Spectral asymptotics on negatively curved surfaces and hyperbolic dynamics

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References:

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Counting function:

$$N(\lambda) = \#\{\lambda_i < \lambda\} = \int_M N_{x,x}(\lambda)$$

Weyl's law

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$$N_{x,x}(\lambda) = \frac{\lambda^n}{(4\pi)^{\frac{n}{2}}\,\Gamma(\frac{n}{2}+1)} + R_x(\lambda),$$

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Remainder estimates are sharp and attained on a round sphere.

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When the curvature is negative, hyperbolic dynamics allows to prove finer estimates.

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Moreover, it is shown in **L.–P.–S.** that the spectral function grows *on average* as $\lambda^{\frac{n-1}{2}}$ on any manifold.

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The latter bound is simpler and proved using different techniques than the estimate for surfaces.

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We give a positive answer to this question in the **negatively curved** case.

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It is an **Anosov flow**, i.e. there exists natural splitting of $T_\xi(SX)$ into a direct sum of DG^t-invariant subspaces

$$T_\xi(SX) = E_\xi^u \oplus E_\xi^o \oplus E_\xi^s.$$

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Here E_ξ^u is the unstable (exponentially expanding) subspace of dimension $(n-1)$, E_ξ^s is the stable (exponentially contracting) subspace of dimension $(n-1)$, and E_ξ^o is a one-dimensional subspace tangent to the flow.

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Example $\quad P(-\mathcal{H}) = 0.$

Variational principle: $P(0) = h$, h — **topological entropy** of G^t.

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Margulis's theorem:

$$\nu(T) = \frac{e^{hT}}{hT}(1 + o(1)),$$

$\nu(T)$ — number of closed geodesics on X of length $\leq T$.

Local results: off–diagonal

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Theorem A (Jakobson–P.) If X is negatively curved then for any $\delta > 0$ and $x \neq y$

$$N_{x,y}(\lambda) \neq o\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{\frac{P(-\mathcal{H}/2)}{h} - \delta}\right)$$

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Get a log–improvement compared to the general case!

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If $n \geq 4$, we do not get anything new compared to the general case:

$$\frac{1}{\lambda} \int_0^\lambda |\, R_x(\mu)|\; d\,\mu >> \lambda^{n-2}.$$

Remainder estimate on surfaces

Theorem C (Jakobson–P.–Toth) Let X be a compact surface of negative curvature. Then for any $\delta > 0$

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Our approach: wave trace asymptotics for long times, thermodynamic formalism, small–scale microlocalization.

Theorem C is in agreement with the following

Conjecture (*folklore*) *On a* *generic* *negatively curved surface*

$$R(\lambda) = O(\lambda^{\varepsilon})$$

for any $\varepsilon > 0$.

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Example On the unit circle

$$\sum_{k=1}^{\infty} \cos\, k\, t = -\frac{1}{2} + \pi \sum_{k=-\infty}^{\infty} \delta(x - 2\, k\, \pi)$$

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Let us sum up the contributions of all closed geodesics of length $L \leq T(\lambda)$, where $T(\lambda) \to \infty$ as $\lambda \to \infty$.

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$\mathcal{P}_\gamma$ — **linearized Poincaré map** corresponding to γ

Proposition 1

Let $T(\lambda) \to \infty$ as $\lambda \to \infty$ with $T(\lambda) \leq \varepsilon \log \lambda$ for some $\varepsilon > 0$ small enough. Then the smoothed Fourier transform of the wave trace

$$w(\lambda, T) = \int_{-\infty}^{\infty} \chi(t, T) e(t) \cos \lambda t \, dt$$

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$$w(\lambda, T) = \sum_{L_\gamma \leq T(\lambda)} \frac{L_\gamma^\sharp \cos(\lambda L_\gamma)\, \chi(L_\gamma, T)}{\sqrt{|\det(I - \mathcal{P}_\gamma)|}} + O(1)\,.$$

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Using Proposition 1 we show that $w(\lambda, T)$ grows sufficiently fast, and then apply Proposition 2 to prove Theorem C by contradiction.

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Instead: **Microlocalize** and separate closed geodesics in the **phase space**.

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Instead: **Microlocalize** and separate closed geodesics in the **phase space**. To do this, we use the following dynamical result.

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Proof uses uniqueness of a closed geodesic in each free homotopy class.

Growth of $w(\lambda, T)$

By Proposition 1

$$w(\lambda, T) = \sum_{L_\gamma \leq T(\lambda)} \frac{L_\gamma^\sharp \cos(\lambda L_\gamma)\, \chi(L_\gamma, T)}{\sqrt{|\det(I - \mathcal{P}_\gamma)|}} + O\left(1\right).$$

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Because of $\cos(\lambda L_\gamma)$, it is an oscillating sum! We need to estimate it from **below**.

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To find the asymptotics of this sum we use the following property of the topological pressure $P(f)$ due to **Parry** and **Pollicott**:

$$\sum_{L_\gamma \le T} L_\gamma \exp\left[\int_\gamma f(\gamma(s), \gamma'(s)) ds\right] \sim \frac{C\, e^{P(f)T}}{P(f)}.$$

Proposition 3

There exists a constant $C_0 > 0$ such that

$$\sum_{L_\gamma \leq T} \frac{L_\gamma}{\sqrt{|\det(I - \mathcal{P}_\gamma)|}} = C_0 e^{P\left(-\frac{\mathcal{H}}{2}\right)\cdot T}(1 + o(1))$$

as $T \to \infty$, where P is the topological pressure and $\mathcal{H}$ is the Sinai-Ruelle-Bowen potential.

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as $T \to \infty$, where P is the topological pressure and $\mathcal{H}$ is the Sinai-Ruelle-Bowen potential.

One can show that $P\left(-\frac{\mathcal{H}}{2}\right) \geq \frac{K_2}{2} > 0$ and hence the sum grows **exponentially** in T.

Proposition 3

There exists a constant $C_0 > 0$ such that

$$\sum_{L_\gamma \leq T} \frac{L_\gamma}{\sqrt{|\det(I - \mathcal{P}_\gamma)|}} = C_0 e^{P\left(-\frac{\mathcal{H}}{2}\right)\cdot T}(1 + o(1))$$

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Now we deal with the oscillations in $w(\lambda, T)$.

Dirichlet box principle

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One can choose λ large enough so that

$$\mathrm{dist}(\lambda L_\gamma, 2\pi \mathbf{Z}) < \frac{1}{10}$$

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This "straightening the phases" argument is well-known in analytic number theory.

By Margulis's theorem,

$$\#\{\gamma \,|\, L_\gamma \leq T\} \sim \frac{e^{hT}}{hT},$$

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Thus, exponential growth in Proposition 3 yields a **logarithmic** lower bound in Theorem C. $\square$