SHAPE OPTIMIZATION FOR LOW NEUMANN AND STEKLOV EIGENVALUES

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Abstract. We give an overview of results on shape optimization for low eigenvalues of the Laplacian on bounded planar domains with Neumann and Steklov boundary conditions. These results share a common feature: they are proved using methods of complex analysis. In particular, we present modernized proofs of the classical inequalities due to Szegö and Weinstock for the first nonzero Neumann and Steklov eigenvalues. We also extend the inequality for the second nonzero Neumann eigenvalue, obtained recently by Nadirashvili and the authors, to non-homogeneous membranes with log-subharmonic densities. In the homogeneous case, we show that this inequality is strict, which implies that the maximum of the second nonzero Neumann eigenvalue is not attained in the class of simply-connected membranes of a given mass. The same is true for the second nonzero Steklov eigenvalue, as follows from our results on the Hersch–Payne–Schiffer inequalities.

1. Introduction and main results

1.1. **Neumann and Steklov eigenvalue problems.** Let Ω be a simply-connected bounded planar domain with Lipschitz boundary. Consider the *Neumann* and *Steklov* eigenvalue problems on Ω:

$$-\Delta u = \mu u \text{ in } \Omega \text{ and } \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega, \tag{1.1.1}$$

$$\Delta u = 0 \text{ in } \Omega \text{ and } \frac{\partial u}{\partial n} = \sigma u \text{ on } \partial\Omega. \tag{1.1.2}$$

Here $\Delta = \partial_x^2 + \partial_y^2$ is the Laplace operator and $\frac{\partial}{\partial n}$ is the outward normal derivative. Both problems have discrete spectra (see [1, p. 7 and p. 113])

$$0 = \mu_0 < \mu_1(\Omega) \le \mu_2(\Omega) \le \mu_3(\Omega) \le \cdots \nearrow \infty,$$

$$0 = \sigma_0 < \sigma_1(\Omega) \le \sigma_2(\Omega) \le \sigma_3(\Omega) \le \cdots \nearrow \infty,$$

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starting with the simple eigenvalues $\mu_0 = 0$ and $\sigma_0 = 0$, which correspond to constant eigenfunctions. The eigenvalues μ_k and σ_k satisfy the following variational characterizations:

$$\mu_k(\Omega) = \inf_{U_k} \sup_{0 \neq u \in U_k} \frac{\int_\Omega |\nabla u|^2 \, dz}{\int_\Omega u^2 \, dz}, \quad k = 1, 2, \ldots \tag{1.1.3}$$

$$\sigma_k(\Omega) = \inf_{E_k} \sup_{0 \neq u \in E_k} \frac{\int_\Omega |\nabla u|^2 \, dz}{\int_{\partial\Omega} u^2 \, ds}, \quad k = 1, 2, \ldots \tag{1.1.4}$$

The infima are taken over all k–dimensional subspaces U_k and E_k of the Sobolev space $H^1(\Omega)$ which are orthogonal to constants on Ω and $\partial\Omega$, respectively.

Remark 1.1.5. Here and further on we identify $\mathbf{R}^2$ with the complex plane $\mathbf{C}$ and set $z = (x, y)$. We write $dz = dx\, dy$ for the Lebesgue measure.

1.2. **Shape optimization.** Both Neumann and Steklov eigenvalue problems describe the vibration of a free membrane. In the Neumann case the membrane is homogeneous, while in the Steklov case the whole mass of the membrane is uniformly distributed on $\partial\Omega$. Therefore, we may define the *mass* of the membrane Ω by setting

$$M(\Omega) = \begin{cases} \mathrm{Area}(\Omega) & \text{in the Neumann case,} \\ \mathrm{Length}(\partial\Omega) & \text{in the Steklov case.} \end{cases} \tag{1.2.1}$$

In this survey we focus on the following shape optimization problem.

Question 1.2.2. *How large can μ_k and σ_k be on a membrane of a given mass?*

In 1954, this problem was solved by G. Szegö for μ_1 and by R. Weinstock for σ_1. Let $\mathbf{D} = \{z \in \mathbf{C} \,|\, |z| < 1\}$ be the open unit disk.

Theorem 1.2.3. ([2]) *Let Ω be a simply-connected bounded planar domain with Lipschitz boundary. Then*

$$\mu_1(\Omega) M(\Omega) \leq \mu_1(\mathbf{D})\pi \approx 3.39\pi, \tag{1.2.4}$$

with equality if and only if Ω is a disk.

Szegö's inequality was later generalized by H. Weinberger [3] to arbitrary (not necessarily simply–connected) domains in any dimension.

Theorem 1.2.5. ([4]) *Let Ω be a simply-connected bounded planar domain with Lipschitz boundary. Then*

$$\sigma_1(\Omega)\, M(\Omega) \leq 2\pi, \tag{1.2.6}$$

with equality if and only if Ω is a disk.

Many results were motivated by Weinstock's inequality: see, for instance, [5—11].

Recently, analogues of Theorems 1.2.3 and 1.2.5 for the second nonzero Neumann and Steklov eigenvalues were proved in [12] and [13].

Theorem 1.2.7. **([12])** *(i) Let Ω be a simply-connected bounded planar domain with Lipschitz boundary. Then*

$$\mu_2(\Omega)M(\Omega) < 2\,\mu_1(\mathbf{D})\,\pi \approx 6.78\pi. \tag{1.2.8}$$

(ii) There exists a family of simply-connected bounded Lipschitz domains $\Omega_\varepsilon \subset \mathbf{R}^2$, degenerating to the disjoint union of two identical disks as $\varepsilon \to 0+$, such that

$$\lim_{\varepsilon\to 0+} \mu_2(\Omega_\varepsilon)\,M(\Omega_\varepsilon) = 2\,\mu_1(\mathbf{D})\,\pi.$$

Note that inequality (1.2.8) is *strict*, and hence Theorem 1.2.7 is a slight improvement upon [12, Theorem 1.1.3].

Theorem 1.2.7 implies the Pólya conjecture [14] for the second nonzero Neumann eigenvalue of a simply-connected bounded planar domain:

$$\mu_2(\Omega)\mathrm{Area}(\Omega) \le 8\pi.$$

The best previous estimate on μ_2 was obtained in [15]:

$$\mu_2(\Omega)\mathrm{Area}(\Omega) \le 16\pi.$$

Theorem 1.2.9. **([13])** *(i) Let Ω be a simply-connected bounded planar domain with Lipschitz boundary. Then*

$$\sigma_2(\Omega)\,M(\Omega) < 4\pi. \tag{1.2.10}$$

(ii) There exists a family of simply-connected bounded Lipschitz domains $\Omega_\varepsilon \subset \mathbf{R}^2$, degenerating to the disjoint union of two identical disks as $\varepsilon \to 0+$, such that

$$\lim_{\varepsilon\to 0+} \sigma_2(\Omega_\varepsilon)\,M(\Omega_\varepsilon) = 4\pi.$$

The proofs of Theorems 1.2.7 and 1.2.9 use similar techniques. Inequality (1.2.10) is a slight sharpening in the case $k = 2$ of the estimate

$$\sigma_k(\Omega)\,M(\Omega) \le 2\pi k, \quad k = 1, 2, \ldots \tag{1.2.11}$$

obtained earlier by Hersch–Payne–Schiffer [10, p. 102] by a completely different method. Our approach allows to show that (1.2.11) is strict for $k = 2$, similarly to (1.2.8). Note that this contrasts with estimates (1.2.4) and (1.2.6). In particular, we have the following

Corollary 1.2.12. *The maximal values of the second nonzero Neumann and Steklov eigenvalues are not attained in the class of simply–connected Lipschitz domains of a given mass.*

Remark 1.2.13. Theorems 1.2.3 and 1.2.5 are analogues of the Faber–Krahn inequality for the first Dirichlet eigenvalue ([16, 17], [1, section 3.2]), while Theorems 1.2.7 and 1.2.9 are similar to Krahn's inequality for the second Dirichlet eigenvalue ([18], [1, section 4.1]). Note that the equalities in the estimates of Faber–Krahn and Krahn are also attained for the disk and the disjoint union of two identical disks, respectively.

1.3. **Higher eigenvalues.** One could ask whether μ_k and σ_k are maximized in the limit by the disjoint union of k identical disks for all $k \geq 1$. In [13] we show that this is indeed true for all Steklov eigenvalues, and that the Hersch–Payne–Schiffer inequality (1.2.11) is *sharp* for all $k \geq 1$. This gives an almost complete answer to Question 1.2.2 in the Steklov case. It remains to check whether (1.2.11) is always a *strict* inequality.

For Neumann eigenvalues the situation is more complicated. Indeed, if all μ_k are maximized in the limit by the disjoint union of k identical disks, then for any simply-connected domain Ω and each integer $k \geq 1$,

$$\mu_k(\Omega)M(\Omega) \leq k\,\mu_1(\mathbf{D})\,\pi \approx 3.39k\pi.$$

However, this is false for *any* domain Ω, because

$$\mu_k(\Omega)M(\Omega) \sim 4k\pi \quad \text{ as } k \to \infty$$

according to Weyl's law [19, p. 31].

1.4. **Non-homogeneous membranes.**

Neumann problem. Let $\rho \in L^\infty(\Omega)$ be a positive function representing the density of a non-homogeneous membrane Ω of mass

$$M(\Omega) = \int_\Omega \rho(z)\,dz.$$

In this context, the Neumann eigenvalue problem becomes

$$-\Delta u = \mu\,\rho\,u \;\text{ in } \Omega \;\text{ and }\; \frac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega. \tag{1.4.1}$$

It also has a discrete spectrum

$$0 = \mu_0 < \mu_1(\Omega,\rho) \leq \mu_2(\Omega,\rho) \leq \mu_3(\Omega,\rho) \leq \cdots \nearrow \infty.$$

The following result shows that inequalities (1.2.4) and (1.2.8) can be generalized to this setting.

Theorem 1.4.2. *Let Ω be a simply-connected domain with Lipschitz boundary and density $\rho \in C^2(\overline{\Omega})$. If $\Delta \log \rho \geq 0$, then*

$$\mu_1(\Omega,\rho)M(\Omega) \leq \mu_1(\mathbf{D})\pi, \tag{1.4.3}$$

$$\mu_2(\Omega,\rho)M(\Omega) \leq 2\,\mu_1(\mathbf{D})\,\pi. \tag{1.4.4}$$

Inequality (1.4.3) was proved by C. Bandle ([20], [21, p. 121-128]). To the best of our knowledge, estimate (1.4.4) is new. Its proof is very similar to that of (1.2.8). We believe that inequality (1.4.4) is strict, and it would be interesting to establish this fact.

Remark 1.4.5. Recall that the Gaussian curvature of the Riemannian metric $g = \rho(x, y)(dx^2 + dy^2)$ is given by the well known formula [22, Theorem 13.1.3]:

$$K_g = -\frac{1}{2\rho}\Delta \log \rho. \tag{1.4.6}$$

It follows that the condition $\Delta \log \rho \geq 0$ is equivalent to $K_g \leq 0$. In other words, *log-subharmonic* densities correspond to nonpositively curved membranes.

Remark 1.4.7. If we impose no restriction on the density ρ, then it is easy to see that maximizing μ_k for simply–connected membranes is equivalent to finding a Riemannian metric on the sphere that maximizes the k-th Laplace-Beltrami eigenvalue. It follows from [23, Corollary 1] that

$$\sup_{\Omega,\rho} \mu_k(\Omega, \rho) M(\Omega) \geq 8k\pi \quad \text{for each } k \geq 1. \tag{1.4.8}$$

For $k = 1, 2$ this is an equality, as was shown in [24] and [25]. Interestingly enough, extremal metrics for the first two eigenvalues on the sphere resemble the extremal domains described in section 1.2: the first eigenvalue is maximized by the round sphere, and the supremum for the second eigenvalue is attained in the limit by a sequence of metrics converging to the disjoint union of two identical round spheres. In fact, in [25] it is conjectured that (1.4.8) is an equality for each $k \geq 1$, and that the supremum is attained in the limit by the densities corresponding to a family of surfaces degenerating to the disjoint union of k identical round spheres. If true, this would be similar to the case of higher Steklov eigenvalues, see section 1.3.

Steklov problem. Let $\rho \in L^\infty(\partial\Omega)$ be a positive function representing the boundary density of the membrane Ω of total mass

$$M(\Omega) = \int_{\partial\Omega} \rho(s)\, ds.$$

The non-homogeneous Steklov eigenvalue problem is given by

$$\Delta u = 0 \ \text{ in } \Omega \ \text{ and } \ \frac{\partial u}{\partial n} = \sigma \rho u \text{ on } \partial\Omega. \tag{1.4.9}$$

It has discrete spectrum

$$0 = \sigma_0 < \sigma_1(\Omega, \rho) \leq \sigma_2(\Omega, \rho) \leq \sigma_3(\Omega, \rho) \leq \cdots \nearrow \infty.$$

Estimates (1.2.6) and (1.2.10) can be generalized to this setting.

Theorem 1.4.10. *Let Ω be a simply-connected bounded Lipschitz planar domain with the density ρ on the boundary. Then*

$$\sigma_1(\Omega, \rho)\, M(\Omega) \leq 2\pi. \tag{1.4.11}$$

$$\sigma_2(\Omega, \rho)\, M(\Omega) < 4\pi. \tag{1.4.12}$$

Inequality (1.4.11) was proved in [4] similarly to (1.2.6). Estimate (1.4.12) was proved in [13], and we refer to this paper for the details of the proof. In fact, it is almost identical to that of (1.2.10). Let us also note that the estimate for higher eigenvalues (1.2.11) is valid in the non-homogeneous case as well [10].

1.5. **Structure of the paper.** In section 2.1, the Riemann mapping theorem is used to transplant the Neumann and Steklov problems to the disk. In section 2.2, we describe Hersch's renormalization ([24], see also [26, p. 144]), which is applied in section 2.3 to prove Theorem 1.2.5. Theorem 1.2.3 and the first part of Theorem 1.4.2 are proved in section 2.5 using some results on subharmonic functions presented in section 2.4.

In the remaining part of the paper we prove estimates on the second non-zero eigenvalues μ_2 and σ_2. In section 3.1 some additional results on Hersch's renormalization are presented. In sections 3.2–3.3 we introduce hyperbolic caps and folded measures (this idea goes back to [25]), which are used to define test functions in section 3.4. In section 3.5 we apply a topological argument to prove the existence of a suitable two-dimensional space of test functions. We use this argument in section 3.6 to prove the first part of Theorem 1.2.9, and in section 3.7 to prove inequality (1.2.8) and the second part of Theorem 1.4.2. Finally, in section 3.8 we show that inequalities (1.2.8) and (1.2.10) are sharp.

2. Shape optimization for μ_1 and σ_1

2.1. **Application of the Riemann mapping theorem.** The results presented in the introduction share a common feature: they are proved using methods of complex analysis. The following application of the Riemann mapping theorem plays a key role in the proofs. We formulate it in such a way that it works for both Neumann and Steklov eigenvalue problems simultaneously.

Let Ω be a simply-connected bounded planar domain with Lipschitz boundary. Consider a conformal diffeomorphism $\phi : \mathbf{D} \to \Omega$. Here and further on we denote a conformal map and its extension to the boundary by the same symbol. Let dz be the Lebesgue measure on Ω, ds be the arc-length measure on $\partial\Omega$ and let $\nu = \phi^*(dz)$ or $\nu = \phi^*(ds)$ be the pullback of either of these measures to $\overline{\mathbf{D}}$.

Remark 2.1.1. To simplify notation, we write ν instead of $d\nu$ unless we integrate over this measure. The same convention applies to other measures defined later on.

Set

$$\lambda_k(\nu) = \inf_E \sup_{0\neq f\in E} \frac{\int_{\overline{\mathbf{D}}} |\nabla f|^2\, dz}{\int_{\overline{\mathbf{D}}} f^2\, d\nu}, \quad k = 1, 2, \ldots \tag{2.1.2}$$

The infimum is taken over all k-dimensional subspaces E of the Sobolev space $H^1(\mathbf{D})$ such that

$$\int_{\overline{\mathbf{D}}} f\, d\nu = 0 \quad \text{for all } f \in E. \tag{2.1.3}$$

Proposition 2.1.4.

- *For $\nu = \phi^*(dz)$, $\lambda_k(\nu) = \mu_k(\Omega)$ is the k-th eigenvalue of the Neumann problem.*
- *For $\nu = \phi^*(ds)$, $\lambda_k(\nu) = \sigma_k(\Omega)$ is the k-th eigenvalue of the Steklov problem.*

Proof. It is well known that the Dirichlet energy of a function f is a conformal invariant in dimension two. The result then follows from the variational characterizations of μ_k and σ_k given by (1.1.3) and (1.1.4). $\square$

The eigenvalue problems themselves can be pulled back to the disk. The Neumann and Steklov problems on Ω are, respectively, equivalent to the following ones:

$$-\Delta u = \mu\, |\phi'(z)|^2\, u \text{ in } \mathbf{D} \text{ and } \left.\frac{\partial u}{\partial r}\right|_{\mathbf{S}^1} = 0, \tag{2.1.5}$$

$$\Delta u = 0 \text{ in } \mathbf{D} \text{ and } \frac{\partial u}{\partial r} = \sigma|\phi'(z)|u \text{ on } \mathbf{S}^1. \tag{2.1.6}$$

This will be useful when treating the case of equality in Szego's and Weinstock's inequalities.

Subharmonic functions. Recall that a function $\delta : \Omega \to \mathbf{R}$ is called *subharmonic* if $\Delta\delta \geq 0$, and *log-subharmonic* if $\Delta \log \delta \geq 0$. We state here some simple facts about subharmonic functions that will be used repeatedly.

Lemma 2.1.7. *Let $\delta \in C^2(\Omega)$ be a positive function.*

(i) If δ is log-subharmonic, then it is subharmonic.

(ii) If δ is log–subharmonic and $\Delta\delta \leq 0$, then δ is constant.

Proof. (i) It follows from

$$\Delta \log \delta = \frac{\delta\Delta\delta - |\nabla\delta|^2}{\delta^2} \tag{2.1.8}$$

that $\Delta\delta = \delta\,\Delta \log \delta + \frac{|\nabla\delta|^2}{\delta} \geq 0$.

(ii) From $|\nabla\delta|^2 = \delta\Delta\delta - \delta^2\,\Delta \log \delta \leq 0$ it follows that $|\nabla\delta| = 0$, and hence δ is constant. □

Lemma 2.1.9. *Let $\rho \in C^2(\Omega)$ be a positive function and $\phi : \mathbf{D} \to \Omega$ be a conformal map. Consider the density $\rho(z)$ on Ω, and let*

$$\delta(z) = \rho(\phi(z))|\phi'(z)|^2$$

be its pullback to the unit disk. Then the function $\log \delta$ is (sub)harmonic iff the function $\log \rho$ is (sub)harmonic.

In particular, if ρ is constant then $\log \delta$ is harmonic.

Proof. The Gaussian curvature of the Riemannian metric $ds^2 = \rho(dx^2 + dy^2)$ is given by (1.4.6). The pullback of g by ϕ is $\delta(dx^2 + dy^2)$ where $\delta(z) = \rho(\phi(z))|\phi'(z)|^2$. Therefore, $K_{\phi^*g}(z) = -\frac{1}{2\delta}\Delta \log \delta$. The result now follows from the formula $K_{\phi^*g}(z) = K_g(\phi(z))$. □

2.2. **Hersch's renormalization.** Let $\Psi : \overline{\mathbf{D}} \to \overline{\mathbf{D}}$ be a diffeomorphism such that $\Psi(z) = z$ for each $z \in \partial\mathbf{D} = \mathbf{S}^1$. The center of mass relative to Ψ of a finite measure ν on the closed disk $\overline{\mathbf{D}}$ is defined by

$$\overline{\mathbf{D}} \ni \mathrm{C}(\nu) = \frac{1}{M(\nu)} \int_{\overline{\mathbf{D}}} z\, d\,\Psi_*\nu$$

where $M(\nu) = \int_{\overline{\mathbf{D}}} d\nu$ is the mass of the measure ν. Note that for $\nu = \phi^*(dz)$ and $\nu = \phi^*(ds)$ the mass of ν coincides with the mass of Ω defined by (1.2.1). For example, the center of mass of the Dirac mass δ_p (where $p \in \overline{\mathbf{D}}$) is $\mathrm{C}(\delta_p) = \Psi(p)$. Given $t \in \mathbf{R}^2$, define $X_t : \overline{\mathbf{D}} \to \mathbf{R}$ by

$$X_t(z) = \langle \Psi(z),\, t\rangle. \tag{2.2.1}$$

Note that:

– For $\Psi = \mathrm{id}$, the functions X_t are the eigenfunctions corresponding to the double eigenvalue $\sigma_1(\mathbf{D}) = \sigma_2(\mathbf{D}) = 1$ of the Steklov problem.
– Let J_1 be the first Bessel function of the first kind, and let $\zeta \approx 1.84$ be the smallest positive zero of the derivative J_1'. Set
$$f(r) = J_1(\zeta r)/J_1(\zeta).$$
For $\Psi(re^{i\theta}) = f(r)e^{i\theta}$, the functions X_t are the eigenfunctions corresponding to the double eigenvalue $\mu_1(\mathbf{D}) = \mu_2(\mathbf{D})$ of the Neumann problem.

Given $\xi \in \mathbf{D}$, define the automorphism d_ξ of $\mathbf{D}$ by
$$d_\xi(z) = \frac{z+\xi}{\bar{\xi}z+1}.$$
Observe that for any $p \in \partial\mathbf{D}$ and $-p \neq z \in \overline{\mathbf{D}}$, we have $\lim_{\xi\to p} d_\xi(z) = p$. Then for any point $p \in \partial\mathbf{D}$,
$$\lim_{\xi\to p}(d_\xi)_*\nu = \delta_p, \tag{2.2.2}$$
if the measure ν is absolutely continuous with respect to the Lebesgue measure on $\mathbf{D}$ and with respect to the arc–length measure on $\partial\mathbf{D}$. Note that both measures defined in the beginning of section 2.1 satisfy these conditions.

Remark 2.2.3. We use the weak topology on the space of measures: a sequence of measure (ν_k) converges to ν iff for each continuous function f
$$\lim_{k\to\infty}\int_{\overline{\mathbf{D}}} f\,d\nu_k = \int_{\overline{\mathbf{D}}} f\,d\nu.$$
In particular, (2.2.2) means that
$$\lim_{\xi\to p}\int_{\overline{\mathbf{D}}} f\,(d_\xi)_*\nu = f(p) \quad \text{for each } f \in C^0(\overline{\mathbf{D}}).$$

Proposition 2.2.4. (cf. [24, 26, 27, 12]) *Let ν be a finite measure on the closed disk $\overline{\mathbf{D}}$ satisfying (2.2.2). Then there exists a point $\xi \in \mathbf{D}$ such that*
$$\int_{\overline{\mathbf{D}}} X_t \circ d_\xi\,d\nu = 0 \quad \textit{for each } t \in \mathbf{R}^2.$$

Proof. Define the map $\Gamma : \overline{\mathbf{D}} \to \overline{\mathbf{D}}$ by
$$\Gamma(\xi) = \begin{cases} \mathrm{C}\big((d_\xi)_*\nu\big) & \text{for } \xi \in \mathbf{D}, \\ \xi & \text{for } \xi \in \partial\mathbf{D}. \end{cases}$$

It follows from (2.2.2) that for $p \in \partial\mathbf{D}$,

$$\lim_{\xi \to p} \mathrm{C}\big((d_\xi)_*\nu\big) = \mathrm{C}(\delta_p) = \Psi(p) = p,$$

so that Γ is continuous. Moreover, its restriction to $\partial\mathbf{D}$ is the identity map. It follows by the standard topological argument that Γ is onto: there exists $\xi \in \mathbf{D}$ such that

$$\mathrm{C}\big((d_\xi)_*\nu\big) = 0.$$

Therefore, for any $t \in \mathbf{R}^2$,

$$\int_{\overline{\mathbf{D}}} X_t \circ d_\xi \, d\nu = \int_{\overline{\mathbf{D}}} \langle \Psi \circ d_\xi(z), t\rangle \, d\nu = M(\nu) \left\langle \mathrm{C}\big((d_\xi)_*\nu\big), t \right\rangle = 0.$$

$\square$

2.3. **Proof of Weinstock's theorem.** The goal of this section is to prove Theorem 1.2.5. Let $\phi : \mathbf{D} \to \Omega$ be a conformal equivalence. We will use the variational characterization (2.1.2) with the measure $\nu = \phi^*(ds)$. This measure is supported on $\mathbf{S}^1$. We use the test functions X_t introduced in (2.2.1) with $\Psi(z) = z$, that is $X_t(z) = \langle z, t\rangle$. Applying Proposition 2.2.4, we may assume that

$$\int_{\mathbf{S}^1} X_t \, d\nu = 0 \qquad \text{for all } t \in \mathbf{R}^2.$$

Choose $s, t \in \mathbf{S}^1$ such that $\langle s, t\rangle = 0$. Observe that for any $z \in \mathbf{S}^1$,

$$X_s^2(z) + X_t^2(z) = 1.$$

Switching s and t if necessary, we may assume that

$$\int_{\mathbf{S}^1} X_t^2 \, d\nu \geq \frac{1}{2} \int_{\mathbf{S}^1} d\nu = \frac{M(\Omega)}{2}. \tag{2.3.1}$$

Recall that X_t is a Steklov eigenfunction corresponding to the double eigenvalue $\sigma_1(\mathbf{D}) = 1$. Therefore,

$$\int_{\mathbf{D}} |\nabla X_t|^2 \, dz = \int_{\mathbf{S}^1} X_t^2 \, ds = \pi.$$

Inequality (1.2.6) then follows from the variational characterization (2.1.2).

Case of equality. Let Ω be such that $\sigma_1(\Omega)\, M(\Omega) = 2\pi$. We may assume wihout loss of generality that $M(\Omega) = 2\pi$ (this can always be achieved using a dilation). For t satisfying (2.3.1) we have

$$1 = \sigma_1(\Omega) \leq \frac{\int_{\mathbf{D}} |\nabla X_t|^2 \, dz}{\int_{\mathbf{S}^1} X_t^2 \, d\nu} \leq 1 = \sigma_1(\mathbf{D}).$$

It follows that X_t is an eigenfunction of problem (2.1.6) with eigenvalue 1:

$$\Delta X_t = 0 \text{ in } \mathbf{D} \text{ and } \frac{\partial X_t}{\partial r} = |\phi'(z)| X_t \text{ on } \mathbf{S}^1. \tag{2.3.2}$$

However, by definition, X_t also satisfies $\partial_r X_t = X_t$ so that $|\phi'(z)| = 1$ for each $z \in \mathbf{S}^1$. By Lemma 2.1.9 the function $\log|\phi'(z)|$ is harmonic. Because $\log|\phi'(z)| = 0$ on $\mathbf{S}^1$, it is also identically 0 on $\mathbf{D}$. Therefore, $|\phi'(z)| = 1$ for each $z \in \mathbf{D}$. It follows that $\phi : \mathbf{D} \to \Omega$ is an isometry.

2.4. **Growth of subharmonic functions.** Given a measure $\nu = \delta(z)dz$ on $\mathbf{D}$, define

$$G(r) = \int_{B(0,r)} d\nu. \tag{2.4.1}$$

Lemma 2.4.2. *i) Let δ be a positive subharmonic function on $\mathbf{D}$ such that $G(1) = \pi$. Then*

$$G(r) \leq \pi r^2$$

for each $r \in [0,1]$.
ii) The function δ is harmonic iff $G(r) = \pi r^2$ for each $r \in [0,1]$.

Remark 2.4.3. Let $\phi : \mathbf{D} \to \Omega$ and $\delta(z) = |\phi'(z)|^2$. Then Lemma 2.4.2 states that

$$\text{Area}(\phi(B_r(0))) \leq \text{Area}(B_r(0)).$$

Proof. i) Let us write

$$G(r) = \int_{B(0,r)} \delta(z)\, dz = \int_0^r W(\rho)\rho\, d\rho,$$

where

$$W(\rho) = \int_0^{2\pi} \delta(\rho e^{i\theta})\, d\theta.$$

The function W is non-decreasing on $[0,1]$ (see [28]). Indeed, define $\widetilde{W} : \mathbf{D} \to \mathbf{R}$ by

$$\widetilde{W}(z) = \int_0^{2\pi} \delta(z e^{i\theta})\, d\theta.$$

This function is subharmonic and satisfies $\widetilde{W}(z) = W(|z|)$. It follows from the maximum principle that for any z with $|z| = \rho$,

$$W(\rho) = \widetilde{W}(z) \geq \max_{|z|\leq\rho} \widetilde{W}(z) = \max_{s\leq\rho} W(s).$$

Therefore,

$$G(r) = r^2 \int_0^1 W(r\,\rho)\rho\,d\rho \leq r^2 \int_0^1 W(\rho)\,\rho\,d\rho = G(1)r^2 = \pi r^2.$$

ii) If δ is harmonic, then the value of δ at zero is equal to the average over any circle centered at zero:

$$\delta(0) = \frac{1}{2\pi} W(\rho).$$

Hence, $G(r) = \int_0^r W(\rho)\rho\,d\rho = \pi\,\delta(0)\,r^2$, and since $G(1) = \pi$, we get $\delta(0) = 1$ and $G(r) = \pi r^2$.

In the opposite direction, if $G(r) = \pi r^2$, then $2\pi r = G'(r) = W(r)r$, so that the function W is constant:

$$2\pi = W(r) = \int_0^{2\pi} \delta(re^{i\theta})\,d\theta.$$

Using a version of Jensen's formula (see [28, p.47]) we get:

$$\delta(0) + \int_{\mathbf{D}} \log\left(\frac{1}{|z|}\right) \Delta\delta(z)\,dz = \frac{W(r)}{2\pi} = 1.$$

Since the right–hand side is the average of δ over a circle of radius r, we get

$$\delta(0) = \lim_{r\to 0} \frac{W(r)}{2\pi} = 1.$$

It follows that

$$\int_{\mathbf{D}} \log\left(\frac{1}{|z|}\right) \Delta\delta(z)\,dz = 0.$$

Since $\log\left(\frac{1}{|z|}\right) > 0$ and $\Delta\,\delta \geq 0$, this implies $\Delta\,\delta = 0$. $\square$

Lemma 2.4.4. *Let $\delta(z)$ be a subharmonic function on $\mathbf{D}$, $\nu = \delta(z)dz$ be the corresponding measure, and $h : [0,1] \to \mathbf{R}$ be a smooth strictly increasing function with $h(0) = 0$. Suppose that $G(1) = \pi$, where $G(r)$ is given by* (2.4.1)*. Then*

$$\int_{\mathbf{D}} h(|z|)\,d\nu \geq \int_{\mathbf{D}} h(|z|)\,dz.$$

Moreover, equality holds iff δ is harmonic.

Proof. Using Lemma 2.4.2 and integration by parts we obtain:

$$\begin{aligned}\int_{\mathbf{D}} h(|z|)\,d\nu &= \int_{\mathbf{D}} h(|z|)\delta(z)\,dz = \int_0^1 h(r)G'(r)\,dr \\ &= h(1)G(1) - \int_0^1 \frac{d}{dr}\big(h(r)\big)G(r)\,dr \\ &\geq h(1)G(1) - \pi\int_0^1 \frac{d}{dr}\big(h(r)\big)r^2\,dr \\ &= 2\pi\int_0^1 h(r)r\,dr = \int_{\mathbf{D}} h(|z|)\,dz.\end{aligned}$$

If $\int_{\mathbf{D}} h(|z|)\,d\nu = \int_{\mathbf{D}} h(|z|)\,dz$, then from the computation above we deduce that

$$\int_0^1 \frac{d}{dr}\big(h(r)\big)\left(G(r) - \pi r^2\right)\,dr = 0.$$

By Lemma 2.4.2 (i), we have $G(r) \leq \pi r^2$. Since h is strictly increasing, we get $G(r) = \pi r^2$, which implies that δ is harmonic by Lemma 2.4.2 (ii). ☐

2.5. **Proof of Szegö's theorem.** The goal of this section is to prove Theorem 1.2.3. Let $\phi : \mathbf{D} \to \Omega$ be a conformal equivalence. Let $\delta(z) = |\phi'(z)|^2$. It follows from Lemma 2.1.9 that $\log\delta$ is harmonic, and hence δ is subharmonic. Applying a rescaling if necessary, we may assume without loss of generality that $M(\Omega) = \pi$. We will use the variational characterization (2.1.2) with the measure $\nu = \phi^*(dz) = \delta dz$ and with the test functions

$$X_t(z) = \langle \Psi(z), t\rangle,$$

where $\Psi(re^{i\theta}) = f(r)e^{i\theta}$, with $f(r) = \frac{J_1(\zeta r)}{J_1(\zeta)}$. By Proposition 2.2.4, we may assume

$$\int_{\mathbf{D}} X_t\,d\nu = 0 \qquad \text{for all } t \in \mathbf{R}^2.$$

Choose $s, t \in \mathbf{S}^1$ such that $\langle s, t\rangle = 0$. Observe that for any $z \in \mathbf{D}$,

$$X_s^2(z) + X_t^2(z) = f^2(|z|).$$

Switching s and t if necessary, we may assume that

$$\int_{\mathbf{D}} X_t^2\,d\nu \geq \frac{1}{2}\int_{\mathbf{D}} f^2(|z|)\,d\nu(z) \geq \frac{1}{2}\int_{\mathbf{D}} f^2(|z|)\,dz, \tag{2.5.1}$$

where the last inequality follows from Lemma 2.4.4, because $f(r)$ is strictly increasing on $[0,1]$. Recall that the functions X_t are the Neumann eigenfunctions corresponding to the eigenvalue $\mu_1(\mathbf{D})$. Therefore,

$$\int_{\mathbf{D}} |\nabla X_t|^2\,dz = \mu_1(\mathbf{D})\int_{\mathbf{D}} X_t^2\,dz = \frac{\mu_1(\mathbf{D})}{2}\int_{\mathbf{D}} f^2(|z|)\,dz \tag{2.5.2}$$

The proof of inequality (1.2.4) now follows from (2.5.1), (2.5.2) and the variational characterization (2.1.2).

Remark 2.5.3. This argument is motivated by [12, section 2.7] and is a modification of the proof given in [26, p. 138]. As indicated in [12, Remark 2.7.12], the novelty of our approach is that it uses the properties of subharmonic functions.

Remark 2.5.4. The first part of Theorem 1.4.2 is proved in a similar way. To obtain inequality (1.4.3), we take $\nu = \phi^*(\rho dz)$. By Lemma 2.1.9 this measure is also of the form $\delta\,dz$, where $\Delta\log\delta \geq 0$. It follows from Lemma 2.1.7 that δ is subharmonic. The rest of the proof is unchanged.

Case of equality. Let us show that the equality in (1.2.4) implies that Ω is a disk. We will give two proofs of this fact.

First proof. Suppose that $\mu_1(\Omega) = \mu_1(\mathbf{D})$ and $M(\Omega) = \pi$. For the specific choice of t made in (2.5.1) we have

$$\mu_1(\mathbf{D}) = \mu_1(\Omega) \leq \frac{\int_{\mathbf{D}} |\nabla X_t|^2\,dz}{\int_{\mathbf{D}} X_t^2\,d\nu} \leq \mu_1(\mathbf{D})$$

It follows that the function X_t is a first eigenfunction of problem (2.1.5):

$$-\Delta X_t = \delta\mu_1(\mathbf{D})X_t \text{ in } \mathbf{D}. \tag{2.5.5}$$

Because $-\Delta X_t = \mu_1(\mathbf{D})X_t$, we deduce that $1 = \delta = |\phi'(z)|^2$, so that the conformal equivalence $\phi : \mathbf{D} \to \Omega$ is an isometry. $\square$

Our second proof is a bit more involved, but it can be adapted to the case of μ_2: in section 3.7 we use a similar idea to prove that inequality (1.2.8) is strict.

Second proof. Suppose that $\mu_1(\Omega) = \mu_1(\mathbf{D})$ and $M(\Omega) = \pi$. For each $t \in \mathbf{S}^1$ we have

$$\mu_1(\mathbf{D}) \leq \frac{\int_{\mathbf{D}} |\nabla X_t|^2\,dz}{\int_{\mathbf{D}} X_t^2\,d\nu} = \mu_1(\mathbf{D})\frac{\int_{\mathbf{D}} X_t^2\,dz}{\int_{\mathbf{D}} X_t^2\,d\nu}.$$

It follows that for each t

$$\int_{\mathbf{D}} X_t^2\,d\nu \leq \int_{\mathbf{D}} X_t^2\,dz$$

Let $s, t \in \mathbf{S}^1$ be such that $\langle s, t\rangle = 0$. It follows from $X_t^2(z) + X_s^2(z) = f^2(|z|)$ and the above inequality that

$$\int_{\mathbf{D}} f^2(|z|)\, d\nu = \int_{\mathbf{D}} (X_t^2 + X_s^2)\, d\nu \le \int_{\mathbf{D}} (X_t^2 + X_s^2)\, dz = \int_{\mathbf{D}} f^2(|z|)\, dz.$$

From Lemma 2.4.4 we get $\int_{\mathbf{D}} f^2(|z|)\, d\nu = \int_{\mathbf{D}} f^2(|z|)\, dz$ and $\Delta\delta = 0$. By construction of δ (see Lemma 2.1.9) we have $\Delta \log \delta = 0$, so that by Lemma 2.1.7 δ is a constant. Therefore, since $M(\Omega) = \pi = M(\mathbf{D})$, we have $\delta(z) = |\phi'(z)|^2 = 1$, and hence $\phi : \mathbf{D} \to \Omega$ is an isometry. ☐

3. Shape optimization for μ_2 and σ_2

3.1. **Hersch's method revisited.**

In order to apply the Hersch method to the second nonzero Neumann and Steklov eigenvalues, more control is needed on the point ξ obtained in Proposition 2.2.4.

Proposition 3.1.1. *Let ν be a finite measure on the closed disk $\overline{\mathbf{D}}$ satisfying (2.2.2). The renormalizing point ξ is unique and depends continuously on ν.*

Proof. We give the proof for $\Psi(z) = z$ only. For more details on the general case, see [12]. First, suppose that $\mathrm{C}(\nu) = 0$ and let $\xi \neq 0$. Let $s = \frac{\xi}{|\xi|}$. An easy computation shows that $\langle d_\xi(z), s\rangle > \langle z, s\rangle$. It follows that

$$\left\langle \mathrm{C}\big((d_\xi)_*\nu\big), s\right\rangle = \frac{1}{M(\nu)} \int_{\overline{\mathbf{D}}} \langle d_\xi(z), s\rangle\, d\nu > \frac{1}{M(\nu)} \int_{\overline{\mathbf{D}}} \langle z, s\rangle\, d\nu = \left\langle \mathrm{C}\big(\nu\big), s\right\rangle.$$

In other words, if the center of mass of the measure ν is the origin, then $\xi = 0$.

Now, let ν be an arbitrary finite measure and suppose that it is renormalized by d_ξ and d_η. By explicit computation one gets

$$d_\eta \circ d_{-\xi} = \frac{1 - \eta\bar{\xi}}{1 - \bar{\eta}\xi} d_\alpha,$$

where $\alpha = d_{-\xi}(\eta)$ and $\left|\frac{1-\eta\bar{\xi}}{1-\bar{\eta}\xi}\right| = 1$. Moreover,

$$(d_\eta)_*\nu = (d_\eta \circ d_{-\xi})_* \left(d_\xi\right)_* \nu = \frac{1 - \eta\bar{\xi}}{1 - \bar{\eta}\xi} (d_\alpha)_* \left(d_\xi\right)_* \nu.$$

This implies that d_α renormalizes the measure $\left(d_\xi\right)_* \nu$ whose center of mass is already at the origin. It follows from the previous case that $\alpha = 0$, which in turn implies $\eta = \xi$.

Let us prove continuity. Let (ν_k) be a sequence of measures converging to the measure ν. Without loss of generality suppose that ν is renormalized. Let $\xi_k \in \mathbf{D} \subset \overline{\mathbf{D}}$ be the unique element such that d_{ξ_k}

renormalizes ν_k. Let (ξ_{k_j}) be a convergent subsequence, say to $\xi \in \overline{\mathbf{D}}$. Now, by definition of ξ_k there holds

$$0 = \lim_{j\to\infty} \left| \int_{\overline{\mathbf{D}}} z\ (d_{\xi_{k_j}})_* d\nu_{k_j} \right| = \left| \int_{\overline{\mathbf{D}}} z\ (d_\xi)_* d\nu \right|,$$

and hence d_ξ renormalizes ν. Since we assumed that ν is normalized, by uniqueness we get $\xi = 0$. Therefore, 0 is the unique accumulation point of the sequence (ξ_k) in $\overline{\mathbf{D}}$ and hence by compactness we get $\xi_k \to 0$. $\square$

3.2. **Hyperbolic caps.** Let γ be a geodesic in the Poincaré disk model, that is a diameter or the intersection of the disk with a circle, which is orthogonal to $\mathbf{S}^1$. Each connected component of $\mathbf{D} \setminus \gamma$ is called a

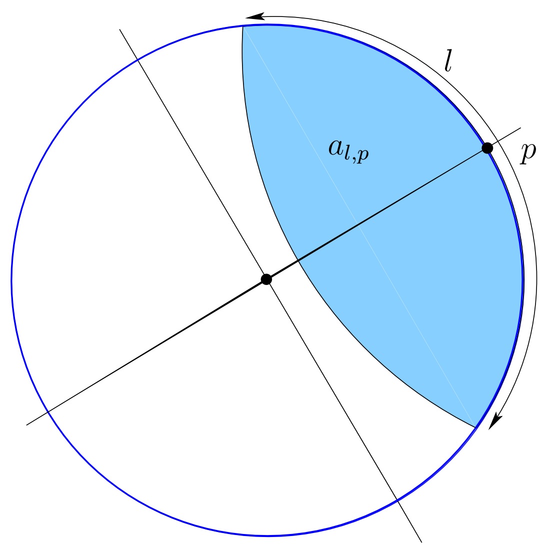

FIGURE 1. The hyperbolic cap $a_{l,p}$

hyperbolic cap [12]. Given $p \in \mathbf{S}^1$ and $l \in (0, 2\pi)$, let $a_{l,p}$ be the hyperbolic cap such that the circular segment $\partial a_{l,p} \cap \mathbf{S}^1$ has length l and is centered at p (see Figure 1). This gives an identification of the space $\mathcal{HC}$ of all hyperbolic caps with the cylinder $(0, 2\pi) \times \mathbf{S}^1$. Given a cap $a \in \mathcal{HC}$, let $\tau_a : \overline{\mathbf{D}} \to \overline{\mathbf{D}}$ be the reflection across the hyperbolic geodesic bounding a. That is, τ_a is the unique non-trivial conformal involution of $\mathbf{D}$ leaving every point of the geodesic $\partial a \cap \mathbf{D}$ fixed. In particular the cap adjacent to a is $a^* = \tau_a(a)$.

3.3. **Folded measure.** The *lift* of a function $u : \overline{a} \to \mathbf{R}$ is the function $\tilde{u} : \overline{\mathbf{D}} \to \mathbf{R}$ defined by

$$\tilde{u}(z) = \begin{cases} u(z) & \text{if } z \in \overline{a}, \\ u(\tau_a z) & \text{if } z \in \overline{a^*}. \end{cases} \tag{3.3.1}$$

As before, let ν be a finite measure on the closed disk $\overline{\mathbf{D}}$ which is absolutely continuous with respect to the Lebesgue measure on $\mathbf{D}$ and with respect to the arc–length measure on $\partial\mathbf{D}$. Observe that

$$\begin{aligned} \int_{\overline{\mathbf{D}}} \tilde{u}\, d\nu &= \int_{\overline{a}} u\, d\nu + \int_{\overline{a^*}} u \circ \tau_a\, d\nu \\ &= \int_{\overline{a}} u\, (d\nu + \tau_a^* d\nu). \end{aligned} \tag{3.3.2}$$

The measure

$$d\nu_a = \begin{cases} d\nu + \tau_a^* d\nu & \text{on } \overline{a}, \\ 0 & \text{on } \overline{a^*} \end{cases} \tag{3.3.3}$$

is called the *folded measure*. Equation (3.3.2) can be rewritten as

$$\int_{\overline{\mathbf{D}}} \tilde{u}\, d\nu = \int_{\overline{\mathbf{D}}} u\, d\nu_a.$$

3.4. **Test functions.** Let $a \in \mathcal{HC}$ be a hyperbolic cap and let $\phi_a : \mathbf{D} \to a$ be a conformal equivalence. For each $t \in \mathbf{R}^2$, define $u_a^t : \overline{a} \to \mathbf{R}$ by

$$u_a^t(z) = X_t \circ \phi_a^{-1}(z). \tag{3.4.1}$$

For each cap $a \in \mathcal{HC}$ we will use the two-dimensional space of test functions

$$E_a = \left\{ \tilde{u}_a^t \ : \ t \in \mathbf{R}^2 \right\}$$

in the variational characterization (2.1.2). It follows from the conformal invariance of the Dirichlet energy that

$$\begin{aligned} \int_{\mathbf{D}} |\nabla \tilde{u}_a^t|^2\, dz &= \int_a |\nabla u_a^t|^2\, dz + \int_{a^*} |\nabla (u_a^t \circ \tau_a)|^2\, dz \\ &= 2\int_a |\nabla u_a^t|^2\, dz = 2\int_{\mathbf{D}} |\nabla X_t|^2\, dz. \end{aligned} \tag{3.4.2}$$

Observe that the denominator in (2.1.2) can be rewritten as

$$\int_{\overline{\mathbf{D}}} (\tilde{u}_a^t)^2\, d\nu = \int_{\overline{\mathbf{D}}} X_t^2\, d\, \phi_a^* \nu_a. \tag{3.4.3}$$

We call $\zeta_a = \phi_a^* \nu_a$ the *rearranged measure.* Taking (3.4.2) and (3.4.3) into account, we obtain from the variational characterization (2.1.2) that

$$\lambda_2(\nu) \leq 2 \sup_{t \in \mathbf{S}^1} \frac{\int_{\overline{\mathbf{D}}} |\nabla X_t|^2 \, dz}{\int_{\overline{\mathbf{D}}} X_t^2 \, d\zeta_a} \tag{3.4.4}$$

provided that the functions $\tilde{u}_a^t$ satisfy the admissibility condition (2.1.3). This condition can be rewritten in terms of the rearranged measure ζ_a:

$$\int_{\overline{\mathbf{D}}} \tilde{u}_a^t \, d\nu = \int_{\overline{\mathbf{D}}} X_t \, d\zeta_a = 0.$$

In other words, (2.1.3) is satisfied by the function $\tilde{u}_a^t$ iff the rearranged measure ζ_a is renormalized.

Note that we are free to choose the conformal equivalences $\phi_a : \mathbf{D} \to a$ in our construction of test functions.

Lemma 3.4.5. *There exists a family of conformal equivalences $\{\phi_a : \mathbf{D} \to a\}_{a \in \mathcal{HC}}$ such that the rearranged measure ζ_a depends continuously on the cap $a \in \mathcal{HC}$ and satisfies*

$$\int_{\overline{\mathbf{D}}} X_t \, d\zeta_a = 0, \tag{3.4.6}$$

$$\lim_{a \to \mathbf{D}} \zeta_a = \nu, \tag{3.4.7}$$

$$\lim_{a \to p} \zeta_a = R_p^* d\nu, \tag{3.4.8}$$

where $p \in \mathbf{S}^1$ and $R_p(x) = x - 2\langle x, p \rangle$ is the reflection with respect to the diameter orthogonal to the vector p.

From now on, we fix the family of conformal maps ϕ_a defined in Lemma 3.4.5.

Proof of Lemma 3.4.5. Let us give an outline of the proof, for more details, see [12, Section 2.5]. Start with any continuous family of conformal maps $\{\psi_a : \mathbf{D} \to a\}_{a \in \mathcal{HC}}$, such that $\lim_{a \to \mathbf{D}} \psi_a = \mathrm{id}$. The maps ϕ_a are defined by composing the ψ_a's on both sides with automorphisms of the disk appearing in the Hersch renormalization procedure. In particular, (3.4.6) is automatically satisfied. As the cap a converges to the full disk $\mathbf{D}$, the conformal equivalences ϕ_a converge to the identity map on $\mathbf{D}$, which implies (3.4.7). Finally, setting $n = 1$ in [12, Lemma 4.3.2] one gets (3.4.8). $\square$

3.5. **Maximization of the moment of inertia.** The *moment of inertia* of a finite measure ν on the closed disk $\overline{\mathbf{D}}$ is the quadratic form $V_\nu : \mathbf{R}^2 \to \mathbf{R}$ defined by

$$V_\nu(t) = \int_{\overline{\mathbf{D}}} X_t^2 \, d\nu,$$

where X_t is defined by (2.2.1). When $\Psi = \mathrm{id}$ and $t \in \mathbf{S}^1$ this corresponds to the usual definition given in mechanics for the moment of inertia of ν with respect to the axis orthogonal to t.

Let $\mathbf{R}P^1 = \mathbf{S}^1/\mathbf{Z}_2$ be the projective line. We denote by $[t] \in \mathbf{R}P^1$ the element of the projective line corresponding to the pair of points $\pm t \in \mathbf{S}^1$. We say that $[t] \in \mathbf{R}P^1$ is a *maximizing direction* for the measure ν if $V_\nu(t) \geq V_\nu(s)$ for any $s \in \mathbf{S}^1$. The measure ν is called *simple* if there is a unique maximizing direction. Otherwise, it is said to be *multiple*.

Lemma 3.5.1. *A measure ν is multiple if and only if $V_\nu(t)$ does not depend on $t \in \mathbf{S}^1$.*

Proof. This follows from the fact that V_ν is quadratic, see [12, Lemma 2.6.1]. □

Proposition 3.5.2. *If the measure ν is simple, then there exists a cap $a \in \mathcal{HC}$ such that the rearranged measure ζ_a is multiple.*

Proposition 3.5.2 is proved by contradiction. Assume that the measure ν, as well as the rearranged measures ζ_a for all $a \in \mathcal{H}C$, are simple. Let $m(\nu)$ be the unique maximizing direction for the measure ν, and $m(\zeta_a) \in \mathbf{R}P^1$ be the unique maximizing direction for the measure ζ_a. The next lemma describes the behavior of $m(\zeta_a)$ as the cap a degenerates either to the full disk or to a point.

Lemma 3.5.3. *Let the measure ν as well as the rearranged measures ζ_a for all $a \in \mathcal{H}C$ be simple. Then*

$$\lim_{a \to \mathbf{D}} m(\zeta_a) = m(\nu) \tag{3.5.4}$$

$$\lim_{a \to e^{i\theta}} m(\zeta_a) = [e^{2i\theta}]. \tag{3.5.5}$$

Proof. Without loss of generality, assume $m[\nu] = [e_1]$. First, note that formula (3.5.4) immediately follows from (3.4.7). Let us prove (3.5.5). Set $p = e^{i\theta}$. Formula (3.4.8) implies

$$\lim_{a \to p} \int_{\overline{\mathbf{D}}} X_t^2 \, d\zeta_a = \int_{\overline{\mathbf{D}}} X_t^2 \, R_p^* d\nu = \int_{\overline{\mathbf{D}}} X_t^2 \circ R_p \, d\nu = \int_{\overline{\mathbf{D}}} X_{R_p t}^2 \, d\nu. \tag{3.5.6}$$

Since ν is simple, $m(\nu) = [e_1]$ is the unique maximizing direction for ν and the right hand side of (3.5.6) is maximal for $R_p t = \pm e_1$. Applying R_p on both sides we get $t = \pm e^{2i\theta}$. $\square$

Proof of Proposition 3.5.2. Suppose that for each hyperbolic cap $a \in \mathcal{HC}$, the rearranged measure ζ_a is simple. Recall that the space $\mathcal{HC}$ is identified with the open cylinder $(0, 2\pi) \times \mathbf{S}^1$. Define $h : (0, 2\pi) \times S^1 \to \mathbf{R}P^1$ by $h(l, p) = m(a_{l,p})$. The maximizing direction depends continuously on the cap a. Therefore, it follows from Lemma 3.5.3 that h extends to a continuous map on the closed cylinder $[0, 2\pi] \times S^1$ such that

$$h(0, e^{i\theta}) = [e_1],\ h(2\pi, e^{i\theta}) = [e^{2i\theta}].$$

This means that h is a homotopy between a trivial loop and a non-contractible loop in $\mathbf{R}P^1$. This is a contradiction. $\square$

3.6. **Estimate on σ_2.** In this section we prove Theorem 1.2.9. Consider the functions X_t introduced in (2.2.1) with $\Psi(z) = z$, that is $X_t(z) = \langle z, t\rangle$. The measure $\nu = \phi^*(ds)$ is supported on $\mathbf{S}^1$. We provide details only in the case when the measure ν is simple. If the measure ν is multiple the proof is easier, see [13].

Let $a \in \mathcal{HC}$ be a cap such that the rearranged measure ζ_a is multiple. Using (3.4.4) and taking into account that the functions X_t are eigenfunctions corresponding to $\sigma_1(\mathbf{D}) = 1$, we get

$$\sigma_2(\Omega) \le 2\frac{\int_{\overline{\mathbf{D}}} |\nabla X_t|^2\, dz}{\int_{\mathbf{S}^1} X_t^2\, d\zeta_a} = 2\frac{\int_{\mathbf{S}^1} X_t^2\, ds}{\int_{\mathbf{S}^1} X_t^2\, d\zeta_a} = \frac{2\pi}{\int_{\mathbf{S}^1} X_t^2\, d\zeta_a} \tag{3.6.1}$$

Given $t \in \mathbf{S}^1$, choose $s \in \mathbf{S}^1$ such that $\langle t, s\rangle = 0$. Multiplicity of the rearranged measure ζ_a and $X_t^2 + X_s^2 = 1$ on $\mathbf{S}^1$ implies

$$\int_{\mathbf{S}^1} X_t^2\, d\zeta_a = \frac{1}{2}\int_{\overline{\mathbf{D}}} (X_t^2 + X_s^2)\, d\zeta_a(z) = \frac{1}{2} M(\Omega) \tag{3.6.2}$$

This proves that $\sigma_2(\Omega) \le \frac{4\pi}{M(\Omega)}$.

The inequality is strict. Let $w_a^t \in C^\infty(\mathbf{D})$ be the unique harmonic extension of $\tilde{u}_a^t\big|_{\mathbf{S}^1}$, that is

$$\begin{cases} \Delta w_a^t = 0 & \text{in } \mathbf{D}, \\ w_a^t = \tilde{u}_a^t & \text{on } \mathbf{S}^1. \end{cases} \tag{3.6.3}$$

These functions are smooth while the original test functions u_a^t are not smooth along the geodesic bounding the hyperbolic cap a (see [13, Lemma 3.4.1]). Therefore, $w_a^t \neq \tilde{u}_a^t$ in $H^1(\mathbf{D})$. It is well-known that a harmonic function, such as w_a^t, is the unique minimizer of the Dirichlet energy among all functions with the same boundary data (see [29, p. 157]). Therefore,

$$\int_{\mathbf{D}} |\nabla w_a^t|^2 \, dz < \int_{\mathbf{D}} |\nabla \tilde{u}_a^t|^2 \, dz. \tag{3.6.4}$$

Let us take the functions w_a^t as test functions instead of $\tilde{u}_a^t$ in section 3.4. Their admissibility follows from (3.4.6), because $w_a^t = \tilde{u}_a^t$ on $\mathbf{S}^1$. For the same reason, the denominator in the Rayleigh quotient calculated in (3.6.2) remains unchanged. Together with (3.6.4) this implies that inequality (3.6.1) is strict.

3.7. **Estimate on μ_2.** We use the measure $\nu = \phi^*(dz)$ and the functions $X_t(z) = \langle \Psi(z), t\rangle$, where $\Psi(re^{i\theta}) = f(r)e^{i\theta}$, with $f(r) = \frac{J_1(\zeta r)}{J_1(\zeta)}$.

Lemma 3.7.1. *The rearranged measure ζ_a on $\mathbf{D}$ can be represented as $\zeta_a = \delta(z)dz$, where $\delta : \mathbf{D} \to \mathbf{R}$ is a subharmonic function.*

Proof. The rearranged measure $\zeta_a = \phi_a^*(\nu_a)$ can be rewritten as

$$\zeta_a = (\phi \circ \phi_a)^* dz + (\phi \circ \tau_a \circ \phi_a)^*(dz) = \alpha(z)dz + \beta(z)dz$$

where $\alpha(z) = |(\phi\circ\phi_a)'(z)|^2$ and $\beta(z) = |(\phi\circ\tau_a\circ\phi_a)'(z)|^2$. It follows from Lemma 2.1.9 that $\log\alpha$ and $\log\beta$ are harmonic functions. Therefore, $\alpha(z)$ and $\beta(z)$ are subharmonic by Lemma 2.1.7. □

Proof of inequality (1.2.8). We provide details only in the case when the measure ν is simple. If the measure ν is multiple, then the proof is easier, see [12].

Without loss of generality, suppose that $M(\Omega) = \pi$. Let $a \in \mathcal{HC}$ be a cap such that the rearranged measure ζ_a is multiple. Using (3.4.4) and taking into account that the functions X_t are eigenfunctions corresponding to $\mu_1(\mathbf{D})$, we get

$$\mu_2(\Omega) \le 2\frac{\int_{\mathbf{D}} |\nabla X_t|^2 \, dz}{\int_{\mathbf{D}} X_t^2 \, d\zeta_a} = 2\mu_1(\mathbf{D})\frac{\int_{\mathbf{D}} X_t^2 \, dz}{\int_{\mathbf{D}} X_t^2 \, d\zeta_a} = \mu_1(\mathbf{D})\frac{\int_{\mathbf{D}} f^2(|z|) \, dz}{\int_{\mathbf{D}} X_t^2 \, d\zeta_a}$$

Given $t \in \mathbf{S}^1$, choose $s \in \mathbf{S}^1$ such that $\langle t, s\rangle = 0$. Multiplicity of the rearranged measure ζ_a implies

$$\int_{\mathbf{D}} X_t^2 \, d\zeta_a = \frac{1}{2}\int_{\mathbf{D}} (X_t^2 + X_s^2) \, d\zeta_a = \frac{1}{2}\int_{\mathbf{D}} f^2(|z|) \, d\zeta_a$$

This leads to

$$\mu_2(\Omega) \le 2\mu_1(\mathbf{D})\frac{\int_{\mathbf{D}} f^2(|z|) \, dz}{\int_{\mathbf{D}} f^2(|z|) \, d\zeta_a} \tag{3.7.2}$$

By Lemma 3.7.1, one can apply Lemma 2.4.4 to the measure ζ_a. Hence, (3.7.2) implies

$$\mu_2(\Omega) \leq 2\mu_1(\mathbf{D}). \tag{3.7.3}$$

Proof of Theorem 1.4.2. Inequality (1.4.3) was already proved in Remark 2.5.4. The proof of inequality 1.4.4 is almost identical to the one above. We use the measure $\nu = \phi^*(\rho dz)$. By Lemma 2.1.9 this measure is also of the form δdz for some subharmonic function δ. The rearranged measure $\zeta_a = \phi_a^*(\nu_a)$ can be rewritten

$$\zeta_a = (\phi \circ \phi_a)^*(\rho\, dz) + (\phi \circ \tau_a \circ \phi_a)^*(\rho\, dz) = \alpha(z)dz + \beta(z)dz$$

where (by Lemma 2.1.9) $\alpha(z)$ and $\beta(z)$ are subharmonic. Hence, the statement of Lemma 3.7.1 holds in this case as well. The rest of the proof is unchanged. □

The inequality (3.7.3) *is strict.* Suppose that $\mu_2(\Omega) = 2\mu_1(\mathbf{D})$. Then, by (3.7.2) we get

$$\int_{\mathbf{D}} f^2(|z|)\, d\zeta_a \leq \int_{\mathbf{D}} f^2(|z|)\, dz.$$

Recall that according to Lemma 3.7.1, $\zeta_a = \delta(z)dz$ for some subharmonic function δ. It follows from Lemma 2.4.4 that $\int_{\mathbf{D}} f^2(|z|)\, d\zeta_a = \int_{\mathbf{D}} f^2(|z|)\, dz$ and that $\Delta\delta = 0$. Now, by construction of δ in the proof of Lemma 3.7.1 we have $\delta = \alpha + \beta$ with $\Delta \log \alpha = 0$ and $\Delta \log \beta = 0$. It follows from $\Delta\alpha \geq 0$, $\Delta\beta \geq 0$ and from $0 = \Delta\delta = \Delta\alpha + \Delta\beta$ that $\Delta\alpha = 0$ and $\Delta\beta = 0$. Hence, by Lemma 2.1.7 (ii) the functions α and β are constant. Now, from the proof of Lemma 3.7.1 we see that $\alpha(z) = |(\phi \circ \phi_a)'(z)|^2$ and $\beta(z) = |(\phi \circ \tau_a \circ \phi_a)'(z)|^2$. This implies that $\phi \circ \phi_a$ and $\phi \circ \tau_a \circ \phi_a$ are dilations. Recall that $a = \phi_a(\mathbf{D})$ and $a^* = \tau_a \circ \phi_a(\mathbf{D})$, where a^* is the cap adjacent to a. Hence, $\phi(a)$ and $\phi(a^*)$ are disjoint disks. We get a contradiction, because

$$\overline{\Omega} = \overline{\phi(a)} \cup \overline{\phi(a^*)}$$

is a connected set. This completes the proof of the first part of Theorem 1.2.7. □

3.8. **The inequalities for μ_2 and σ_2 are sharp.** The goal of this section is to prove the second parts of Theorems 1.2.7 and 1.2.9.

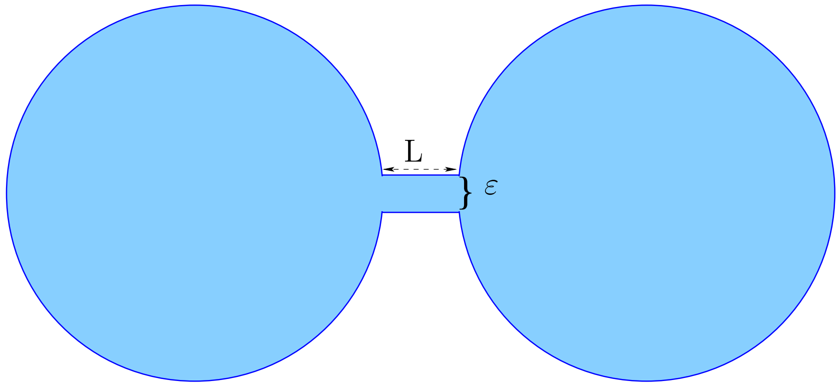

FIGURE 2. The domain Ω_ε

Neumann boundary conditions. The family Ω_ε is constructed by joining two disks using a thin passage. More precisely, let $\Omega_\varepsilon = \mathbf{D}_1 \cup P_\varepsilon \cup \mathbf{D}_2$, where $\mathbf{D}_1$ and $\mathbf{D}_2$ are two copies of the unit disk joined by a rectangular passage P_ε of length L and width ε. It follows from [30] (see also [31, 32]) that the Neumann spectrum of Ω_ε converges to the disjoint union of the Neumann spectra of $\mathbf{D}_1$ and $\mathbf{D}_2$ and the Dirichlet spectrum of the operator $-\frac{d^2}{dx^2}$ acting on the interval $[0, L]$. The first Dirichlet eigenvalue of $[0, L]$ is $\frac{\pi^2}{L^2}$. It follows that for $L < 1$ we have

$$\lim_{\varepsilon\to 0} \mu_0(\Omega_\varepsilon) = 0, \ \lim_{\varepsilon\to 0} \mu_1(\Omega_\varepsilon) = 0,$$
$$\lim_{\varepsilon\to 0} \mu_2(\Omega_\varepsilon) = \mu_1(\mathbf{D}).$$

Since $\lim_{\varepsilon\to 0} M(\Omega_\varepsilon) = 2\pi$, this completes the proof.

Steklov boundary conditions. The details of the proof can be found in [13]. Let us mention that simply joining two disks by a thin passage does not work in the case of Steklov eigenvalues. In fact, it was proved in [13] that for the domains Ω_ε defined above, the Steklov spectrum is collapsing:

$$\lim_{\varepsilon\to 0} \sigma_k(\Omega_\varepsilon) = 0 \quad \text{for each } k = 1, 2, \dots. \tag{3.8.1}$$

Instead, we use a family of domains Σ_ε , $\varepsilon \to 0+$, obtained by "pulling two disks apart" as shown on Figure 3. Similarly, taking k disks pulled apart, we show in [13] that the Hersch–Payne–Schiffer inequality (1.2.11) is sharp for all $k \geq 1$.

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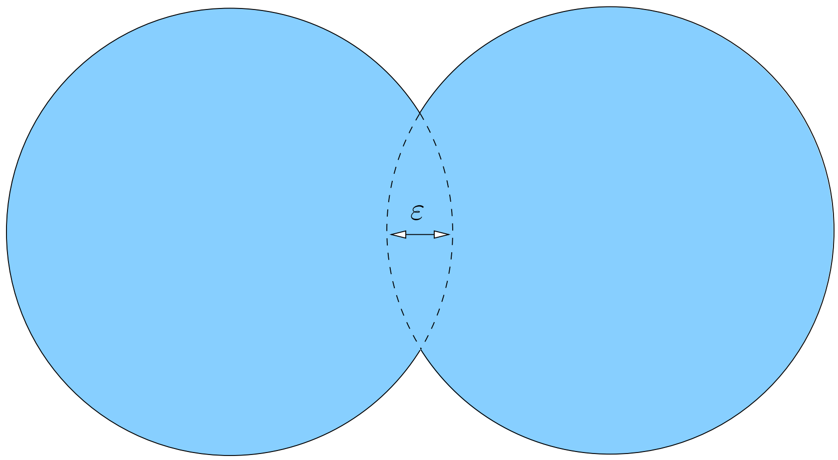

FIGURE 3. The domain Σ_ε

us his illuminating insights. We are also thankful to Michael Levitin, Marco Marletta and Mikhail Sodin for useful discussions.

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