Dynamical aspects

of spectral asymptotics

Iosif Polterovich

(Université de Montréal)

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1. Introduction

M^n — smooth compact manifold

g_{ij} — Riemannian metric

Laplace operator: $\Delta = -\text{div grad}$

In local coordinates $(x_1, \ldots, x_n)$:

$$\Delta f = -\frac{1}{\sqrt{g}} \sum_{i,j=1}^{n} \frac{\partial(\sqrt{g} g^{ij} (\partial f / \partial x_i))}{\partial x_j},$$

where $(g^{ij}) = (g_{ij})^{-1}$, $g = \det g_{ij}$.

If the metric g_{ij} is **Euclidean**, then

$$\Delta f = -\sum_{i=1}^{n} \frac{\partial^2 f}{\partial x_i^2}$$

The Laplace operator on a *compact* manifold has **discrete** spectrum:

$$\Delta \phi_i = \lambda_i^2 \phi_i$$

$\{\phi_i\}$ — orthonormal basis of **eigenfunctions**

$0 < \lambda_1^2 \leq \lambda_2^2 \leq \ldots$ — **eigenvalues**

Spectral function:

$$N_{x,y}(\lambda) = \sum_{\lambda_i < \lambda} \phi_i(x)\phi_i(y)$$

Counting function:

$$N(\lambda) = \#\{\lambda_i < \lambda\} = \int_M N_{x,x}(\lambda)$$

Spectral function bounds

Off-diagonal:

$$N_{x,y}(\lambda) = O(\lambda^{n-1}), \;\; x \neq y. \quad (1)$$

On-diagonal (pointwise Weyl's law):

$$N_{x,x}(\lambda) = \frac{\lambda^n}{(4\pi)^{\frac{n}{2}}\,\Gamma(\frac{n}{2}+1)} + R_x(\lambda),$$

where

$$R_x(\lambda) = O(\lambda^{n-1}). \quad (2)$$

Estimates are **sharp** and attained on a round sphere (**Avakumovič** ('52,'56), **Levitan** ('52), **Hörmander** ('68)).

Eigenvalue asymptotics (Weyl's law):

Integrating the pointwise Weyl's law over the manifold M we get:

$$N(\lambda) = \frac{\mathrm{Vol}(M)\,\lambda^n}{(4\pi)^{\frac{n}{2}}\,\Gamma(\frac{n}{2}+1)} + R(\lambda),$$

where

$$R(\lambda) = O(\lambda^{n-1}). \quad (3)$$

This bound is **sharp** and also attained on a round sphere.

Example: Weyl's law on a flat square 2-torus

Eigenfunctions on a torus:

$\phi_j(x) = e^{2\pi i(n_1 x_1 + n_2 x_2)},\ x = (x_1, x_2),\ n_1, n_2 \in \mathbb{Z}$

Eigenvalues: $\lambda_j^2 = 4\pi^2(n_1^2 + n_2^2)$

Weyl's law:

$$N(\lambda) = \frac{\lambda^2}{4\pi} + R(\lambda),\ \ R(\lambda) = O(\lambda).$$

We recover **Gauss's** asymptotics of the number of **integer points** in a **circle** of radius $\frac{\lambda}{2\pi}$.

Gauss's circle problem: find optimal bound on $R(\lambda)$.

Question: Can one **improve** the **universal estimates** (1-3) under some conditions on M?

Bohr's correspondence principle in quantum mechanics: **spectral data** at high energies "feels" the **dynamics** of the geodesic flow.

Problem: understand the influence of **dynamics** on the **error terms** $R_x(\lambda)$ and $R(\lambda)$, and the **spectral function**.

The standard tool to study this problem is the **wave equation**:

$$\Delta u = \frac{\partial^2 u}{\partial t^2}.$$

2. Dynamical improvements

Consider the (even part) of the **wave kernel** on M:

$$e(t,x,y) \;=\; \sum_{j=0}^{\infty} \cos(\lambda_j\, t)\phi_j(x)\phi_j(y)$$

Let ρ be an even non-negative Schwartz function with $\operatorname{supp}\widehat{\rho} \subset [-1,1]$ and let T be a (large) parameter. The following formula holds:

$$\sum_j \rho(T(\lambda-\lambda_j))\phi_j(x)\phi_j(y) = \frac{2}{T}\left(\widehat{\rho}\left(\frac{t}{T}\right)e(t,x,y)\right)^{\vee}(\lambda) + O(\lambda^{-\infty}),$$

where $^{\vee}$ denotes the inverse Fourier transform. This shows that the growth of the **spectral function** is related to the **singularities** of the **wave kernel**.

Singularities of the wave equation propagate along **geodesics**. One can observe that the following spectral and dynamical objects are closely related as $\lambda \to \infty$:

$N_{x,y}(\lambda) \longleftrightarrow$ **geodesic segments** joining x and y

$R_x(\lambda) \longleftrightarrow$ **geodesic loops** at x

$R(\lambda) \longleftrightarrow$ **closed geodesics**

The next result illustrates this correspondence.

Duistermaat-Guillemin ('75), **Safarov** ('87):

Let the measure of points corresponding to

• *geodesic segments* from x to y

•• *geodesic loops* at x

• • • *closed geodesics*

be **zero** in the unit cotangent space/bundle.

Then

• $N_{x,y}(\lambda) = o(\lambda^{n-1})$

•• $R_x(\lambda) = o(\lambda^{n-1})$

• • • $R(\lambda) = o(\lambda^{n-1})$

Euclidean domains

The results of Duistermaat–Guillemin and Safarov could be generalized to manifolds **with boundary**.

In particular, we have the following important

Conjecture 1. (**Ivrii, Safarov–Vassiliev**) Let D be a Euclidean domain with piecewise–smooth boundary. Then the **measure** of points in the unit cotangent bundle S^*D corresponding to **closed billiard trajectories** is **zero**.

Conjecture 1 holds for certain classes of domains: *convex analytic* domains, domains with *piecewise concave* boundary, *polyhedra*, etc.

Ivrii and **Melrose** proved independently in 1980 that Conjecture 1 implies the **two–term** asymptotic formula for the **counting function** conjectured by **Weyl** in 1911:

$$N(\lambda) = \frac{\mathrm{Vol}_n(D)\,\lambda^n}{(4\pi)^{\frac{n}{2}}\,\Gamma(\frac{n}{2}+1)} \pm \frac{\mathrm{Vol}_{n-1}(\partial D)\,\lambda^{n-1}}{(2\sqrt{\pi})^{n-1}\,\Gamma(\frac{n+1}{2})} + o(\lambda^{n-1}).$$

Here "$-$" in front of the second term corresponds to the **Dirichlet** boundary conditions and "$+$" — to the **Neumann** boundary conditions.

3. Spectral asymptotics in the integrable case

Conjecture 2. (cf. **Steiner** '94) Let M be a Liouville torus or a *generic* surface of revolution. Then

$$R(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$$

for any $\varepsilon > 0$.

On a flat square torus, Conjecture 2 is the celebrated **Hardy conjecture** (1915) for the **Gauss's circle problem**.

In the **integrable** case one can *approximate* the **eigenvalue** count by the **lattice** count.

Generically, one gets $R(\lambda) = O(\lambda^{2/3})$. For *flat* tori, **Huxley** '03 slightly improved this bound.

Hardy–Landau lower bound

and its generalizations

The exponent $\frac{1}{2} + \varepsilon$ in Conjecture 2 can not be improved. For flat tori this follows from the classical **Hardy–Landau** lower bound (1915):

$$R(\lambda) = \Omega\left(\sqrt{\lambda}\ (\log\lambda)^{1/4}\right),$$

where $f_1(x) = \Omega(f_2(x))$ means:

$$\limsup_{x\to\infty}\left|\frac{f_1(x)}{f_2(x)}\right| > 0.$$

Sarnak '95 suggested the following dynamical interpretation and generalization of the Hardy–Landau bound.

Let M be a surface and G^t be **the geodesic flow** on S^*M. Assume that there exists $T > 0$ such that

$$\dim\{v \in S^*M \mid G^T v = v\} = 2.$$

In other words, there is a 2-**dim** family of trajectories in S^*M with a *common* period. Then

$$\frac{1}{\lambda}\int_{\lambda}^{2\lambda} |R(\mu))| d\mu >> \lambda^{1/2}, \quad (4)$$

where $f_1(x) >> f_2(x)$ means that there exists a constant $c_0 > 0$ and a number x_0, such that $f_1(x) > c_0 f_2(x)$ for any $x > x_0$.

Estimate (4) follows from the **Duistermaat–Guillemin** wave trace formula. A 2-dimensional family of trajectories with a common period produces a *bigger singularity* in the wave trace. Such a family always exists on a surface with **integrable** geodesic flow.

The Hardy-Landau bound can be interpreted as a lower bound not only on $R(\lambda)$, but also on $R_x(\lambda)$, since $R(\lambda) = R_x(\lambda)$ on a flat torus.

Lapointe–P.–Safarov '08 proved that on **any** surface

$$\frac{1}{\lambda}\int_{\lambda}^{2\lambda} |R_x(\mu))| d\mu >> \lambda^{1/2}, \quad (5)$$

provided that x is not conjugate to itself along any shortest geodesic. This assumption is purely technical and, most likely, (5) is *always* true.

4. Spectral asymptotics on negatively curved manifolds

Geodesic flow on a manifold of negative curvature is **ergodic** which is dynamically "opposite" to the integrable case.

Conjecture 3. (cf. **Steiner** '94) On a *generic* negatively curved surface $R(\lambda) = O(\lambda^{\varepsilon})$ for any $\varepsilon > 0$.

Conjecture 3 looks quite surprising if compared with the pointwise lower bound (5) on $R_x(\lambda)$. The two estimates are compatible only if substantial **cancellations** occur when $R_x(\lambda)$ is integrated over a negatively curved surface.

The genericity assumption can **not** be omitted from the formulation of Conjecture 2. Indeed, on **arithmetic** surfaces of constant negative curvature

$$R(\lambda) = \Omega\left(\frac{\sqrt{\lambda}}{\log\lambda}\right)$$

(**Selberg, Hejhal** '76). **Randol** '81 conjectured that on *any* negatively curved surface

$$R(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$$

for any $\varepsilon > 0$. Note the same exponent as in Hardy's conjecture.

Arithmetic surfaces have exceptionally high **multiplicities** in the length spectrum. From the dynamical viewpoint this explains the faster growth of the remainder.

The following estimate due to **Jakobson–P.–Toth** '07 is in agreement with Conjecture 3.

Let M be a surface of negative curvature K satisfying $-K_1^2 \le K \le -K_2^2$. Then

$$R(\lambda) = \Omega\left((\log\lambda)^{C-\delta}\right), \quad (6)$$

for any $\delta > 0$, where

$$C = \frac{P\left(-\mathcal{H}/2\right)}{h} \ge \frac{K_2}{2K_1}$$

The exponent C is expressed in terms of **dynamical** characteristics of the geodesic flow: *topological entropy* h, *topological pressure* P and *Sinai–Ruelle–Bowen potential* $\mathcal{H}$.

On surfaces of **constant** negative curvature ($C = 1/2$) this bound was proved in 1976 independently by **Randol** and **Hejhal** using methods of analytic number theory.

The "dynamical" exponent C appears also in the pointwise lower bounds. **Jakobson-P.** '07 proved that at any point x on a negatively curved surface

$$R_x(\lambda) = \Omega\left(\sqrt{\lambda}\,(\log\lambda)^{C-\delta}\right) \qquad (7)$$

for any $\delta > 0$. This estimate sharpens the lower bound (5) in the negatively curved case.

It is *quite likely* that (7) is sharp and

$$R_x(\lambda) = O(\lambda^{\frac{1}{2}+\varepsilon})$$

for any ε. Note that this prediction is consistent with **Randol's** conjecture for $R(\lambda)$.

We are still *very* far from establishing conjectured upper bounds for $R(\lambda)$ and $R_x(\lambda)$ in the negatively curved case. The best result known is due to **Berard** '77:

Let M be a manifold of negative curvature of dimension n. Then

$$R_x(\lambda) = O\left(\frac{\lambda^{n-1}}{\log \lambda}\right), \quad R(\lambda) = O\left(\frac{\lambda^{n-1}}{\log \lambda}\right). \qquad (8)$$

The proofs of estimates (6-8) use wave kernel (or wave trace) asymptotics for **long times**. Recall the formula

$$\sum_j \rho(T(\lambda-\lambda_j))\phi_j(x)\phi_j(y) = \frac{2}{T}\left(\widehat{\rho}\left(\frac{t}{T}\right)e(t,x,y)\right)^{\vee}(\lambda)+O(\lambda^{-\infty}),$$

Here ρ is a Schwartz function, and T is a parameter.

The idea is to take $T\to\infty$ as $\lambda\to\infty$. This allows to capture a **growing** number of **singularities** on the right–hand side as $\lambda\to\infty$.

For instance, to prove lower bounds (6) and (7) we take $T\sim\log\log\lambda$.

How fast can T grow?

The number of closed geodesics (geodesic loops at x, geodesic segments joining x and y) of length $\leq T$ grows **exponentially** in T on a negatively curved manifold (**Margulis** '69).

Therefore, the formula on the previous slide captures $O(\exp(O(T))$ singularities.

Each singularity can be studied using the *stationary phase method*, yielding an **error** which is **polynomial** in λ. We get **accumulation** of errors!

This problem forces us to choose $T \leq \alpha \log \lambda$ for a small enough constant α. In particular, the **Ehrenfest time** scale $T \sim \log \lambda$ gives Berard's bound (8). Getting over the logarithmic time barrier would allow to obtain better remainder estimates.

5. Average growth and almost periodic properties of the spectral function

The **spectral function** and the **error terms** in Weyl's law are **oscillating** functions of λ. Our goal is to describe their behavior as $\lambda \to \infty$.

As we have seen, it is often hard to obtain good bounds on individual **amplitudes** of the oscillations.

Let us estimate the amplitudes **on average** and find the **frequencies** of the oscillations.

In **Lapointe–P.–Safarov** '08 we study **average** growth of the spectral function **off** the diagonal. The following result holds:

For every **finite** measure ν on $\mathbb{R}_+$ and each fixed $x \in M$, there exists a subset $M_{x,\nu} \subset M$ of **full** measure such that

$$\int_0^\infty \left| \frac{N_{x,y}(\lambda)}{1+\lambda^{\frac{n-1}{2}}} \right|^2 d\nu(\lambda) < \infty, \quad \forall y \in M_{x,\nu}. \qquad (9)$$

Note that (9) does **not** imply $N_{x,y}(\lambda) = O\left(\lambda^{\frac{n-1}{2}}\right)$ for all x and almost all y: indeed, for **all** x, y on a negatively curved manifold,

$$N_{x,y}(\lambda) = \Omega\left(\lambda^{\frac{n-1}{2}} (\log \lambda)^{C-\delta}\right), \; x \neq y$$

for any $\delta > 0$ (**Jakobson-P.** '07).

What are the **frequencies** of oscillations of the **spectral function**?

As **Kosygin–Minasov–Sinai** '93 and **Bleher** '94, who studied $R(\lambda)$ in the *integrable* case, we use the notion of **almost periodic functions**.

The space B^p of **Besicovitch** almost periodic functions is the completion of the space of all finite trigonometric sums

$$\sum_{k=1}^{N} a_k e^{i\omega_k x},$$

$a_k \in \mathbb{C}, \omega_k \in \mathbb{R}$, with respect to the seminorm

$$||f||_{B^p} = \limsup_{T\to\infty} \left(\frac{1}{2T}\int_{-T}^{T} |f(x)|^p dx\right)^{1/p}, \quad p \geq 1$$

We write $f \sim g$ if $||f-g||_{B^p} = 0$.

In **Lapointe—P.–Safarov** '08 we formulate the following **question**:

Let M be a **surface**. Does there exist $p \geq 1$, such that for all $x \in M$ and almost all $y \in M$,

$$\frac{N_{x,y}(\lambda)}{1+\lambda^{\frac{1}{2}}} \sim \frac{2}{(2\pi)^{\frac{3}{2}}} \sum_{\gamma \in \Gamma_{x,y}} \frac{\sin(\lambda\, l(\gamma) - \frac{\pi}{4} - m(\gamma)\frac{\pi}{2})}{l(\gamma)\sqrt{|J(l(\gamma)|}} \qquad (10)$$

in the Besicovitch space B^p ?

Here $\Gamma_{x,y}$ is the set of **all** geodesic segments between x and y, $l(\gamma)$ is the **length** of γ, $m(\gamma)$ is the **Morse index** of γ and $J(t)$ is the orthogonal **Jacobi field** along γ with the initial conditions $J(0) = 0$, $J'(0) = 1$.

Formula (10) suggests that the **frequencies** of the **spectral function** are the **lengths** of geodesic segments joining x and y.

Remarks

1. Formula (10) holds for **round spheres** and **flat tori** with $p = 2$. We believe the same is true for *surfaces of revolution* and *Liouville tori*, as well as for *negatively curved* surfaces: in the latter case, just a *tiny* improvement of the **Ehrenfest** time scale would give the result.

2. The right–hand side of (10) is well-defined for all x and y that are **not conjugate** along **any** geodesic segment joining them, which is true for all x and almost all y on any surface.

3. A positive answer to our question implies that the **rescaled spectral function** has a *limit distribution*.

4. A similar question could be formulated for manifolds of **any** dimension.