\frac{p^{|f'(p)-g'(p)|}}{(1-\mathbb{1}_{f'(p)=g'(p)\geqslant 1}/p)^2(1-1/p^{1+\frac{\tau}{4}})^3}\\
&\ll_\tau (\delta')^{2+\tau}\mu(\mathcal{V}')\mu(\mathcal{W}')\cdot\frac{ab}{\gcd(a,b)^2}\cdot\frac{ab}{\varphi(a)\varphi(b)}\\
&= (\delta')^{1+\tau}\mu(\mathcal{E}')\cdot\frac{ab}{\gcd(a,b)^2}\cdot\frac{ab}{\varphi(a)\varphi(b)}. \qquad (9.6)
\end{aligned}
$$

It remains to bound $\mu(\mathcal{E}')$.

We have that

$$\mathcal{E}'\subseteq\mathcal{E}=\{(v,w)\in\mathcal{S}\times\mathcal{S}: D(v,w)\leqslant y\},$$

where we recall that $D(v,w)=\max\{v\psi(w),w\psi(v)\}/\gcd(v,w)$. Since $\gcd(v,w)=\gcd(a,b)$ for all $(v,w)\in\mathcal{E}'$, we infer that

$$\psi(v)\leqslant\frac{y\cdot\gcd(a,b)}{w}\quad\text{and}\quad\psi(w)\leqslant\frac{y\cdot\gcd(a,b)}{v}\quad\text{for all}\quad(v,w)\in\mathcal{E}'.$$

The vertex sets $\mathcal{V}',\mathcal{W}'$ are finite sets of positive integers. For each $v\in\mathcal{V}'$, let $w_{\max}(v)$ be the largest integer in $\mathcal{W}'$ such that $(v,w_{\max}(v))\in\mathcal{E}'$. (This quantity is well-defined in virtue of property (b) above. In addition, we emphasise to the reader that 'largest' refers to the size of elements as positive integers, and does not depend on the measure μ.) Similarly, for each $w\in\mathcal{W}'$, let $v_{\max}(w)$ be the largest element of $\mathcal{V}'$ such that $(v_{\max}(w),w)\in\mathcal{E}'$. Consequently,

$$\psi(v)\leqslant\frac{y\cdot\gcd(a,b)}{w_{\max}(v)}\quad\text{and}\quad\psi(w)\leqslant\frac{y\cdot\gcd(a,b)}{v_{\max}(w)}\quad\text{for all}\quad v\in\mathcal{V}',\ w\in\mathcal{W}'. \qquad (9.7)$$

Now, let w_0 be the largest integer in $\mathcal{W}'$ and $\mathcal{E}''=\{(v,w)\in\mathcal{E}':(v,w_0)\in\mathcal{E}'\}$. We then have

$$w_{\max}(v)=w_0\quad\text{for all}\quad(v,w)\in\mathcal{E}''. \qquad (9.8)$$

In addition, by properties (b) and (c) of G', we have

$$\mu(\mathcal{E}'')=\sum_{v\in\Gamma_{G'}(w_0)}\mu(v)\mu(\Gamma_{G'}(v))\geqslant\mu(\Gamma_{G'}(w_0))\cdot\frac{\delta'\mu(\mathcal{W}')}{2}\geqslant\left(\frac{\delta'}{2}\right)^2\mu(\mathcal{V}')\mu(\mathcal{W}')=\frac{\delta'\mu(\mathcal{E}')}{4}.$$

Substituting this bound into (9.6), we find

$$q(G')\ll_\tau(\delta')^\tau\mu(\mathcal{E}'')\cdot\frac{ab}{\varphi(a)\varphi(b)}\cdot\frac{ab}{\gcd(a,b)^2}\leqslant\mu(\mathcal{E}'')\cdot\frac{ab}{\gcd(a,b)^2}\cdot\frac{ab}{\varphi(a)\varphi(b)}, \qquad (9.9)$$

where we used the trivial bound $\delta'\leqslant 1$ in the second inequality and our assumption that $\tau>0$.

In addition,

$$\mu(\mathcal{E}'')=\sum_{(v,w)\in\mathcal{E}''}\frac{\psi(v)\varphi(v)}{v}\cdot\frac{\psi(w)\varphi(w)}{w}.$$

Since $a|v$ and $b|w$, we have $\varphi(v)/v\leqslant\varphi(a)/a$ and $\varphi(w)/w\leqslant\varphi(b)/b$. Therefore

$$\mu(\mathcal{E}'')\leqslant\frac{\varphi(a)\varphi(b)}{ab}\sum_{(v,w)\in\mathcal{E}''}\psi(v)\psi(w).$$

Together with (9.7), (9.8) and (9.9), this implies that

$$q(G') \ll y^2 ab \sum_{(v,w)\in\mathcal{E}''} \frac{1}{v_{\max}(w)w_0} \leqslant y^2 ab \sum_{(v,w)\in\mathcal{E}'} \frac{1}{v_{\max}(w)w_0}. \tag{9.10}$$

Writing $v = v'a$ and $w = w'b$, we find that

$$\begin{aligned}\sum_{(v,w)\in\mathcal{E}'} \frac{1}{v_{\max}(w)w_0} &\leqslant \sum_{w'\leqslant w_0/b} \frac{1}{w_0 v_{\max}(bw')} \sum_{v'\leqslant v_{\max}(bw')/a} 1 \\ &\leqslant \sum_{w'\leqslant w_0/b} \frac{1}{w_0 v_{\max}(bw')} \cdot \frac{v_{\max}(bw')}{a} \\ &\leqslant \frac{1}{ab}.\end{aligned}$$

Together with (9.10), this implies that

$$q(G') \ll y^2.$$

as needed. This completes the proof of Proposition 7.1. □

Proof of Proposition 7.2. If $t \leqslant C_6 s$, then the result follows immediately by Proposition 7.1. Let us assume now that $t \geqslant C_6 s$.

Let $\mathcal{S} = \mathbb{Z}\cap[1,Q]$, $\mu(q) = \varphi(q)\psi(q)/q$ and $\mathcal{E} = \{(q,r)\in\mathcal{S}^2 : D(q,r)\leqslant y,\ L_t(q,r)\geqslant 1/s\}$, and consider the GCD graph $G = (\mu,\mathcal{S},\mathcal{S},\mathcal{E},\emptyset,f_\emptyset,g_\emptyset)$ with trivial set of primes. Let $\delta = \delta(G)$ be its edge density. As in the proof of Proposition 7.1, it suffices to show that

$$q(G) \ll_{\tau,C} y^2 e^{-Ct/s}. \tag{9.11}$$

We may assume that $\delta > 0$; otherwise, (9.11) holds trivially. To proceed, we adapt the argument leading to Propositions 6.3 and 7.1 of [24].

We shall apply the results of Section 8 on GCD graphs with the parameter $M = e^{4C}$. First, we use Proposition 8.3 to find a GCD subgraph $G^{(0)} = (\mu,\mathcal{V}^{(0)},\mathcal{W}^{(0)},\mathcal{E}^{(0)},\mathcal{P}^{(0)},f^{(0)},g^{(0)})$ of G of density $\delta^{(0)} > 0$ such that:

(a) $\mathcal{P}^{(0)} \subseteq \{p \leqslant C_6\}$;

(b) $\mathcal{R}(G^{(0)}) \subseteq \{p > C_6\}$;

(c) $q(G^{(0)}) \gg_{\tau,C} q(G)$.

Then, we repeatedly apply Proposition 8.1 in an iterative fashion till we arrive at a GCD subgraph $G^{(1)} = (\mu,\mathcal{V}^{(1)},\mathcal{W}^{(1)},\mathcal{E}^{(1)},\mathcal{P}^{(1)},f^{(1)},g^{(1)})$ of $G^{(0)}$ of density $\delta^{(1)} > 0$ such that:

(a) $G^{(1)}$ is maximal;

(b) $\mathcal{R}^\flat(G^{(1)}) = \emptyset$;

(c) $q(G^{(1)}) \geqslant e^{4CN} q(G^{(0)})$ with $N = \#\{p\in\mathcal{P}^{(1)} : p > C_6,\ f^{(1)}(p)\neq g^{(1)}(p)\}$;

(d) $p^{f^{(1)}(p)}\|v$ and $p^{g^{(1)}(p)}\|w$ for all $(v,w)\in\mathcal{V}^{(1)}\times\mathcal{W}^{(1)}$ and all primes $p > C_6$ lying in $\mathcal{P}^{(1)}$.

We then separate two cases.

Case 1: $N \geqslant 0.25t/s$. We then iterate Propositions 8.1 and 8.2, and we subsequently apply Lemma 8.6 to arrive at a GCD subgraph $G' = (\mu,\mathcal{V}',\mathcal{W}',\mathcal{E}',\mathcal{P}',f',g')$ of $G^{(1)}$ of density $\delta' > 0$ such that:

(a) $\mathcal{R}(G') = \emptyset$;

(b) For all $v \in \mathcal{V}'$, we have $\mu(\Gamma_{G'}(v)) \geqslant \frac{1+\tau}{2+\tau}\delta'\mu(\mathcal{W}') \geqslant 0.5\delta'\mu(\mathcal{W}')$;

(c) For all $w \in \mathcal{W}'$, we have $\mu(\Gamma_{G'}(w)) \geqslant \frac{1+\tau}{2+\tau}\delta'\mu(\mathcal{V}') \geqslant 0.5\delta'\mu(\mathcal{V}')$;

(d) $q(G') \geqslant q(G^{(1)})$.

In particular, we see that $q(G') \gg_\tau e^{4CN}q(G) \geqslant e^{Ct/s}q(G)$. In addition, arguing as in the proof of Proposition 7.1, we have that $q(G') \ll_\tau y^2$. Hence, (9.11) holds in this case.

Case 2: $N < 0.25t/s$. In this case we must exploit the condition $L_t(q,r) \geqslant 1/s$ to obtain the necessary savings. If we blindly apply Proposition 8.2, we might lose track of this condition. So we first perform some cosmetic surgery to our graph. Applying Lemma 8.4, we find that there exists a GCD subgraph $G^{(2)} = (\mu, \mathcal{V}^{(2)}, \mathcal{W}^{(2)}, \mathcal{E}^{(2)}, \mathcal{P}^{(2)}, f^{(2)}, g^{(2)})$ of $G^{(1)}$ with

$$\mathcal{P}^{(2)} = \mathcal{P}^{(1)}, \tag{9.12}$$

$$q(G^{(2)}) \geqslant \frac{q(G^{(1)})}{2}, \tag{9.13}$$

and such that

$$\sum_{\substack{p \mid vw/\gcd(v,w)^2 \\ p>t,\ p\notin\mathcal{R}(G^{(1)})}} \frac{1}{p} \geqslant \frac{1}{2s} \quad \text{whenever} \quad (v,w) \in \mathcal{E}^{(2)}. \tag{9.14}$$

We claim that an inequality of the form (9.14) holds even if we remove from consideration the primes lying in $\mathcal{P}^{(1)}$. Indeed, if $p > C_6$ lies in $\mathcal{P}^{(1)}$ and $(v,w) \in \mathcal{E}^{(2)} \subseteq \mathcal{E}^{(1)}$, then we use property (d) of $G^{(1)}$ to find that $f^{(1)}(p) \neq g^{(1)}(p)$. Hence,

$$\sum_{\substack{p \mid vw/\gcd(v,w)^2 \\ p>t,\ p\in\mathcal{P}^{(1)}}} \frac{1}{p} < \frac{\#\{p \in \mathcal{P}^{(1)} : f^{(1)}(p) \neq g^{(1)}(p),\ p > t\}}{t} \leqslant \frac{N}{t} < \frac{1}{4s}$$

by our assumption on N, and because $t \geqslant C_6 s \geqslant C_6$ here. As a consequence, for all $(v,w) \in \mathcal{E}^{(2)}$ we have

$$\sum_{\substack{p \mid vw/\gcd(v,w)^2 \\ p>t,\ p\notin\mathcal{R}(G^{(1)})\cup\mathcal{P}^{(1)}}} \frac{1}{p} \geqslant \frac{1}{4s} \tag{9.15}$$

Next, we iterate Propositions 8.1 and Proposition 8.2, and we also apply Lemma 8.6 to find a GCD subgraph $G' = (\mu, \mathcal{V}', \mathcal{W}', \mathcal{E}', \mathcal{P}', f', g')$ of $G^{(2)}$ of density $\delta' > 0$ such that:

(a) $\mathcal{R}(G') = \emptyset$;

(b) For all $v \in \mathcal{V}'$, we have $\mu(\Gamma_{G'}(v)) \geqslant \frac{1+\tau}{2+\tau}\delta'\mu(\mathcal{W}') \geqslant 0.5\delta'\mu(\mathcal{W}')$;

(c) For all $w \in \mathcal{W}'$, we have $\mu(\Gamma_{G'}(w)) \geqslant \frac{1+\tau}{2+\tau}\delta'\mu(\mathcal{V}') \geqslant 0.5\delta'\mu(\mathcal{V}')$;

(d) $q(G') \geqslant q(G^{(2)})$;

(e) (9.15) holds for all $(v,w) \in \mathcal{E}'$.

Hence, Proposition 7.2 will follow in this case too if we show that

$$q(G') \ll_{\tau,C} y^2 e^{-Ct/s}. \tag{9.16}$$

To do this, we largely follow the proof of Propositions 6.3 and 7.1 in [24]. We give all details for the ease of the reader.

As in the proof of Proposition 7.1 above, we define a and b by (9.3) and take note of relations (9.4) and (9.5). In addition, relation (9.6) implies that

$$q(G') \ll_\tau (\delta')^{1+\tau}\mu(\mathcal{E}') \cdot \frac{ab}{\gcd(a,b)^2} \cdot \frac{ab}{\varphi(a)\varphi(b)}. \tag{9.17}$$

It remains to bound $\mathcal{E}'$. First of all, note that

$$\mathcal{P}' \subseteq \mathcal{R}(G^{(2)}) \cup \mathcal{P}^{(2)} \subseteq \mathcal{R}(G^{(1)}) \cup \mathcal{P}^{(1)},$$

where the second relation follows by (9.12). Let $(v,w) \in \mathcal{E}'$ and let us write $v = av'$ and $w = bw'$. Since $\gcd(v,w) = \gcd(a,b)$, we must have that $\gcd(v',w') = 1$. Hence, if $p|vw$ but $p \nmid v'w'$, then $p|ab$, and thus $p \in \mathcal{P}' \subseteq \mathcal{R}(G^{(1)}) \cup \mathcal{P}^{(1)}$. Together with (9.15), this implies that $L_t(v',w') \geqslant 1/(4s)$. We conclude that

$$\mathcal{E}' \subseteq \left\{ (v,w) = (av',bw') \in \mathcal{V}' \times \mathcal{W}' : \begin{array}{l} \gcd(v,w) = \gcd(a,b),\ D(v,w) \leqslant y, \\ L_t(v',w') \geqslant \frac{1}{4s} \end{array} \right\}.$$

Next, let us define $w_{\max}(v)$, $v_{\max}(w)$, w_0 and $\mathcal{E}''$ as in the proof of Proposition 7.1. We find that

$$q(G') \ll y^2 ab \sum_{(v,w)\in\mathcal{E}''} \frac{1}{v_{\max}(w)w_0} \leqslant y^2 ab \sum_{(v,w)\in\mathcal{E}'} \frac{1}{v_{\max}(w)w_0}. \tag{9.18}$$

If $v = v'a$ and $w = w'b$ and $(v,w) \in \mathcal{E}'$, then we know that $L_t(v',w') \geqslant 1/(4s)$. In particular, we see that either

$$\sum_{p|v',\, p>t} \frac{1}{p} \geqslant \frac{1}{8s} \quad \text{or} \quad \sum_{p|w',\, p>t} \frac{1}{p} \geqslant \frac{1}{8s}$$

whenever $(v,w) \in \mathcal{E}'$. Consequently,

$$\sum_{(v,w)\in\mathcal{E}'} \frac{1}{v_{\max}(w)w_0} \leqslant S_1 + S_2,$$

where

$$S_1 := \sum\sum_{\substack{w'\leqslant w_0/b,\ v'\leqslant v_{\max}(bw')/a \\ \sum_{p|v',\,p>t} 1/p \geqslant 1/(8s)}} \frac{1}{v_{\max}(bw')w_0},$$

$$S_2 := \sum\sum_{\substack{w'\leqslant w_0/b,\ v'\leqslant v_{\max}(bw')/a \\ \sum_{p|w',\,p>t} 1/p \geqslant 1/(8s)}} \frac{1}{v_{\max}(bw')w_0}.$$

For S_1, we note that

$$\begin{aligned} S_1 &\leqslant \sum_{w'\leqslant w_0/b} \frac{1}{w_0 v_{\max}(bw')} \sum_{\substack{v'\leqslant v_{\max}(bw')/a \\ \sum_{p|v',\,p>t} 1/p \geqslant 1/(8s)}} 1 \\ &\ll_C \sum_{w'\leqslant w_0/b} \frac{1}{w_0 v_{\max}(bw')} \cdot \frac{v_{\max}(bw')/a}{e^{Ct/s}} \\ &\leqslant \frac{1}{abe^{Ct/s}} \end{aligned}$$

by Lemma 6.2. Similarly for S_2, we find that

$$\begin{aligned} S_2 &\leqslant \sum_{\substack{w' \leqslant w_0/b \\ \sum_{p\mid w',\, p>t} 1/p \geqslant 1/(8s)}} \frac{1}{w_0 v_{\max}(bw')} \sum_{v' \leqslant v_{\max}(bw')/a} 1 \\ &\leqslant \sum_{\substack{w' \leqslant w_0/b \\ \sum_{p\mid w',\, p>t} 1/p \geqslant 1/(8s)}} \frac{1}{aw_0} \\ &\ll_C \frac{1}{abe^{Ct/s}}, \end{aligned}$$

by applying Lemma 6.2 once again. Substituting these bounds into (9.18) establishes (9.16), thus completing the proof of Proposition 7.2. □

Proof of Proposition 7.3. We use a very similar argument to the one employed in the previous proof. We highlight only the important changes.

- We may assume that t is large enough in terms of κ and C; otherwise, the result follows from Proposition 7.1.
- We use the results of Section 8 with $M = e^{4C/\kappa}$.
- We separate Cases 1 and 2 according to whether $N > 0.25\kappa \log t$ or $N \leqslant 0.25\kappa \log t$.
- In Case 2, we aim to show that

$$q(G') \ll_{\tau,\kappa,C} y^2 t^{-C}. \tag{9.19}$$

- In Case 2, we use Lemma 8.5 with $U = \kappa \log t$ in place of Lemma 8.4 to find $G^{(2)}$ such that $\#\{p | vw/\gcd(v,w)^2 : C_6 < p \leqslant t,\ p \notin \mathcal{R}(G^{(1)})\} \geqslant 0.5\kappa \log t$ whenever $(v,w) \in \mathcal{E}^{(2)}$.
- In Case 2, we note that $\#\{p | vw/\gcd(v,w)^2 : p > C_6,\ p \in \mathcal{P}^{(1)}\} \leqslant N \leqslant 0.25\kappa \log t$, where we used property (d) of the graph $G^{(1)}$. Thus $\#\{p | vw/\gcd(v,w)^2 : C_6 < p \leqslant t,\ p \notin \mathcal{R}(G^{(1)}) \cup \mathcal{P}^{(1)}\} \geqslant 0.25\kappa \log t$.
- To prove (9.19) and conclude the proof, we apply Lemma 6.3 in place of 6.2.

We leave the details for how to complete the proof of Proposition 7.3 to the reader. □

10. Proof of Lemma 8.6

In this section we establish Lemma 8.6.

Lemma 10.1 (Quality increment or all vertices have high degree)**.** *Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph with edge density $\delta > 0$. For each $v \in \mathcal{V}$ and for each $w \in \mathcal{W}$, we let*

$$\Gamma_G(v) := \{w \in \mathcal{W} : (v,w) \in \mathcal{E}\} \quad \text{and} \quad \Gamma_G(w) := \{v \in \mathcal{V} : (v,w) \in \mathcal{E}\}$$

be the sets of their neighbours. Then one of the following holds:

(a) For all $v \in \mathcal{V}$ and for all $w \in \mathcal{W}$, we have

$$\mu(\Gamma_G(v)) \geqslant \frac{1+\tau}{2+\tau} \cdot \delta\mu(\mathcal{W}) \quad \text{and} \quad \mu(\Gamma_G(w)) \geqslant \frac{1+\tau}{2+\tau} \cdot \delta\mu(\mathcal{V}).$$

(b) There is a GCD subgraph $G' = (\mu, \mathcal{V}', \mathcal{W}', \mathcal{E}', \mathcal{P}, f, g)$ of G with quality $q(G') > q(G)$.

Proof. Assume that (a) fails. Then either its first or its second inequality fails. Assume that the first one fails for some $v \in \mathcal{V}$; the other case is entirely analogous. Let $\mathcal{E}'$ be the set of edges between the vertex sets $\mathcal{V} \setminus \{v\}$ and $\mathcal{W}$. Note that

$$\mu(\mathcal{E}') = \mu(\mathcal{E}) - \mu(v)\mu(\Gamma_G(v)) > \Big(1 - \frac{1+\tau}{2+\tau}\Big)\mu(\mathcal{E}) > 0$$

because $\mu(\Gamma_G(v)) < \frac{1+\tau}{2+\tau}\delta\mu(\mathcal{W})$, $\mu(v) \leqslant \mu(\mathcal{V})$, and $\mu(\mathcal{E}) > 0$ by the assumption that $\delta > 0$. In particular, we have $\mu(\mathcal{W}), \mu(\mathcal{V} \setminus \{v\}) > 0$. We then consider $G' = (\mu, \mathcal{V} \setminus \{v\}, \mathcal{W}, \mathcal{E}', \mathcal{P}, f, g)$, which is a GCD subgraph of G. Let G' have edge density δ'. We claim that $\delta' \geqslant \delta$ and $q(G') \geqslant q(G)$.

Indeed, we have

$$\begin{aligned}\mu(\mathcal{E}') = \mu(\mathcal{E}) - \mu(v)\mu(\Gamma_G(v)) &> \delta\mu(\mathcal{V})\mu(\mathcal{W}) - \frac{(1+\tau)\delta}{2+\tau}\mu(v)\mu(\mathcal{W}) \\ &= \delta\big(\mu(\mathcal{V}) - \mu(v)\big)\mu(\mathcal{W}) \cdot \Big(1 + \frac{\mu(v)}{\mu(\mathcal{V}) - \mu(v)} \cdot \frac{1}{2+\tau}\Big).\end{aligned}$$

Thus the edge density δ' of G' satisfies

$$\delta' = \frac{\mu(\mathcal{E}')}{\mu(\mathcal{V} \setminus \{v\})\mu(\mathcal{W})} = \frac{\mu(\mathcal{E}')}{\big(\mu(\mathcal{V}) - \mu(v)\big)\mu(\mathcal{W})} > \delta \cdot \Big(1 + \frac{\mu(v)}{\mu(\mathcal{V}) - \mu(v)} \cdot \frac{1}{2+\tau}\Big).$$

Consequently,

$$(\delta')^{2+\tau}\mu(\mathcal{V} \setminus \{v\})\mu(\mathcal{W}) > \delta^{2+\tau}\big(\mu(\mathcal{V}) - \mu(v)\big)\mu(\mathcal{W})\Big(1 + \frac{\mu(v)}{\mu(\mathcal{V}) - \mu(v)}\Big) = \delta^{2+\tau}\mu(\mathcal{V})\mu(\mathcal{W}).$$

This proves the claim that $q(G') > q(G)$, thus completing the proof of the lemma. □

Proof of Lemma 8.6. Let G' be a $(\mathcal{P}, f, g)$-maximal GCD subgraph of G, and let us apply Lemma 10.1 to G'. By its maximality, conclusion (a) of Lemma 10.1 must hold. This completes the proof. □

11. LEMMAS ON GCD GRAPHS

Lemma 11.1 (Quality variation for special GCD subgraphs)**.** *Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph, $p \in \mathcal{R}(G)$ and $k, \ell \in \mathbb{Z}_{\geqslant 0}$. In addition, if G is non-trivial and $\mu(\mathcal{V}_{p^k}), \mu(\mathcal{W}_{p^\ell}) > 0$, then we have*

$$\frac{q(G_{p^k,p^\ell})}{q(G)} = \Big(\frac{\mu(\mathcal{E}_{p^k,p^\ell})}{\mu(\mathcal{E})}\Big)^{2+\tau}\Big(\frac{\mu(\mathcal{V})}{\mu(\mathcal{V}_{p^k})}\Big)^{1+\tau}\Big(\frac{\mu(\mathcal{W})}{\mu(\mathcal{W}_{p^\ell})}\Big)^{1+\tau}\frac{p^{|k-\ell|}}{(1 - \mathbb{1}_{k=\ell\geqslant 1}/p)^2(1 - 1/p^{1+\tau/4})^3}.$$

Proof. This follows from directly the definitions. □

Lemma 11.2 (One subgraph must have limited quality loss)**.** *Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph with edge density $\delta > 0$, and let $\mathcal{V} = \mathcal{V}_1 \sqcup \cdots \sqcup \mathcal{V}_I$ and $\mathcal{W} = \mathcal{W}_1 \sqcup \cdots \sqcup \mathcal{W}_J$ be partitions of $\mathcal{V}$ and $\mathcal{W}$. Then there is a GCD subgraph $G' = (\mu, \mathcal{V}', \mathcal{W}', \mathcal{E}', \mathcal{P}, f, g)$ of G with edge density $\delta' > 0$ such that*

$$q(G') \geqslant \frac{q(G)}{(IJ)^{2+\tau}}, \qquad \delta' \geqslant \frac{\delta}{IJ},$$

and with $\mathcal{V}' \in \{\mathcal{V}_1, \ldots, \mathcal{V}_I\}$, $\mathcal{W}' \in \{\mathcal{W}_1, \ldots, \mathcal{W}_J\}$, and $\mathcal{E}' = \mathcal{E} \cap (\mathcal{V}' \times \mathcal{W}')$.

Proof. This is Lemma 11.2 in [24] with the obvious changes to account for the modified definition of $q(G)$. □

Lemma 11.3 (Few edges between unbalanced sets, I)**.** *Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph with edge density $\delta > 0$. Let $p \in \mathcal{R}(G)$, $r \in \mathbb{Z}_{\geqslant 1}$ and $k \in \mathbb{Z}_{\geqslant 0}$ be such that $p^r > C_4$ and*

$$\frac{\mu(\mathcal{W}_{p^k})}{\mu(\mathcal{W})} \geqslant 1 - \frac{C_2}{p}.$$

(In particular, if $p \leqslant C_2$, the last hypothesis is vacuous.)

If we set $\mathcal{L}_{k,r} = \{\ell \in \mathbb{Z}_{\geqslant 0} : |\ell - k| \geqslant r + 1\}$, then one of the following holds:

(a) There is $\ell \in \mathcal{L}_{k,r}$ such that $q(G_{p^k,p^\ell}) > M \cdot q(G)$.

(b) $\sum_{\ell \in \mathcal{L}_{k,r}} \mu(\mathcal{E}_{p^k,p^\ell}) \leqslant \mu(\mathcal{E})/(4p^{1+\tau/4})$.

Proof. Assume that conclusion (b) does not hold, so $\sum_{\ell \in \mathcal{L}_{k,r}} \mu(\mathcal{E}_{p^k,p^\ell}) > \mu(\mathcal{E})/(4p^{1+\tau/4})$ and we wish to establish (a). Then there must exist some $\ell \in \mathcal{L}_{k,r}$ such that

$$\mu(\mathcal{E}_{p^k,p^\ell}) > \frac{\mu(\mathcal{E})}{300 \cdot 2^{|k-\ell|/20} p^{1+\tau/4}} > 0,$$

where we used that $\sum_{|j|\geqslant 0} 2^{-|j|/20} \leqslant 2/(1 - 2^{-1/20}) \leqslant 60$. In particular, G_{p^k,p^ℓ} is a non-trivial GCD graph. Since $\mu(\mathcal{W}_{p^k}) \geqslant (1 - C_2/p)\mu(\mathcal{W})$ and $\ell \neq k$, we have that $\mu(\mathcal{W}_{p^\ell}) \leqslant C_2\mu(\mathcal{W})/p$. Consequently,

$$\begin{aligned}\frac{q(G_{p^k,p^\ell})}{q(G)} &= \left(\frac{\mu(\mathcal{E}_{p^k,p^\ell})}{\mu(\mathcal{E})}\right)^{2+\tau} \left(\frac{\mu(\mathcal{V})}{\mu(\mathcal{V}_{p^k})}\right)^{1+\tau} \left(\frac{\mu(\mathcal{W})}{\mu(\mathcal{W}_{p^\ell})}\right)^{1+\tau} \frac{p^{|k-\ell|}}{(1-1/p^{1+\tau/4})^3} \\ &\geqslant \left(\frac{1}{300 \cdot 2^{|k-\ell|/20} p^{1+\tau/4}}\right)^{2+\tau} \left(\frac{p}{C_2}\right)^{1+\tau} p^{|k-\ell|} \\ &\geqslant \frac{1}{300^{2+\tau} C_2^{1+\tau}} \cdot \left(\frac{p}{2^{\frac{2+\tau}{20}}}\right)^{|k-\ell|} \cdot p^{-1-\tau/2-\tau^2/4}\end{aligned}$$

Note that $0 < \tau \leqslant 0.01$, so $1 + \tau/2 + \tau^2/4 \leqslant 1.01$ and $(2+\tau)/20 \leqslant 1/9$. Since $|k - \ell| \geqslant r + 1 \geqslant 0.99r + 1.01$ and $p/2^{1/9} \geqslant p^{8/9}$, we have

$$\left(\frac{p}{2^{\frac{2+\tau}{20}}}\right)^{|k-\ell|} \cdot p^{-1-0.5\tau-0.25\tau^2} \geqslant p^{\frac{88}{100}r} \cdot \frac{1}{2^{\frac{101}{900}}}$$

Therefore

$$\frac{q(G_{p^k,p^\ell})}{q(G)} \geqslant \frac{1}{300^{2.01} C_2^{1.01}} \cdot p^{\frac{88}{100}r} \cdot \frac{1}{2^{\frac{101}{900}}} > M$$

by our assumption that $p^r > C_4$ (recall (8.1)). This completes the proof of the lemma. □

Clearly, the symmetric version of Lemma 11.3 also holds:

Lemma 11.4 (Few edges between unbalanced sets, II)**.** *Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph with edge density $\delta > 0$. Let $p \in \mathcal{R}(G)$, $r \in \mathbb{Z}_{\geqslant 1}$ and $\ell \in \mathbb{Z}_{\geqslant 0}$ be such that $p^r > C_4$ and*

$$\frac{\mu(\mathcal{V}_{p^\ell})}{\mu(\mathcal{V})} \geqslant 1 - \frac{C_2}{p},$$

and set $\mathcal{K}_{\ell,r} = \{k \in \mathbb{Z}_{\geqslant 0} : |\ell - k| \geqslant r + 1\}$. Then one of the following holds:

(a) There is $k \in \mathcal{K}_{\ell,r}$ such that $q(G_{p^k,p^\ell}) > M \cdot q(G)$.

(b) $\sum_{k \in \mathcal{K}_{\ell,r}} \mu(\mathcal{E}_{p^k,p^\ell}) \leqslant \mu(\mathcal{E})/(4p^{1+\tau/4})$.

Lemma 11.5 (Few edges between small sets)**.** *Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph with edge density $\delta > 0$ and let $\eta \in (0, 1]$. Then one of the following holds:*

(a) For all sets $\mathcal{A} \subseteq \mathcal{V}$ and $\mathcal{B} \subseteq \mathcal{W}$ such that $\mu(\mathcal{A}) \leqslant \eta \cdot \mu(\mathcal{V})$ and $\mu(\mathcal{B}) \leqslant \eta \cdot \mu(\mathcal{W})$, we have $\mu(\mathcal{E} \cap (\mathcal{A} \times \mathcal{B})) \leqslant \eta^{(2+2\tau)/(2+\tau)} \cdot \mu(\mathcal{E})$.

(b) There is a GCD subgraph $G' = (\mu, \mathcal{V}', \mathcal{W}', \mathcal{E}', \mathcal{P}, f, g)$ of G such that $q(G') > q(G)$, $\mathcal{V}' \subsetneq \mathcal{V}$ and $\mathcal{W}' \subsetneq \mathcal{W}$.

Proof. This is Lemma 11.5 in [24] with the obvious changes to account for the modified definition of $q(G)$. □

Lemma 11.6 (Subgraph with few edges between all small sets)**.** *Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph with edge density $\delta > 0$, and let $\eta \in (0, 1]$. Then there is a GCD subgraph $G' = (\mu, \mathcal{V}', \mathcal{W}', \mathcal{E}', \mathcal{P}, f, g)$ of G with edge density $\delta' > 0$ such that both of the following hold:*

(a) $q(G') \geqslant q(G) > 0$.

(b) For all sets $\mathcal{A} \subseteq \mathcal{V}'$ and $\mathcal{B} \subseteq \mathcal{W}'$ such that $\mu(\mathcal{A}) \leqslant \eta \cdot \mu(\mathcal{V}')$ and $\mu(\mathcal{B}) \leqslant \eta \cdot \mu(\mathcal{W}')$, we have $\mu(\mathcal{E}' \cap (\mathcal{A} \times \mathcal{B})) \leqslant \eta^{(2+2\tau)/(2+\tau)} \mu(\mathcal{E}')$.

Proof. By iterating Lemma 11.5. □

Lemma 11.7. *Let $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ be a GCD graph with $\mathcal{R}^\flat(G) = \emptyset$ and edge density $\delta > 0$. Let $p > C_6$ be a prime in $\mathcal{R}(G)$. There exists an integer $k \geqslant 0$ such that if we let $\mathcal{V}' = \mathcal{V} \setminus \mathcal{V}_{p^k}$ and $\mathcal{W}' = \mathcal{W} \setminus \mathcal{W}_{p^k}$, then one of the following holds:*

(a) We have

$$\mu\Big(\mathcal{E} \cap \big((\mathcal{V} \times \mathcal{W}') \cup (\mathcal{V}' \times \mathcal{W})\big)\Big) \leqslant \frac{C_6}{10^{10} p} \cdot \mu(\mathcal{E}).$$

(b) The GCD subgraph $G' = (\mu, \mathcal{V}', \mathcal{W}', \mathcal{E}', \mathcal{P}, f, g)$ of G with $\mathcal{E}' = \mathcal{E} \cap (\mathcal{V}' \times \mathcal{W}')$ satisfies $q(G') > q(G)$.

Proof. Let $p > C_6$ be a prime in $\mathcal{R}(G)$. Since $\mathcal{R}^\flat(G) = \emptyset$, we must have $p \in \mathcal{R}^\sharp(G)$. Therefore there exists an integer $k \geqslant 0$ such that

$$\frac{\mu(\mathcal{V}_{p^k})}{\mu(\mathcal{V})} \geqslant 1 - \frac{C_2}{p} \quad \text{and} \quad \frac{\mu(\mathcal{W}_{p^k})}{\mu(\mathcal{W})} \geqslant 1 - \frac{C_2}{p}, \tag{11.1}$$

and

$$q(G_{p^a,p^b}) < M \cdot q(G) \quad \text{for all } (a,b) \in \mathbb{Z}_{\geqslant 0}^2 \text{ with } a \neq b. \tag{11.2}$$

In particular, we have

$$\frac{\mu(\mathcal{V}')}{\mu(\mathcal{V})} \leqslant \frac{C_2}{p} \quad \text{and} \quad \frac{\mu(\mathcal{W}')}{\mu(\mathcal{W})} \leqslant \frac{C_2}{p}.$$

By the proof of Lemma 11.5 applied with $\eta = \min\{1, C_2/p\}$ we see that one of the following holds:

(a) $\mu(\mathcal{E}') \leqslant \mu(\mathcal{E}) \cdot (C_2/p)^{\frac{2+2\tau}{2+\tau}}$;

(b) $q(G') > q(G)$.

If (b) holds, we are done, so let us assume that (a) holds. In particular, we then find that

$$\sum\sum_{a,b\in\mathbb{Z}_{\geqslant 0}\setminus\{k\}} \mu(\mathcal{E}_{p^a,p^b}) = \mu(\mathcal{E}') \leqslant \mu(\mathcal{E})\cdot\Big(\frac{C_2^{2+2\tau}}{p^\tau}\Big)^{\frac{1}{2+\tau}}\cdot\frac{1}{p} \leqslant \mu(\mathcal{E})\cdot\frac{1}{10^{10}p} \tag{11.3}$$

since $p > C_6 \geqslant C_2^{10/\tau}$ and $\tau \leqslant 0.01$ (recall (8.1)). Moreover, since $p > C_6 \geqslant C_4$, we may apply Lemmas 11.3 and 11.4 with $r = 1$. Relation (11.2) tells us that we must in case (b) of these two lemmas. We must therefore have that

$$\sum_{\substack{b\geqslant 0\\ |k-b|\geqslant 2}} \mu(\mathcal{E}_{p^k,p^b}) \leqslant \frac{\mu(\mathcal{E})}{4p} \quad\text{and}\quad \sum_{\substack{a\geqslant 0\\ |a-k|\geqslant 2}} \mu(\mathcal{E}_{p^a,p^k}) \leqslant \frac{\mu(\mathcal{E})}{4p}. \tag{11.4}$$

In addition, if we take $(a,b) = (k,k+1)$ in (11.2) and we use Lemma 11.1, we find that

$$M > \frac{q(G_{p^k,p^{k+1}})}{q(G)} \geqslant \Big(\frac{\mu(\mathcal{E}_{p^k,p^{k+1}})}{\mu(\mathcal{E})}\Big)^{2+\tau}\Big(\frac{\mu(\mathcal{V})}{\mu(\mathcal{V}_{p^k})}\Big)^{1+\tau}\Big(\frac{\mu(\mathcal{W})}{\mu(\mathcal{W}_{p^{k+1}})}\Big)^{1+\tau} p.$$

Using (11.1), we find that $\mu(\mathcal{W})/\mu(\mathcal{W}_{p^{k+1}}) \geqslant p/C_2$. In addition, we have the trivial lower bound $\mu(\mathcal{V})/\mu(\mathcal{V}_{p^k}) \geqslant 1$. Therefore we conclude that

$$M > \Big(\frac{\mu(\mathcal{E}_{p^k,p^{k+1}})}{\mu(\mathcal{E})}\Big)^{2+\tau}\frac{p^{2+\tau}}{C_2^{1+\tau}},$$

whence

$$\mu(\mathcal{E}_{p^k,p^{k+1}}) \leqslant \mu(\mathcal{E})\cdot\frac{(MC_2^{1+\tau})^{\frac{1}{2+\tau}}}{p} \leqslant \mu(\mathcal{E})\cdot\frac{C_6}{10^{11}p} \tag{11.5}$$

since $C_6 \geqslant C_4 = 10^{10}M^2C_2^2 \geqslant 10^{12}$ from (8.1). Similarly, we find that

$$\mu(\mathcal{E}_{p^k,p^{k-1}}),\ \mu(\mathcal{E}_{p^{k+1},p^k}),\ \mu(\mathcal{E}_{p^{k-1},p^k}) \leqslant \mu(\mathcal{E})\cdot\frac{C_6}{10^{11}p}, \tag{11.6}$$

with the convention that $\mathcal{E}_{p^0,p^{-1}} = \mathcal{E}_{p^{-1},p^0} = \emptyset$. Combining (11.3), (11.4), (11.5) and (11.6) completes the proof of the lemma. ☐

Corollary 11.8. *Let $G = (\mu,\mathcal{V},\mathcal{W},\mathcal{E},\mathcal{P},f,g)$, p and G' be as in Lemma 11.7. Then one of the following holds:*

(a) $\mu\Big(\Big\{(v,w)\in\mathcal{E} : p|\frac{vw}{\gcd(v,w)^2}\Big\}\Big) \leqslant \frac{C_6}{10^{10}p}\cdot\mu(\mathcal{E})$;

(b) $q(G') > q(G)$.

Proof. Let $k\in\mathbb{Z}_{\geqslant 0}$, $\mathcal{V}'$ and $\mathcal{W}'$ be as in Lemma 11.7. Assume that conclusion (b) is false. Then conclusion (a) of Lemma 11.7 must be true. In addition, note that if $p|vw/\gcd(v,w)^2$, then $p^j\|v$ and $p^\ell\|w$ for some $j\neq\ell$. In particular we cannot have $p^k\|v$ and $p^k\|w$. Thus

$$\mu\Big(\Big\{(v,w)\in\mathcal{E} : p|\frac{vw}{\gcd(v,w)^2}\Big\}\Big) \leqslant \mu\Big(\mathcal{E}\cap\big((\mathcal{V}\times\mathcal{W}')\cup(\mathcal{V}'\times\mathcal{W})\big)\Big) \leqslant \frac{C_6}{10^{10}p}\cdot\mu(\mathcal{E}),$$

as claimed. ☐

12. Proof of the anatomical Lemmas 8.4 and 8.5

Proof of Lemma 8.4. For brevity, let

$$L_t'(v,w)=\sum_{\substack{p|vw/\gcd(v,w)^2\\ p\in\mathcal{R}(G),\ p>t}}\frac{1}{p}.$$

Since we have assumed that G is maximal, conclusion (b) of Corollary 11.8 cannot hold, and thus

$$\mu\Big(\Big\{(v,w)\in\mathcal{E}:p|\frac{vw}{\gcd(v,w)^2}\Big\}\Big)\leqslant\frac{C_6}{10^{10}p}\cdot\mu(\mathcal{E}) \tag{12.1}$$

for every prime $p>C_6$ lying in $\mathcal{R}(G)$. Consequently,

$$\begin{aligned}\sum_{(v,w)\in\mathcal{E}}\mu(v)\mu(w)L_t'(v,w)&=\sum_{\substack{p\in\mathcal{R}(G)\\ p>t}}\frac{1}{p}\cdot\mu\Big(\Big\{(v,w)\in\mathcal{E}:p|\frac{vw}{\gcd(v,w)^2}\Big\}\Big)\\ &\leqslant\sum_{p>t}\frac{C_6\mu(\mathcal{E})}{10^{10}p^2}<\frac{C_6\mu(\mathcal{E})}{10^{10}t}\leqslant\frac{\mu(\mathcal{E})}{10^{10}s}\end{aligned}$$

by Corollary 11.8 and our assumption that $t\geqslant C_6s\geqslant C_6$ (recall (8.1)). Now, let us define

$$\mathcal{E}':=\{(v,w)\in\mathcal{E}:L_t'(v,w)\leqslant 1/(2s)\}.$$

Evidently, we have that

$$\mu(\mathcal{E}\setminus\mathcal{E}')=\sum_{\substack{(v,w)\in\mathcal{E}\\ L_t'(v,w)>1/(2s)}}\mu(v)\mu(w)<2s\sum_{(v,w)\in\mathcal{E}}\mu(v)\mu(w)L_t'(v,w)<\frac{\mu(\mathcal{E})}{1000}.$$

Thus $\mu(\mathcal{E}')\geqslant\frac{999}{1000}\mu(\mathcal{E})$. We then take $G':=(\mu,\mathcal{V},\mathcal{W},\mathcal{E}',\mathcal{P},f,g)$ and note that

$$\frac{q(G')}{q(G)}=\left(\frac{\mu(\mathcal{E}')}{\mu(\mathcal{E})}\right)^{2+\tau}\geqslant\frac{1}{2}.$$

Hence, since $\mathcal{E}'\subseteq\mathcal{E}\subseteq\{(v,w)\in\mathcal{V}\times\mathcal{W}:L_t(v,w)\geqslant 1/s\}$, for any $(v,w)\in\mathcal{E}'$ we have

$$\sum_{\substack{p|vw/\gcd(v,w)^2\\ p\notin\mathcal{R}(G),\ p>t}}\frac{1}{p}=L_t(v,w)-L_t'(v,w)\geqslant\frac{1}{s}-\frac{1}{2s}=\frac{1}{2s}.$$

This completes the proof. □

Proof of Lemma 8.5. For brevity, let

$$\omega_t'(v,w)=\sum_{\substack{p|vw/\gcd(v,w)^2\\ p\in\mathcal{R}(G),\ C_6<p\leqslant t}}1.$$

As before, using the maximality of G yields (12.1) for each prime $p > C_6$ lying in $\mathcal{R}(G)$. Hence,

$$\begin{aligned}\sum_{(v,w)\in\mathcal{E}} \mu(v)\mu(w)\omega_t'(v,w) &= \sum_{\substack{p\in\mathcal{R}(G)\\ C_6<p\leqslant t}} 1\cdot\mu\Big(\Big\{(v,w)\in\mathcal{E} : p\Big|\frac{vw}{\gcd(v,w)^2}\Big\}\Big)\\ &\leqslant \sum_{C_6<p\leqslant t}\frac{C_6\mu(\mathcal{E})}{10^{10}p}\\ &\leqslant \Big(\log\log t - \log\log C_6 + \frac{1}{2(\log t)^2} + \frac{1}{2(\log C_6)^2}\Big)\frac{C_6\mu(\mathcal{E})}{10^{10}}\\ &\leqslant \frac{C_6\log\log t}{10^{10}}\cdot\mu(\mathcal{E}),\end{aligned}$$

where we used [28, Theorem 5] in the third line and the fact $C_6 > 10^{10}$ in the final line. Now, let us define

$$\mathcal{E}' := \Big\{(v,w)\in\mathcal{E} : \omega_t'(v,w)\leqslant \frac{C_6}{10^9}\log\log t\Big\}.$$

Evidently, we have that

$$\mu(\mathcal{E}\setminus\mathcal{E}') = \sum_{\substack{(v,w)\in\mathcal{E}\\ \omega_t'(v,w)>C_6 10^{-9}\log\log t}} \mu(v)\mu(w) < \sum_{(v,w)\in\mathcal{E}}\mu(v)\mu(w)\cdot\frac{\omega_t'(v,w)}{C_6 10^{-9}\log\log t}\leqslant\frac{\mu(\mathcal{E})}{10}.$$

Thus $\mu(\mathcal{E}')\geqslant 9\mu(\mathcal{E})/10$. We then take $G' := (\mu,\mathcal{V},\mathcal{W},\mathcal{E}',\mathcal{P},f,g)$ and note that

$$\frac{q(G')}{q(G)} = \Big(\frac{\mu(\mathcal{E}')}{\mu(\mathcal{E})}\Big)^{2+\tau}\geqslant\frac{1}{2}.$$

By assumption we have $\mathcal{E}\subseteq\{(v,w)\in\mathcal{V}\times\mathcal{W} : \omega_t(v,w)\geqslant U\}$. Thus, since $\mathcal{E}'\subseteq\mathcal{E}$, for any $(v,w)\in\mathcal{E}'$ we have

$$\sum_{\substack{p|vw/\gcd(v,w)^2\\ p\notin\mathcal{R}(G),\ C_6<p\leqslant t}} 1\geqslant\omega_t(v,w)-\omega_t'(v,w)-C_6\geqslant U-\frac{C_6}{10^9}\log\log t - C_6\geqslant\frac{U}{2}$$

since we have assumed that $U\geqslant 2C_6\log\log t\geqslant 2C_6^2$. This completes the proof. ☐

13. Proof of Proposition 8.1

Lemma 13.1 (Bounds on edge sets)**.** *Consider a GCD graph $G = (\mu,\mathcal{V},\mathcal{W},\mathcal{E},\mathcal{P},f,g)$ with edge density $\delta > 0$ and a prime $p\in\mathcal{R}(G)$. For each $k,\ell\in\mathbb{Z}_{\geqslant 0}$, let*

$$\alpha_k = \frac{\mu(\mathcal{V}_{p^k})}{\mu(\mathcal{V})}\quad\text{and}\quad\beta_\ell=\frac{\mu(\mathcal{W}_{p^\ell})}{\mu(\mathcal{W})}.$$

Then there exist $k,\ell\in\mathbb{Z}_{\geqslant 0}$ such that $\alpha_k,\beta_\ell > 0$ and

$$\frac{\mu(\mathcal{E}_{p^k,p^\ell})}{\mu(\mathcal{E})}\geqslant\begin{cases}(\alpha_k\beta_k)^{(1+\tau)/(2+\tau)}, & \text{if } k=\ell,\\ \dfrac{\alpha_k(1-\beta_k)+\beta_k(1-\alpha_k)+\alpha_\ell(1-\beta_\ell)+\beta_\ell(1-\alpha_\ell)}{2^{|k-\ell|/20}\times C_1}, & \text{if } k\neq\ell.\end{cases}$$

Proof. Let $\mathcal{X} = \{(k,\ell) \in \mathbb{Z}_{\geqslant 0}^2 : \alpha_k, \beta_\ell > 0\}$. Note that if $(k,\ell) \in \mathbb{Z}_{\geqslant 0}^2 \setminus \mathcal{X}$, then $\mu(\mathcal{E}_{p^k,p^\ell}) \leqslant \mu(\mathcal{V}_{p^k})\mu(\mathcal{W}_{p^\ell}) = 0$. Thus $\sum_{(k,\ell)\in\mathcal{X}} \mu(\mathcal{E}_{p^k,p^\ell}) = \mu(\mathcal{E})$. Hence, if we assume that the inequality in the statement of the lemma does not hold for any pair $(k,\ell) \in \mathcal{X}$, we must have

$$1 = \sum_{(k,\ell)\in\mathcal{X}} \frac{\mu(\mathcal{E}_{p^k,p^\ell})}{\mu(\mathcal{E})} < S_1 + S_2,$$

where

$$S_1 := \sum_{k=0}^{\infty} (\alpha_k\beta_k)^{(1+\tau)/(2+\tau)}$$

and

$$S_2 := \sum_{\substack{k,\ell\geqslant 0\\ k\neq\ell}}\sum \frac{\alpha_k(1-\beta_k)+\beta_k(1-\alpha_k)+\alpha_\ell(1-\beta_\ell)+\beta_\ell(1-\alpha_\ell)}{2^{|k-\ell|/20}\times C_1}.$$

Thus, to arrive at a contradiction, it suffices to show that

$$S_1 + S_2 \leqslant 1.$$

First of all, note that $\sum_{|j|\geqslant 1} 2^{-|j|/20} = 2/(2^{1/20}-1) \leqslant 100$ and recall from (8.1) that $C_1 = 10^4/\tau$, whence

$$\begin{aligned} S_2 &\leqslant \frac{\tau}{100}\bigg(\sum_{k=0}^{\infty}\alpha_k(1-\beta_k) + \sum_{k=0}^{\infty}\beta_k(1-\alpha_k) + \sum_{\ell=0}^{\infty}\alpha_\ell(1-\beta_\ell) + \sum_{\ell=0}^{\infty}\beta_\ell(1-\alpha_\ell)\bigg) \\ &= \frac{\tau}{50}\bigg(\sum_{k=0}^{\infty}\alpha_k(1-\beta_k) + \sum_{\ell=0}^{\infty}\beta_\ell(1-\alpha_\ell)\bigg). \end{aligned}$$

Observing that

$$1-\beta_k = \sum_{\ell\geqslant 0,\ \ell\neq k}\beta_\ell \quad\text{and}\quad 1-\alpha_\ell = \sum_{k\geqslant 0,\ k\neq \ell}\alpha_k,$$

we conclude that

$$S_2 \leqslant \frac{\tau}{25}\sum_{\substack{k,\ell\geqslant 0\\ k\neq\ell}}\sum \alpha_k\beta_\ell = \frac{\tau}{25}\bigg(1-\sum_{k=0}^{\infty}\alpha_k\beta_k\bigg).$$

Let us now study S_1. Since α_k, β_ℓ are non-negative reals which sum to 1, there exists some $k_0 \geqslant 0$ such that

$$\gamma := \max_{k\geqslant 0}\alpha_k\beta_k = \alpha_{k_0}\beta_{k_0}.$$

We thus find that

$$S_1 = \sum_{k=0}^{\infty}(\alpha_k\beta_k)^{\frac{1+\tau}{2+\tau}} \leqslant \gamma^{\frac{\tau}{2(2+\tau)}}\sum_{k=0}^{\infty}(\alpha_k\beta_k)^{\frac12} \leqslant \gamma^{\frac{\tau}{2(2+\tau)}}\bigg(\sum_{k=0}^{\infty}\alpha_k\bigg)^{\frac12}\bigg(\sum_{\ell=0}^{\infty}\beta_\ell\bigg)^{\frac12} = \gamma^{\frac{\tau}{2(2+\tau)}}$$

where we used the Cauchy–Schwarz inequality to bound $\sum_k(\alpha_k\beta_k)^{1/2}$ from above. We also find that

$$S_2 \leqslant \frac{25}{\tau}\bigg(1-\sum_{k=0}^{\infty}\alpha_k\beta_k\bigg) \leqslant \frac{\tau}{25}(1-\gamma).$$

As a consequence,

$$S_1 + S_2 \leqslant \gamma^{\frac{\tau}{2(2+\tau)}} + \frac{\tau}{25}(1-\gamma).$$

The function $x \mapsto x^{\frac{\tau}{2(2+\tau)}} + \frac{\tau}{25}(1-x)$ is increasing for $0 \leqslant x \leqslant 1$, and so maximized at $x = 1$. Thus we infer that $S_1 + S_2 \leqslant 1$ as required, completing the proof of the lemma. □

Lemma 13.2 (Quality increment unless a prime power divides almost all)**.** *Consider a GCD graph $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ with edge density $\delta > 0$ and a prime $p \in \mathcal{R}(G)$ with $p > C_2$. Then one of the following holds:*

(a) There is a GCD subgraph G' of G with multiplicative data $(\mathcal{P}', f', g')$ and edge density $\delta' > 0$ such that:
- *(i) G' is $(\mathcal{P}', f', g')$-maximal;*
- *(ii) $\mathcal{P}' = \mathcal{P} \cup \{p\}$;*
- *(iii) $\mathcal{R}(G') \subseteq \mathcal{R}(G) \setminus \{p\}$;*
- *(iv) $q(G') \geqslant M^{\mathbb{1}_{f'(p) \neq g'(p)}} \cdot q(G)$;*
- *(v) $p^{f'(p)} \| v$ and $p^{g'(p)} \| w$ for all $(v, w) \in \mathcal{V} \times \mathcal{W}$.*

(b) There is some $k \in \mathbb{Z}_{\geqslant 0}$ such that

$$\frac{\mu(\mathcal{V}_{p^k})}{\mu(\mathcal{V})} \geqslant 1 - \frac{C_2}{p} \quad \textit{and} \quad \frac{\mu(\mathcal{W}_{p^k})}{\mu(\mathcal{W})} \geqslant 1 - \frac{C_2}{p}.$$

Proof. Let α_k and β_ℓ be defined as in the statement of Lemma 13.1. Consequently, there are $k, \ell \in \mathbb{Z}_{\geqslant 0}$ such that $\alpha_k, \beta_\ell > 0$ and

$$\frac{\mu(\mathcal{E}_{p^k,p^\ell})}{\mu(\mathcal{E})} \geqslant \begin{cases} (\alpha_k\beta_k)^{(1+\tau)/(2+\tau)}, & \text{if } k = \ell, \\ \dfrac{\alpha_k(1-\beta_k) + \beta_k(1-\alpha_k) + \alpha_\ell(1-\beta_\ell) + \beta_\ell(1-\alpha_\ell)}{2^{|k-\ell|/20} \times C_1}, & \text{if } k \neq \ell \end{cases} \tag{13.1}$$

In particular, $\mu(\mathcal{E}_{p^k,p^\ell}) > 0$, so that G_{p^k,p^ℓ} is a non-trivial GCD subgraph of G.

Let $(\mathcal{P}', f', g')$ be the multiplicative data of G_{p^k,p^ℓ}, and let G' be a $(\mathcal{P}', f', g')$-maximal GCD subgraph of G_{p^k,p^ℓ}. We claim that either $q(G') \geqslant M^{\mathbb{1}_{k\neq\ell}} q(G)$, so that G' satisfies conclusion (a) of the lemma, or that conclusion (b) holds.

We separate two cases, according to whether $k = \ell$ or not.

Case 1: $k = \ell$. Lemma 11.1 and our lower bound $\mu(\mathcal{E}_{p^k,p^k})/\mu(\mathcal{E}) \geqslant (\alpha_k\beta_k)^{(1+\tau)/(2+\tau)}$ imply that

$$\frac{q(G')}{q(G)} \geqslant \frac{q(G_{p^k,p^\ell})}{q(G)} = \left(\frac{\mu(\mathcal{E}_{p^k,p^k})}{\mu(\mathcal{E})}\right)^{2+\tau} \cdot \frac{1}{(\alpha_k\beta_k)^{1+\tau}} \cdot \frac{1}{(1 - \mathbb{1}_{k\geqslant 1}/p)^2(1 - 1/p^{1+\tau/4})^3} \geqslant 1.$$

This establishes conclusion (a) in this case, since $f'(p) = g'(p) = k$ and thus $\mathbb{1}_{f'(p)\neq g'(p)} = 0$.

Case 2: $k \neq \ell$. As before, we use Lemma 11.1 and our lower bound on $\mu(\mathcal{E}_{p^k,p^\ell})$ to find that

$$\begin{aligned} \frac{q(G')}{q(G)} &\geqslant \left(\frac{\mu(\mathcal{E}_{p^k,p^\ell})}{\mu(\mathcal{E})}\right)^{2+\tau} (\alpha_k\beta_\ell)^{-1-\tau} \frac{p^{|k-\ell|}}{(1 - 1/p^{1+\tau/4})^3} \\ &\geqslant \frac{S^{2+\tau}}{C_1^{2+\tau}(\alpha_k\beta_\ell)^{1+\tau}} \cdot \left(\frac{p}{2^{\frac{2+\tau}{20}}}\right)^{|k-\ell|}, \end{aligned}$$

where

$$S = \alpha_k(1-\beta_k) + \beta_k(1-\alpha_k) + \alpha_\ell(1-\beta_\ell) + \beta_\ell(1-\alpha_\ell).$$

Note that

$$S \geqslant \alpha_k(1-\beta_k) \geqslant \alpha_k\beta_\ell. \tag{13.2}$$

Indeed, this follows by our assumption that $k \neq \ell$, which implies that $\beta_k + \beta_\ell \leqslant \sum_{j\geqslant 0}\beta_j = 1$. Combining the above, we conclude that

$$\frac{q(G')}{q(G)} \geqslant \frac{S^2}{\alpha_k\beta_\ell}\cdot\frac{1}{C_1^{2+\tau}}\cdot\left(\frac{p}{2^{\frac{2+\tau}{20}}}\right)^{|k-\ell|}. \tag{13.3}$$

If $q(G') \geqslant M\cdot q(G)$, we are done. So let us assume that $q(G') < M\cdot q(G)$. Since $|k-\ell|\geqslant 1$, we must then have that

$$S \leqslant \frac{S^2}{\alpha_k\beta_\ell} \leqslant M\cdot C_1^{2+\tau}\cdot\left(\frac{2}{p}\right)^{|k-\ell|} \leqslant \frac{2MC_1^{2.01}}{p} \leqslant \frac{C_2}{10p} \leqslant \frac{1}{10},$$

where we used our assumption that $p \geqslant C_2 = 10MC_1^3$ for the second-to-last inequality (recall (8.1)). In particular, this gives

$$S \leqslant \frac{C_2}{10p} \quad\text{and}\quad \frac{S^2}{\alpha_k\beta_\ell} \leqslant \frac{1}{10}. \tag{13.4}$$

We note that

$$S \geqslant \alpha_k(1-\beta_k) + \beta_\ell(1-\alpha_\ell) \geqslant (\alpha_k+\beta_\ell)(1-\max\{\alpha_\ell,\beta_k\}). \tag{13.5}$$

Thus by the arithmetic-geometric mean inequality, and relations (13.5) and (13.4), we have

$$(1-\max\{\alpha_\ell,\beta_k\})^2 \leqslant \frac{(\alpha_k+\beta_\ell)^2}{4\alpha_k\beta_\ell}(1-\max\{\alpha_\ell,\beta_k\})^2 \leqslant \frac{S^2}{4\alpha_k\beta_\ell} \leqslant \frac{1}{40}.$$

In particular, $\max\{\alpha_\ell,\beta_k\}\geqslant 1/2$.

We consider the case when $\beta_k \geqslant 1/2$; the case with $\alpha_\ell \geqslant 1/2$ is entirely analogous with the roles of β and α swapped, and the roles of k and ℓ swapped. Thus, to complete the proof of the lemma, it suffices to show that

$$\alpha_k, \beta_k \geqslant 1 - \frac{C_2}{p}. \tag{13.6}$$

The first inequality of (13.4) states that

$$\alpha_k(1-\beta_k) + \beta_k(1-\alpha_k) + \alpha_\ell(1-\beta_\ell) + \beta_\ell(1-\alpha_\ell) \leqslant \frac{C_2}{10p}.$$

Since $\beta_k \geqslant 1/2$ and $p \geqslant C_2$, we infer that

$$1-\alpha_k \leqslant 2\beta_k(1-\alpha_k) \leqslant \frac{2C_2}{10p} \leqslant \frac{1}{5}.$$

In particular, $\alpha_k \geqslant 1 - C_2/p$ and $\alpha_k \geqslant 1/2$, whence

$$1-\beta_k \leqslant 2\alpha_k(1-\beta_k) \leqslant \frac{2C_2}{10p} \leqslant \frac{C_2}{p}.$$

This completes the proof of (13.6) and hence of the lemma. □

Proof of Proposition 8.1. This follows almost immediately from Lemma 13.2. Since $\mathcal{R}(G) \subseteq \{p > C_6\}$ by assumption, if $p\in\mathcal{R}(G)$ then $p > C_6$. We have also assumed that $\mathcal{R}^\flat(G) \neq \emptyset$. Consequently, there is a prime $p \in \mathcal{R}^\flat(G)$ with $p > C_6 > C_2$. We then have two cases:

Case 1: there exists an integer $k \geqslant 0$ such that

$$\frac{\mu(\mathcal{V}_{p^k})}{\mu(\mathcal{V})} \geqslant 1 - \frac{C_2}{p} \quad\text{and}\quad \frac{\mu(\mathcal{W}_{p^k})}{\mu(\mathcal{W})} \geqslant 1 - \frac{C_2}{p}.$$

Since $p \notin \mathcal{R}^\sharp(G)$, there must exist some a pair of distinct integers $a, b \geqslant 0$ such that $q(G_{p^a,p^b}) \geqslant Mq(G)$. We then take G' to be a $(\mathcal{P}\cup\{p\}, f_{p^a}, g_{p^b})$-maximal GCD subgraph of G_{p^a,p^b}. To complete the proof of the proposition, note that the condition $\delta > 0$ implies that $q(G) > 0$. Therefore $q(G') \geqslant q(G_{p^a,p^b}) \geqslant Mq(G) > 0$, which also means that $\delta' > 0$.

Case 2: for every $k \in \mathbb{Z}_{\geqslant 0}$, we have

$$\min\left\{\frac{\mu(\mathcal{V}_{p^k})}{\mu(\mathcal{V})}, \frac{\mu(\mathcal{W}_{p^k})}{\mu(\mathcal{W})}\right\} < 1 - \frac{C_2}{p}. \tag{13.7}$$

We may then apply Lemma 13.2 with this choice of p. By (13.7), conclusion (b) cannot hold, and so conclusion (a) must hold. This then gives the result. □

14. Proof of Proposition 8.3

In this section we prove Proposition 8.3.

Lemma 14.1 (Small quality loss or prime power divides positive proportion)**.** *Consider a GCD graph $G = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}, \mathcal{P}, f, g)$ with edge density $\delta > 0$, and let $p \in \mathcal{R}(G)$ be a prime. Then one of the following holds:*

(a) There is a GCD subgraph G' of G with multiplicative data $(\mathcal{P}', f', g')$ and edge density $\delta' > 0$ such that

$$\mathcal{P}' = \mathcal{P} \cup \{p\}, \quad \mathcal{R}(G') \subseteq \mathcal{R}(G) \setminus \{p\}, \quad q(G') \geqslant q(G)/C_3.$$

(b) There is some $k \in \mathbb{Z}_{\geqslant 0}$ such that

$$\frac{\mu(\mathcal{V}_{p^k})}{\mu(\mathcal{V})} \geqslant \frac{9}{10} \quad \textit{and} \quad \frac{\mu(\mathcal{W}_{p^k})}{\mu(\mathcal{W})} \geqslant \frac{9}{10}.$$

Proof. Assume that conclusion (a) does not hold, so we intend to establish (b). For $k, \ell \in \mathbb{Z}_{\geqslant 0}$, let $\mu(\mathcal{V}_{p^k}) = \alpha_k \mu(\mathcal{V})$ and $\mu(\mathcal{W}_{p^\ell}) = \beta_\ell \mu(\mathcal{W})$. We begin as in the proof of Lemma 13.2, by considering $k, \ell \in \mathbb{Z}_{\geqslant 0}$ satisfying (13.1) and the inequalities $\alpha_k, \beta_\ell > 0$. We do not need to worry about the maximality of G' here, so we shall simply take $G' = G_{p^k,p^\ell}$. In particular, G' is a non-trivial GCD subgraph of G.

We note that the proof of Lemma 13.2 up to relation (13.3) requires no assumption on the size of p. Now, if $k = \ell$, then Case 1 of the proof of Lemma 13.2 shows that conclusion (a) must hold, contradicting our assumption. Therefore we may assume that $k \neq \ell$. Now, arguing as in Case 2 of the proof of Lemma 13.2, and setting

$$S = \alpha_k(1-\beta_k) + \beta_k(1-\alpha_k) + \alpha_\ell(1-\beta_\ell) + \beta_\ell(1-\alpha_\ell),$$

we infer that

$$\frac{1}{C_3} \geqslant \frac{q(G')}{q(G)} \geqslant \frac{S^2}{\alpha_k\beta_\ell} \cdot \frac{1}{C_1^{2+\tau}} \cdot \left(\frac{p}{2^{\frac{2+\tau}{20}}}\right)^{|k-\ell|} \geqslant \frac{S^2}{\alpha_k\beta_\ell} \cdot \frac{1}{C_1^3}.$$

Therefore we have that

$$S \leqslant \frac{S^2}{\alpha_k\beta_\ell} \leqslant \frac{C_1^3}{C_3} \leqslant \frac{1}{10^3}.$$

Since $S \geqslant (\alpha_k + \beta_\ell)(1 - \max\{\alpha_\ell, \beta_k\})$, we have

$$(1 - \max\{\alpha_\ell, \beta_k\})^2 \leqslant \frac{(\alpha_k + \beta_\ell)^2}{4\alpha_k\beta_\ell}(1 - \max\{\alpha_\ell, \beta_k\})^2 \leqslant \frac{S^2}{4\alpha_k\beta_\ell} \leqslant \frac{1}{100},$$

so $\max\{\alpha_\ell,\beta_k\}\geqslant 9/10$. We deal with the case when $\beta_k\geqslant 9/10$; the case with $\alpha_\ell\geqslant 9/10$ is entirely analogous with the roles of k and ℓ and the roles of α and β swapped.

Since $\beta_k\geqslant 9/10$, we have

$$1-\alpha_k\leqslant 2\beta_k(1-\alpha_k)\leqslant 2S\leqslant\frac{2}{10^3}$$

In particular, $\alpha_k\geqslant 9/10$ and so conclusion (b) holds, as required. □

Lemma 14.2 (Adding small primes to $\mathcal{P}$)**.** *Let $G=(\mu,\mathcal{V},\mathcal{W},\mathcal{E},\mathcal{P},f,g)$ be a GCD graph with edge density $\delta>0$. Let $p\in\mathcal{R}(G)$ be a prime.*

Then there is a GCD subgraph G' of G with set of primes $\mathcal{P}'$ and edge density $\delta'>0$ such that

$$\mathcal{P}'=\mathcal{P}\cup\{p\},\quad \mathcal{R}(G')\subseteq\mathcal{R}(G)\setminus\{p\},\quad q(G')\geqslant q(G)/C_5.$$

Proof. We first repeatedly apply Lemma 10.1 until we arrive at a GCD subgraph

$$G^{(1)}=(\mu,\mathcal{V}^{(1)},\mathcal{W}^{(1)},\mathcal{E}^{(1)},\mathcal{P},f,g)$$

of G with edge density $\delta^{(1)}$ such that

$$q(G^{(1)})\geqslant q(G),$$

as well as

$$\mu(\Gamma_{G^{(1)}}(v))\geqslant\frac{1+\tau}{2+\tau}\cdot\delta^{(1)}\mu(\mathcal{W}^{(1)})\qquad\text{for all } v\in\mathcal{V}^{(1)}.$$

(We must eventually arrive at such a subgraph since the vertex sets are strictly decreasing at each stage but can never become empty since the edge density remains bounded away from 0.)

We now apply Lemma 14.1 to $G^{(1)}$. If conclusion (a) of Lemma 14.1 holds, then there is a GCD subgraph $G^{(2)}$ of $G^{(1)}$ satisfying the conclusion of Lemma 14.2, so we are done by taking $G'=G^{(2)}$. Therefore we may assume that instead conclusion (b) of Lemma 14.1 holds, so there is some $k\in\mathbb{Z}_{\geqslant 0}$ such that

$$\frac{\mu(\mathcal{V}^{(1)}_{p^k})}{\mu(\mathcal{V}^{(1)})}\geqslant\frac{9}{10}\quad\text{and}\quad\frac{\mu(\mathcal{W}^{(1)}_{p^k})}{\mu(\mathcal{W}^{(1)})}\geqslant\frac{9}{10}.\tag{14.1}$$

In fact we claim that either the conclusion of Lemma 14.2 holds, or we have the stronger condition

$$\frac{\mu(\mathcal{V}^{(1)}_{p^k})}{\mu(\mathcal{V}^{(1)})}\geqslant\max\Big(\frac{9}{10},1-\frac{C_2}{p}\Big)\quad\text{and}\quad\frac{\mu(\mathcal{W}^{(1)}_{p^k})}{\mu(\mathcal{W}^{(1)})}\geqslant\max\Big(\frac{9}{10},1-\frac{C_2}{p}\Big).\tag{14.2}$$

Relation (14.2) follows immediately from (14.1) if $p\leqslant 10C_2$, so let us assume that $p>10C_2$. We then apply Lemma 13.2 to $G^{(1)}$. If conclusion (a) of Lemma 13.2 holds, then there is a GCD subgraph $G^{(3)}$ of $G^{(1)}$ satisfying the required conditions of Lemma 14.2, so we are done by taking $G'=G^{(3)}$. Therefore we may assume that conclusion (b) of Lemma 13.2 holds, so that there is some $k'\geqslant 0$ such that $\mu(\mathcal{V}^{(1)}_{p^{k'}})/\mu(\mathcal{V}^{(1)})\geqslant 1-C_2/p\geqslant 9/10$ and $\mu(\mathcal{W}^{(1)}_{p^{k'}})/\mu(\mathcal{W}^{(1)})\geqslant 1-C_2/p\geqslant 9/10$. Since there cannot be two disjoint subsets of $\mathcal{V}^{(1)}$ of density $\geqslant 9/10$, we must then have $k'=k$, thus proving (14.2) in this case too.

In conclusion, regardless of the size of p we have established (14.2). Next, we fix an integer $r\leqslant r_0$ such that $p^r>C_4$, where r_0 is the smallest integer such that $2^{r_0}>C_4$, namely $r_0=\lfloor\frac{\log C_4}{\log 2}\rfloor+1$ and we apply Lemma 11.3.

If conclusion (a) of Lemma 11.3 holds, then we take $G'=G^{(1)}_{p^k,p^\ell}$, whose quality satisfies $q(G')\geqslant Mq(G^{(1)})\geqslant Mq(G)>0$. In particular, $\delta'>0$, so the proof is complete in this case.

Thus we may assume that conclusion (b) of Lemma 11.3 holds, so that

$$\sum_{\ell\in\mathcal{L}_{k,r}} \mu(\mathcal{E}^{(1)}_{p^k,p^\ell}) \leqslant \frac{\mu(\mathcal{E}^{(1)})}{4p^{1+\tau/4}} < \frac{\mu(\mathcal{E}^{(1)})}{4},$$

where we recall the notation $\mathcal{L}_{k,r} = \{\ell \in \mathbb{Z}_{\geqslant 0} : |\ell - k| \geqslant r+1\}$. Let

$$\widetilde{\mathcal{W}}^{(1)} = \bigcup_{\ell\geqslant 0\,:\,|\ell-k|\leqslant r} \mathcal{W}^{(1)}_{p^\ell}$$

and let

$$\mathcal{E}^{(2)} = \mathcal{E}^{(1)} \cap (\mathcal{V}^{(1)}_{p^k} \times \widetilde{\mathcal{W}}^{(1)}) \subseteq \mathcal{E}^{(1)}$$

be the set of edges between $\mathcal{V}^{(1)}_{p^k}$ and $\widetilde{\mathcal{W}}^{(1)}$ in $G^{(1)}$. Since $\mu(\mathcal{V}^{(1)}_{p^k}) \geqslant 9\mu(\mathcal{V}^{(1)})/10$ and $\mu(\Gamma_{G^{(1)}}(v)) \geqslant \frac{1+\tau}{2+\tau}\cdot\delta^{(1)}\mu(\mathcal{W}^{(1)})$ for all $v \in \mathcal{V}^{(1)}_{p^k}$, we have

$$\begin{aligned}\mu(\mathcal{E}^{(2)}) \geqslant \mu\big(\mathcal{E}^{(1)} \cap (\mathcal{V}^{(1)}_{p^k} \times \mathcal{W}^{(1)})\big) - \sum_{\ell\in\mathcal{L}_{k,r}} \mu(\mathcal{E}^{(1)}_{p^k,p^\ell}) &\geqslant \sum_{v\in\mathcal{V}^{(1)}_{p^k}} \mu(v)\mu(\Gamma_{G^{(1)}}(v)) - \frac{\mu(\mathcal{E}^{(1)})}{4}\\ &\geqslant \frac{1+\tau}{2+\tau}\cdot\delta^{(1)}\mu(\mathcal{V}^{(1)}_{p^k})\mu(\mathcal{W}^{(1)}) - \frac{\mu(\mathcal{E}^{(1)})}{4}\\ &\geqslant \frac{13\tau+8}{20(\tau+2)}\cdot\mu(\mathcal{E}^{(1)}) \geqslant \frac{\mu(\mathcal{E}^{(1)})}{5} > 0.\end{aligned}$$

Let $G^{(2)} = (\mu, \mathcal{V}^{(1)}_{p^k}, \widetilde{\mathcal{W}}^{(1)}, \mathcal{E}^{(2)}, \mathcal{P}, f, g)$ be the GCD subgraph of $G^{(1)}$ formed by restricting to $\mathcal{V}^{(1)}_{p^k}$ and $\widetilde{\mathcal{W}}^{(1)}$. Since $\mu(\mathcal{E}^{(2)}) > 0$, $G^{(2)}$ is a non-trivial GCD subgraph. In addition, we have that

$$\frac{q(G^{(2)})}{q(G^{(1)})} = \left(\frac{\mu(\mathcal{E}^{(2)})}{\mu(\mathcal{E}^{(1)})}\right)^{2+\tau}\left(\frac{\mu(\mathcal{V}^{(1)})}{\mu(\mathcal{V}^{(1)}_{p^k})}\right)^{1+\tau}\left(\frac{\mu(\mathcal{W}^{(1)})}{\mu(\widetilde{\mathcal{W}}^{(1)})}\right)^{1+\tau} \geqslant \left(\frac{1}{5}\right)^{2+\tau}\cdot 1\cdot 1 \geqslant \frac{1}{5^3}.$$

Finally, we apply Lemma 11.2 to the partition

$$\widetilde{\mathcal{W}}^{(1)} = \bigsqcup_{\ell\geqslant 0\,:\,|\ell-k|\leqslant r} \mathcal{W}^{(1)}_{p^\ell}$$

of $\widetilde{\mathcal{W}}^{(1)}$ into $\leqslant 2\cdot r+1 \leqslant 10\log C_4$ subsets. This produces a GCD subgraph

$$G^{(3)} = (\mu, \mathcal{V}^{(1)}_{p^k}, \mathcal{W}^{(1)}_{p^\ell}, \mathcal{E}^{(1)}_{p^k,p^\ell}, \mathcal{P}, f, g)$$

of $G^{(2)}$ for some $\ell \geqslant 0$ with $|\ell - k| \leqslant r$ such that

$$q(G^{(3)}) \geqslant \frac{q(G^{(2)})}{\big(10\log C_4\big)^{2+\tau}} \geqslant \frac{q(G^{(1)})}{\big(10\log C_4\big)^3\cdot 5^3} \geqslant \frac{q(G)}{\big(50\log C_4\big)^3}.$$

Finally, we note that $G^{(1)}_{p^k,p^\ell}$ is a GCD subgraph of $G^{(3)}$ with set of primes $\mathcal{P}\cup\{p\}$, edge density $\delta^{(1)}_{p^k,p^\ell} = \delta^{(3)}$, and quality $q(G^{(1)}_{p^k,p^\ell}) \geqslant q(G^{(3)})$. Taking $G' = G^{(1)}_{p^k,p^\ell}$ then gives the result on recalling the definition (8.1) of C_5. □

Proof of Proposition 8.3. If $\mathcal{R}(G)\cap\{p\leqslant C_6\}=\emptyset$, then we can simply take $G'=G$.

If $\mathcal{R}(G)\cap\{p\leqslant C_6\}\neq\emptyset$, then we can choose a prime $p\in\mathcal{R}(G)\cap\{p\leqslant C_6\}$ and apply Lemma 14.2. We do this repeatedly to produce a sequence of GCD subgraphs

$$G=:G_1\succeq G_2\succeq\cdots$$

such that

$$q(G_{i+1})\geqslant q(G_i)/C_5. \tag{14.3}$$

for each i. In addition, we let $\mathcal{P}_i$ denote the set of primes associated to G_i, so that $\mathcal{P}=\mathcal{P}_1\subseteq\mathcal{P}_2\subseteq\cdots\subseteq\mathcal{P}\cup\big(\mathcal{R}(G)\cap\{p\leqslant C_6\}\big)$.

At each stage, the set $\mathcal{R}(G_i)\cap\{p\leqslant C_6\}$ is strictly smaller than before. So, after at most C_6 steps we arrive at a GCD subgraph $G^{(1)}=(\mu,\mathcal{V}^{(1)},\mathcal{W}^{(1)},\mathcal{E}^{(1)},\mathcal{P}^{(1)},f^{(1)},g^{(1)})$ of G with

$$\mathcal{P}^{(1)}\subseteq\mathcal{P}\cup\big(\mathcal{R}(G)\cap\{p\leqslant C_6\}\big)\quad\text{and}\quad\mathcal{R}(G^{(1)})\cap\{p\leqslant C_6\}=\emptyset.$$

Iterating (14.3) at most C_6 times, we find that $q(G^{(1)})\geqslant q(G)/C_5^{C_6}=q(G)/C_7$. Thus, taking $G'=G^{(1)}$ gives the result. □

15. Proof of Proposition 8.2

In this section we prove Proposition 8.2.

Lemma 15.1 (Quality increment even when a prime power divides almost all)**.** *Consider a GCD graph $G=(\mu,\mathcal{V},\mathcal{W},\mathcal{E},\mathcal{P},f,g)$ with edge density $\delta>0$ and let $p\in\mathcal{R}(G)$ be a prime with $p\geqslant C_6$. Then there is a GCD subgraph G' of G with set of primes $\mathcal{P}'=\mathcal{P}\cup\{p\}$ such that*

$$\mathcal{R}(G')\subseteq\mathcal{R}(G)\setminus\{p\}\quad\textit{and}\quad q(G')\geqslant q(G)>0.$$

Proof. First of all, we may assume without loss of generality that for all sets $\mathcal{A}\subseteq\mathcal{V}$ and $\mathcal{B}\subseteq\mathcal{W}$, we have that

$$\mu(\mathcal{E}\cap(\mathcal{A}\times\mathcal{B}))\leqslant\frac{\mu(\mathcal{E})}{p^{1+\frac{\tau}{3}}}\quad\text{whenever}\quad\max\left\{\frac{\mu(\mathcal{A})}{\mu(\mathcal{V})},\frac{\mu(\mathcal{B})}{\mu(\mathcal{W})}\right\}\leqslant\frac{C_2}{p}. \tag{15.1}$$

Indeed, if G does not satisfy (15.1), then we apply Lemma 11.6 with $\eta=C_2/p$ to replace G by a non-trivial subgraph $G^{(1)}$ that does have this property (noticing that $(C_2/p)^{(2+2\tau)/(2+\tau)}\leqslant 1/(p^{1+\tau/3})$ for $p\geqslant C_6$ from the definition (8.1)). In addition, $G^{(1)}$ has the same multiplicative data as G and its quality is strictly larger. Hence, we may work with $G^{(1)}$ instead. So, from now on, we assume that (15.1) holds.

We now apply Lemma 13.2. If conclusion (a) of Lemma 13.2 holds, then we are done. Thus we may assume that conclusion (b) holds, that is to say there is some $k\in\mathbb{Z}_{\geqslant 0}$ such that

$$\frac{\mu(\mathcal{V}_{p^k})}{\mu(\mathcal{V})}\geqslant 1-\frac{C_2}{p}\quad\text{and}\quad\frac{\mu(\mathcal{W}_{p^k})}{\mu(\mathcal{W})}\geqslant 1-\frac{C_2}{p}.$$

In particular, by (15.1) we see that

$$\mu\big(\mathcal{E}(\mathcal{V}\setminus\mathcal{V}_{p^k},\mathcal{W}\setminus\mathcal{W}_{p^k})\big)\leqslant\frac{\mu(\mathcal{E})}{p^{1+\frac{\tau}{3}}}. \tag{15.2}$$

Now, set

$$\widetilde{\mathcal{V}}_{p^k}=\mathcal{V}_{p^{k-1}}\cup\mathcal{V}_{p^k}\cup\mathcal{V}_{p^{k+1}}\quad\text{and}\quad\widetilde{\mathcal{W}}_{p^k}=\mathcal{W}_{p^{k-1}}\cup\mathcal{W}_{p^k}\cup\mathcal{W}_{p^{k+1}},$$

with the convention that $\mathcal{V}_{p^{-1}} = \emptyset = \mathcal{W}_{p^{-1}}$. In view of Lemmas 11.3 and 11.4 applied with $r = 1$, we may assume that

$$\mu\big(\mathcal{E}(\mathcal{V} \setminus \widetilde{\mathcal{V}}_{p^k}, \mathcal{W}_{p^k})\big) = \sum_{\substack{i\geqslant 0 \\ |i-k|\geqslant 2}} \mu(\mathcal{E}(\mathcal{V}_{p^i}, \mathcal{W}_{p^k})) \leqslant \frac{\mu(\mathcal{E})}{4p^{1+\frac{\tau}{4}}}, \tag{15.3}$$

and

$$\mu\big(\mathcal{E}(\mathcal{V}_{p^k}, \mathcal{W} \setminus \widetilde{\mathcal{W}}_{p^k})\big) = \sum_{\substack{j\geqslant 0 \\ |j-k|\geqslant 2}} \mu(\mathcal{E}(\mathcal{V}_{p^k}, \mathcal{W}_{p^j})) \leqslant \frac{\mu(\mathcal{E})}{4p^{1+\frac{\tau}{4}}}. \tag{15.4}$$

Hence, if we let

$$\mathcal{E}^* = \mathcal{E}(\mathcal{V}_{p^k}, \widetilde{\mathcal{W}}_{p^k}) \cup \mathcal{E}(\widetilde{\mathcal{V}}_{p^k}, \mathcal{W}_{p^k}),$$

then (15.2)-(15.4) imply that

$$\frac{\mu(\mathcal{E}^*)}{\mu(\mathcal{E})} \geqslant 1 - \frac{1}{2p^{1+\frac{\tau}{4}}} - \frac{1}{p^{1+\frac{\tau}{3}}} \geqslant \Big(1 - \frac{1}{p^{1+\frac{\tau}{4}}}\Big)^{\frac{2}{3}} > 0,$$

where we used our assumption that $p \geqslant C_6$ and the inequality $(1-x)^{2/3} \leqslant 1 - 2x/3$ for $x \in [0, 1]$ that follows from Taylor's theorem. We then consider the non-trivial GCD subgraph $G^* = (\mu, \mathcal{V}, \mathcal{W}, \mathcal{E}^*, \mathcal{P}, f, g)$ of G formed by restricting the edge set to $\mathcal{E}^*$. Note that

$$\frac{q(G^*)}{q(G)} = \Big(\frac{\mu(\mathcal{E}^*)}{\mu(\mathcal{E})}\Big)^{2+\tau} \geqslant \Big(1 - \frac{1}{p^{1+\frac{\tau}{4}}}\Big)^{\frac{4+2\tau}{3}} \geqslant \Big(1 - \frac{1}{p^{1+\frac{\tau}{4}}}\Big)^{2}. \tag{15.5}$$

We set $G^+ = (\mu, \mathcal{V}^+, \mathcal{W}^+, \mathcal{E}^+, \mathcal{P} \cup \{p\}, f^+, g^+)$, where:

$$\mathcal{V}^+ = \mathcal{V}_{p^k} \cup \mathcal{V}_{p^{k+1}}, \quad \mathcal{W}^+ = \mathcal{W}_{p^k} \cup \mathcal{W}_{p^{k+1}}, \quad \mathcal{E}^+ = \mathcal{E}^* \cap (\mathcal{V}^+ \times \mathcal{W}^+),$$

as well as

$$f^+\big|_{\mathcal{P}} = f, \quad f^+(p) = k, \quad g^+\big|_{\mathcal{P}} = g, \quad g^+(p) = k.$$

It is easy to check that G^+ is a GCD subgraph of G^* (and hence of G). Note that $\mu(\mathcal{V}^+) \geqslant \mu(\mathcal{V}_{p^k}) \geqslant 1 - C_2/p > 0$. Similarly, we have $\mu(\mathcal{W}^+) > 0$. Consequently, its quality satisfies the relation

$$\frac{q(G^+)}{q(G^*)} = \Big(\frac{\mu(\mathcal{E}^+)}{\mu(\mathcal{E}^*)}\Big)^{2+\tau} \Big(\frac{\mu(\mathcal{V})}{\mu(\mathcal{V}^+)}\Big)^{1+\tau} \Big(\frac{\mu(\mathcal{W})}{\mu(\mathcal{W}^+)}\Big)^{1+\tau} \Big(1 - \frac{\mathbb{1}_{k\geqslant 1}}{p}\Big)^{-2} \Big(1 - \frac{1}{p^{1+\frac{\tau}{4}}}\Big)^{-3}.$$

(This relation is valid even if $\mu(\mathcal{E}^+) = 0$.) We separate two cases.

Case 1: $k = 0$.

In this case $\mathcal{V}_{p^{k-1}} = \mathcal{W}_{p^{k-1}} = \emptyset$, $\mathcal{E}^+ = \mathcal{E}^*$. And note that $\frac{\mu(\mathcal{V})}{\mu(\mathcal{V}^+)}, \frac{\mu(\mathcal{W})}{\mu(\mathcal{W}^+)} \geqslant 1$. As a consequence,

$$\frac{q(G^+)}{q(G^*)} \geqslant \Big(1 - \frac{1}{p^{1+\frac{\tau}{4}}}\Big)^{-3}.$$

Together with (15.5) this implies that

$$\frac{q(G^+)}{q(G)} \geqslant \Big(1 - \frac{1}{p^{1+\frac{\tau}{4}}}\Big)^{-1} \geqslant 1.$$

In particular, $q(G^+) > 0$. Thus the lemma follows by taking $G' = G^+$.

Case 2: $k \geqslant 1$.

In this case we have

$$\frac{q(G^+)}{q(G^*)} = \left(\frac{\mu(\mathcal{E}^+)}{\mu(\mathcal{E}^*)}\right)^{2+\tau}\left(\frac{\mu(\mathcal{V})}{\mu(\mathcal{V}^+)}\right)^{1+\tau}\left(\frac{\mu(\mathcal{W})}{\mu(\mathcal{W}^+)}\right)^{1+\tau}\left(1-\frac{1}{p}\right)^{-2}\left(1-\frac{1}{p^{1+\frac{\tau}{4}}}\right)^{-3}. \tag{15.6}$$

We also consider the GCD subgraphs $G_{p^k,p^{k-1}}$ and G_{p^{k-1},p^k} of G. Notice that $\mu(\mathcal{V}_{p^k}) \geqslant 1-C_2/p > 0$ for $p \geqslant C_6$. Hence, if $\mu(\mathcal{W}_{p^{k-1}}) > 0$, then Lemma 11.1 implies that

$$\frac{q(G_{p^k,p^{k-1}})}{q(G^*)} = \left(\frac{\mu(\mathcal{E}_{p^k,p^{k-1}})}{\mu(\mathcal{E}^*)}\right)^{2+\tau}\left(\frac{\mu(\mathcal{V})}{\mu(\mathcal{V}_{p^k})}\right)^{1+\tau}\left(\frac{\mu(\mathcal{W})}{\mu(\mathcal{W}_{p^{k-1}})}\right)^{1+\tau}\frac{p}{\left(1-\frac{1}{p^{1+\frac{\tau}{4}}}\right)^3}. \tag{15.7}$$

Similarly, if $\mu(\mathcal{V}_{p^{k-1}}) > 0$, then we have

$$\frac{q(G_{p^{k-1},p^k})}{q(G^*)} = \left(\frac{\mu(\mathcal{E}_{p^{k-1},p^k})}{\mu(\mathcal{E}^*)}\right)^{2+\tau}\left(\frac{\mu(\mathcal{V})}{\mu(\mathcal{V}_{p^{k-1}})}\right)^{1+\tau}\left(\frac{\mu(\mathcal{W})}{\mu(\mathcal{W}_{p^k})}\right)^{1+\tau}\frac{p}{\left(1-\frac{1}{p^{1+\frac{\tau}{4}}}\right)^3}. \tag{15.8}$$

Since $\mu(\mathcal{V}_{p^k}) \geqslant (1-C_2/p)\mu(\mathcal{V})$, we have that $\mu(\mathcal{V}_{p^{k-1}}) \leqslant C_2\mu(\mathcal{V})/p$. Similarly, we have that $\mu(\mathcal{W}_{p^{k-1}}) \leqslant C_2\mu(\mathcal{W})/p$. To this end, let $0 \leqslant A, B \leqslant C_2$ be such that

$$\frac{\mu(\mathcal{V}_{p^{k-1}})}{\mu(\mathcal{V})} = \frac{A}{p} \quad\text{and}\quad \frac{\mu(\mathcal{W}_{p^{k-1}})}{\mu(\mathcal{W})} = \frac{B}{p}. \tag{15.9}$$

We note that this implies that

$$\frac{\mu(\mathcal{V}^+)}{\mu(\mathcal{V})} \leqslant 1-\frac{A}{p} \quad\text{and}\quad \frac{\mu(\mathcal{W}^+)}{\mu(\mathcal{W})} \leqslant 1-\frac{B}{p}.$$

We also note that $\mu(\mathcal{E}_{p^k,p^{k-1}}) \leqslant \mu(\mathcal{V}_{p^k})\mu(\mathcal{W}_{p^{k-1}}) \leqslant B\mu(\mathcal{V})\mu(\mathcal{W})/p$, so if $\mu(\mathcal{E}_{p^k,p^{k-1}}) > 0$ then $B > 0$. Similarly if $\mu(\mathcal{E}_{p^{k-1},p^k}) > 0$ then $A > 0$.

Combining (15.6) and (15.9) with (15.5), we find

$$\frac{q(G^+)}{q(G)} \geqslant \left(\frac{\mu(\mathcal{E}^+)}{\mu(\mathcal{E}^*)}\right)^{2+\tau}\frac{1}{\left(1-\frac{A}{p}\right)^{1+\tau}\left(1-\frac{B}{p}\right)^{1+\tau}\left(1-\frac{1}{p}\right)^2\left(1-\frac{1}{p^{1+\frac{\tau}{4}}}\right)}. \tag{15.10}$$

Similarly, provided $B > 0$, (15.7), (15.9) and (15.5) give

$$\frac{q(G_{p^k,p^{k-1}})}{q(G)} \geqslant \left(\frac{\mu(\mathcal{E}_{p^k,p^{k-1}})}{\mu(\mathcal{E}^*)}\right)^{2+\tau}\frac{p^{2+\tau}}{B^{1+\tau}\left(1-\frac{1}{p^{1+\frac{\tau}{4}}}\right)}\ , \tag{15.11}$$

and, provided $A > 0$, (15.8), (15.9) and (15.5) give

$$\frac{q(G_{p^{k-1},p^k})}{q(G)} \geqslant \left(\frac{\mu(\mathcal{E}_{p^{k-1},p^k})}{\mu(\mathcal{E}^*)}\right)^{2+\tau}\frac{p^{2+\tau}}{A^{1+\tau}\left(1-\frac{1}{p^{1+\frac{\tau}{4}}}\right)}. \tag{15.12}$$

We now claim that at least one of the following inequalities holds:

$$\frac{\mu(\mathcal{E}^+)}{\mu(\mathcal{E}^*)} > \Big(1-\frac{A}{p}\Big)^{\frac{1+\tau}{2+\tau}}\Big(1-\frac{B}{p}\Big)^{\frac{1+\tau}{2+\tau}}\Big(1-\frac{1}{p}\Big)^{\frac{2}{2+\tau}}\Big(1-\frac{1}{p^{1+\frac{\tau}{4}}}\Big)^{\frac{1}{3}}; \tag{15.13}$$

$$\frac{\mu(\mathcal{E}_{p^k,p^{k-1}})}{\mu(\mathcal{E}^*)} > \frac{B^{\frac{1+\tau}{2+\tau}}}{p}\Big(1-\frac{1}{p^{1+\frac{\tau}{4}}}\Big)^{\frac{1}{3}}; \tag{15.14}$$

$$\frac{\mu(\mathcal{E}_{p^{k-1},p^k})}{\mu(\mathcal{E}^*)} > \frac{A^{\frac{1+\tau}{2+\tau}}}{p}\Big(1-\frac{1}{p^{1+\frac{\tau}{4}}}\Big)^{\frac{1}{3}}. \tag{15.15}$$

If (15.13) holds then $q(G^+) \geqslant q(G)$ by (15.10). If (15.14) holds, then $\mu(\mathcal{E}_{p^k,p^{k-1}}) > 0$, so $B > 0$, and so $q(G_{p^k,p^{k-1}}) \geqslant q(G)$ by (15.11) and (15.14). Finally, if (15.15) holds, then $\mu(\mathcal{E}_{p^{k-1},p^k}) > 0$, so $A > 0$, and so $q(G_{p^{k-1},p^k}) \geqslant q(G)$ by (15.12) and (15.15). Therefore this claim would complete the proof by choosing $G' \in \{G^+, G_{p^k,p^{k+1}}, G_{p^{k+1},p^k}\}$ according to which of the inequalities (15.13)-(15.15) hold.

Since $\mu(\mathcal{E}^+)+\mu(\mathcal{E}_{p^k,p^{k-1}})+\mu(\mathcal{E}_{p^{k-1},p^k}) = \mu(\mathcal{E}^*)$, at least one of (15.13)-(15.15) holds if we can prove that

$$S := \left[\Big(1-\frac{A}{p}\Big)^{\frac{1+\tau}{2+\tau}}\Big(1-\frac{B}{p}\Big)^{\frac{1+\tau}{2+\tau}}\Big(1-\frac{1}{p}\Big)^{\frac{2}{2+\tau}} + \frac{B^{\frac{1+\tau}{2+\tau}}}{p} + \frac{A^{\frac{1+\tau}{2+\tau}}}{p}\right]\Big(1-\frac{1}{p^{1+\frac{\tau}{4}}}\Big)^{\frac{1}{3}} < 1.$$

Using the inequality $1-x \leqslant e^{-x}$ three times, we find that

$$S \leqslant \left[\exp\Big(-\frac{(A+B)(1+\tau)+2}{p(2+\tau)}\Big) + \frac{B^{\frac{1+\tau}{2+\tau}}}{p} + \frac{A^{\frac{1+\tau}{2+\tau}}}{p}\right]\Big(1-\frac{1}{p^{1+\frac{\tau}{4}}}\Big)^{\frac{1}{3}}.$$

Since we also have that $e^{-x} \leqslant 1-x+x^2/2$ for $x \geqslant 0$, as well as $0 \leqslant A, B \leqslant C_2$ and $0 < \tau < 1/100$, we conclude that

$$S \leqslant \Big(1-\frac{(A+B)(1+\tau)+2}{p(2+\tau)} + \frac{C_2^2}{p^2} + \frac{B^{\frac{1+\tau}{2+\tau}}}{p} + \frac{A^{\frac{1+\tau}{2+\tau}}}{p}\Big)\Big(1-\frac{1}{p^{1+\frac{\tau}{4}}}\Big)^{\frac{1}{3}}.$$

Using the inequality $\frac{x(1+\tau)+1}{2+\tau} \geqslant x^{\frac{1+\tau}{2+\tau}}$ for $x \geqslant 0$ and $\tau > 0$, we obtain that

$$S \leqslant \Big(1+\frac{C_2^2}{p^2}\Big)\Big(1-\frac{1}{p^{1+\frac{\tau}{4}}}\Big)^{\frac{1}{3}}.$$

Note that $C_2^2/p^2 \leqslant 1/(3p^{1+\tau/4})$ for $p \geqslant C_6$ (recall (8.1)). Since $(1-x)^{1/3} \leqslant 1-x/3$ for $x \in [0,1]$, we must have that $S \leqslant 1-1/(9p^{2+\tau/2}) < 1$ for $p \geqslant C_6$, thus completing the proof of the lemma. □

Proof of Proposition 8.2. This follows almost immediately from Lemma 15.1. Our assumptions that $\mathcal{R}(G) \subseteq \{p > C_6\}$ and $\mathcal{R}^\sharp(G) \neq \emptyset$ imply that there is a prime $p > C_6$ lying in $\mathcal{R}(G)$. Thus we can apply Lemma 15.1 with this choice of p and complete the proof. □

16. Optimality of the exponent $1/2$ in Theorem 1

We shall prove the following result, which is sufficient to prove our claim that the exponent $1/2$ in Theorem 1 cannot be improved in full generality.

Proposition 16.1. *Let $\psi(q) = \mathbb{1}_{q \text{ prime}}/q$. Then*

$$\int_0^1 \big(N(\alpha;Q) - \Psi(Q)\big)^2 \,\mathrm{d}\alpha = \Psi(Q) + O(1).$$

Proof. Arguing as in the proof of Theorem 2 in Section 7, we have

$$\begin{aligned}\int_0^1 \big(N(\alpha;Q) - \Psi(Q)\big)^2 \,\mathrm{d}\alpha &= \sum\sum_{q,r\leqslant Q} \lambda(\mathcal{A}_q \cap \mathcal{A}_r) - \Psi(Q)^2 \\ &= \Psi(Q) + \sum\sum_{\substack{q,r\leqslant Q\\ q\neq r}} \lambda(\mathcal{A}_q \cap \mathcal{A}_r) - \Psi(Q)^2.\end{aligned}$$

Let $q \neq r$ be two primes. We shall compute $\lambda(\mathcal{A}_q \cap \mathcal{A}_r)$ by adapting the proof of Lemma 5.1. In the notation there, we have $\ell = m = 1$ and $n = qr$. Hence,

$$\lambda(\mathcal{A}_q \cap \mathcal{A}_r) = 2 \sum_{\substack{c\geqslant 1\\ (c,qr)=1}} w\Big(\frac{c}{qr}\Big),$$

where w is defined as in Lemma 5.1. Recall also the definition of δ and Δ there. With our definition of ψ, we have $\delta = 1/\max\{q,r\}^2$ and $\Delta = 1/\min\{q,r\}^2$. By Möbius inversion and estimate (5.3), we have

$$\begin{aligned}\lambda(\mathcal{A}_q \cap \mathcal{A}_r) = 2 \sum_{\substack{c\geqslant 1\\ (c,qr)=1}} w\Big(\frac{c}{qr}\Big) &= 2\sum_{d|qr}\mu(d)\sum_{c'\geqslant 1} w\Big(\frac{c'd}{qr}\Big) \\ &= 4\Delta\delta\varphi(q)\varphi(r) + O(\delta) \\ &= \lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r) + O\bigg(\frac{1}{\max\{q,r\}^2}\bigg).\end{aligned}$$

Hence,

$$\begin{aligned}\int_0^1 \big(N(\alpha;Q) - \Psi(Q)\big)^2 \,\mathrm{d}\alpha &= \Psi(Q) + \sum\sum_{\substack{q,r\leqslant Q\\ q\neq r}} \lambda(\mathcal{A}_q)\lambda(\mathcal{A}_r) - \Psi(Q)^2 + O\bigg(\sum\sum_{\substack{q,r \text{ prime}\\ r<q\leqslant Q}} \frac{1}{q^2}\bigg) \\ &= \Psi(Q) - \sum_{q\leqslant Q}\lambda(\mathcal{A}_q)^2 + O\bigg(\sum\sum_{\substack{q,r \text{ prime}\\ r<q\leqslant Q}} \frac{1}{q^2}\bigg).\end{aligned}$$

The sum of $\lambda(\mathcal{A}_q)^2$ is uniformly bounded by the choice of ψ. In addition, by the prime number theorem, the sum inside the big-Oh is $\ll \sum_{q \text{ prime}} 1/(q\log q) \ll 1$. We thus conclude that

$$\int_0^1 \big(N(\alpha;Q) - \Psi(Q)\big)^2 \,\mathrm{d}\alpha = \Psi(Q) + O(1),$$

as claimed. This completes the proof of the proposition. □

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