Coherent states and quantization of the particle motion on the line, on the circle, on 1 + 1-de Sitter space-time and of more general systems

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Abstract

We describe a quantization procedure based on adapted coherent states. This scheme could reveal itself as an efficient tool for quantizing physical systems for which more traditional methods are uneasy to implement. The procedure is illustrated by simple examples like the well-known quantization of the particle motion on the line and the yet problematic quantization of the particle motion on the circle. Related to the latter problem is the quantization of dynamics of a test particle in two-dimensional de Sitter space, the group of symmetry of which is $SO_0(1,2)$. The realization, along our quantization procedure, of the corresponding principal series representation of $SO_0(1,2)$ seems to be a new result. We shall also present some extensions of the method to more elaborate mathematical structures : quantization of Grassmann structures by using Daoud-Kibler coherent states; coherent-state approach to the fuzzy sphere; Ashtekar polymer particle representation.

1 Introduction: a continuous frame for the plane

Everyone is familiar with the usual orthonormal frame of the Euclidean plane $\mathbb{R}^2$. This frame is defined by two vectors (in Dirac ket notations): $|0\rangle$ and $|\frac{\pi}{2}\rangle$ sucht that

$$\langle 0|0\rangle = 1 = \langle \frac{\pi}{2}|\frac{\pi}{2}\rangle,\ \langle 0|\frac{\pi}{2}\rangle = 0,$$

and such that the sum of their corresponding orthogonal projectors *solves the identity*

$$\mathbb{I} = |0\rangle\langle 0| + |\frac{\pi}{2}\rangle\langle\frac{\pi}{2}|. \tag{1}$$

This is a trivial reinterpretation of the matrix identity:

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2}$$

Let us now consider the unit vector with polar angle $\theta \in [0, 2\pi)$:

$$|\theta\rangle = \cos\theta|0\rangle + \sin\theta|\frac{\pi}{2}\rangle. \tag{3}$$

Its corresponding orthogonal projector is given by:

$$|\theta\rangle\langle\theta| = \begin{pmatrix}\cos\theta \\ \sin\theta\end{pmatrix}\begin{pmatrix}\cos\theta & \sin\theta\end{pmatrix} = \begin{pmatrix}\cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta\end{pmatrix}. \tag{4}$$

The θ dependent superposition (3) can also be viewed as a *coherent state* superposition in the sense one understands this word in quantum physics, more precisely in quantum optics, the field in which it was introduced by Glauber [2] (see also [1]). One of the most salient features of coherent states lies in the fact that they solve the identity in the vector space they belong to. Indeed, integrating the matrix elements of (4) over all angles and dividing by π leads to a *continuous* analogue of (1)

$$\frac{1}{\pi}\int_0^{2\pi} d\theta|\theta\rangle\langle\theta| = \mathbb{I}. \tag{5}$$

We thus have obtained a continuous frame for the plane, that is to say the continuous set of unit vectors forming the unit circle, for describing, with an extreme redundancy, the euclidean plane. The operator relation (5) is equally understood through its action on a vector $|v\rangle = \|v\|\,|\phi\rangle$ with polar coordinates $\|v\|, \phi$. By virtue of $\langle\theta|\theta'\rangle = \cos(\theta - \theta')$ we have :

$$|v\rangle = \frac{\|v\|}{\pi}\int_0^{2\pi} d\theta\, \cos(\phi - \theta)|\theta\rangle, \tag{6}$$

a relation which illustrates the *overcompleteness* of the family $\{|\theta\rangle\}$. The vectors of this family are not linearly independent, and their mutual “overlappings” are given by the scalar products $\langle\phi|\theta\rangle = \cos(\phi - \theta)$.

Another way to understand the above continuous frame is to consider the isometric embedding of the euclidean plane into the Hilbert space of real trigonometric Fourier series, *i.e.* the Hilbert space $L^2(S^1)$ of real-valued square-integrable functions on the circle, endowed with the scalar product

$$\langle f|g\rangle_{L^2} = \frac{1}{\pi}\int_0^{2\pi} f(\theta)g(\theta)\, d\theta. \tag{7}$$

Let $|v\rangle$ be a vector with polar coordinates r, ϕ. There corresponds to $|v\rangle$ the function

$$v(\theta) = \langle\theta|v\rangle = r\cos(\phi - \theta), \tag{8}$$

and this is clearly an element of $L^2(S^1)$. We have an isometry since a straightforward application of the resolution of the unity (5) leads to:

$$\langle v_1|v_2\rangle_{L^2} = \frac{1}{\pi}\int_0^{2\pi} v_1(\theta)v_2(\theta)\, d\theta = \frac{1}{\pi}\int_0^{2\pi}\langle v_1|\theta\rangle\langle\theta|v_2\rangle\, d\theta = \langle v_1|v_2\rangle. \tag{9}$$

At this point we can adopt another point of view: starting from the circle S^1 provided with its usual measure $d\theta$, we consider all possible real-valued square-integrable functions forming $L^2(S^1)$. Then we choose the orthonormal system composed of the two fundamental *harmonics*, $\cos\theta$ and $\sin\theta$, we build the superposition (3) and we obtain the resolution (5) which can also be viewed as the orthogonal projector mapping $L^2(S^1)$ onto the two-dimensional subspace spanned by $\cos\theta$ and $\sin\theta$ identified to $|0\rangle$ and $|\frac{\pi}{2}\rangle$ respectively. This example is certainly the simplest one among those we can exhibit in order to become familiar with the formalism of the next sections.

2 A standard example: quantization of the motion of a particle on the line

We now enter Quantum Physics by considering one of its pedagogical models, namely the quantum version of the particle motion on the real line. On the classical level, the corresponding phase space is $\mathbb{R}^2 \simeq \mathbb{C} = \{z = \frac{1}{\sqrt{2}}(q+ip)\}$ (in complex notation and with suitable physical units). Let us provide it with the Gaussian measure $\mu(dz\, d\bar{z}) \equiv \frac{1}{\pi}\, e^{-|z|^2}\, d^2z$ where $d^2z = dx\, dy$ is the Lebesgue measure on the plane $\{z = x+iy\}$. Strictly included in the Hilbert space $L^2(\mathbb{C}, \mu(dz\, d\bar{z}))$ of all complex-valued functions on the complex plane which are square-integrable with respect to this Gaussian measure, there is the so-called *Fock-Bargmann Hilbert* subspace $\mathcal{H}$ of all square integrable functions which are analytical entire. An obvious orthonormal basis of this subspace $\mathcal{H}$ is formed of the normalized powers of the complex variable z, *i.e.* $\phi_n(z) \equiv \frac{z^n}{\sqrt{n!}}$ with $n \in \mathbb{N}$. Then, let us consider the following infinite linear superposition in $\mathcal{H}$

$$|z\rangle = e^{-\frac{|z|^2}{2}} \sum_{n\in\mathbb{N}} \frac{z^n}{\sqrt{n!}} |n\rangle, \tag{10}$$

in which we have adopted the Fock notation $\{|n\rangle \equiv \phi_n\}_{n\in\mathbb{N}}$. This notation is not fortuitous because these states, as eigenstates of the Euler or *number operator*, $N = z\frac{\partial}{\partial z}$, are nothing but the eigenstates of the harmonic oscillator in the Fock-Bargmann representation. From $\sum_n \frac{|z|^2}{n!} = e^{|z|^2}$ and from the orthonormality of the set $\{|n\rangle\}_{n\in\mathbb{N}}$ one easily checks two fundamental features of the states (10), namely normalisation and unity resolution:

$$\langle z\,|z\rangle = 1, \quad \frac{1}{\pi}\int_{\mathbb{C}} |z\rangle\langle z|\, d^2z = \mathbb{I}_{\mathcal{H}}. \tag{11}$$

The states (10) are the well-known Klauder-Glauber-Sudarshan coherent states [2, 3], actually discovered in the early days of Quantum Mechanics by Schrödinger [4] and christened *coherent* by Glauber within the context of quantum optics. Among numerous interesting properties enjoyed by them, a fundamental one is their quality of being *canonical quantizers* [5]. By this we mean that any classical observable f, that is a (usually supposed smooth) function of phase space variables (q,p) or equivalently of $(z,\bar{z})$, is transformed through the operatorial integral

$$\frac{1}{\pi}\int_{\mathbb{C}} f(z,\bar{z})\, |z\rangle\langle z|\, d^2z = A_f, \tag{12}$$

into an operator A_f acting on the Hilbert space $\mathcal{H}$ of quantum states. We thus have for the most basic one,

$$\frac{1}{\pi}\int_{\mathbb{C}} z\, |z\rangle\langle z|\, d^2z = \sum_n \sqrt{n+1}|n\rangle\langle n+1| \equiv a, \tag{13}$$

which is the lowering operator, $a|n\rangle = \sqrt{n}|n-1\rangle$. Its adjoint $a^\dagger$ is obtained by replacing z by $\bar{z}$ in (13), and we get the factorisation $N = a^\dagger a$ for the number operator, together with the commutation rule $[a, a^\dagger] = \mathbb{I}_{\mathcal{H}}$. Also note that $a^\dagger$ and a are realized on $\mathcal{H}$ as multiplication operator and derivation operator respectively, $a^\dagger f(z) = zf(z)$, $af(z) = df(z)/dz$. From $q = \frac{1}{\sqrt{2}}(z+\bar{z})$ et $p = \frac{1}{\sqrt{2}i}(z-\bar{z})$, one easily infers by linearity that the canonical position q and momentum p map to the quantum observables $\frac{1}{\sqrt{2}}(a+a^\dagger) \equiv Q$ and $\frac{1}{\sqrt{2}i}(a-a^\dagger) \equiv P$ respectively. In consequence, the self-adjoint operators Q and P obey the canonical commutation rule $[Q,P] = i\mathbb{I}_{\mathcal{H}}$, and for this reason fully deserve the name of position and momentum operators of the usual (galilean) quantum mechanics, together with all localisation properties specific to the latter. In this context, it is worthy to recall what *quantization of classical mechanics* does mean in a commonly accepted sense (for a recent and complete review see [6]). In the above we have chosen units such that the Planck constant is just put equal to 1. Here we reintroduce it since it parametrizes the link between classical and quantum mechanics.

Van Hove canonical quantization rules [7]

Given a phase space with canonical coordinates $(\boldsymbol{q}, \boldsymbol{p})$

- to the classical observable $f(\boldsymbol{q}, \boldsymbol{p}) = 1$ corresponds the identity operator in the (projective) Hilbert space $\mathcal{H}$ of quantum states,
- the correspondence that assigns to a classical observable $f(\boldsymbol{q}, \boldsymbol{p})$ a (essentially) self-adjoint operator on $\mathcal{H}$ is a linear map,
- to the classical Poisson bracket corresponds, <u>at least at the order</u> $\hbar$, the quantum commutator, multiplied by $i\hbar$:

$$\begin{aligned} &\text{with } f_j(\boldsymbol{q}, \boldsymbol{p}) \rightarrow A_{f_j} \text{ for } j = 1, 2, 3 \\ &\text{we have } \{f_1, f_2\} = f_3 \rightarrow [A_{f_1}, A_{f_2}] = i\hbar A_{f_3} + o(\hbar), \end{aligned}$$

- some conditions of minimality on the resulting observable algebra.

We now have reached the point at which we can think about a general formalism underlying the above coherent state construction and quantization procedure. The approach we are going to follow from now on is mainly based on Refs.[8, 9].

3 Coherent state quantization

Quantum Mechanics and Signal Analysis have many aspects in common. As a departure point of their respective formalism, one finds a *raw* set X of basic parameters or data that we denote by x. This set may be a classical phase space in the former case, like the complex plane for the particle motion on the line, whereas it might be a temporal line or a time-frequency half-plane in the latter one. Actually it can be any set of data accessible to observation, and the minimal significant structure one requires so far is the existence of a measure $\mu(dx)$ on X. As a measure space, X will be given the name of an *observation* set, and the existence of a measure provides us with a statistical reading of the set of all measurable real or complex valued functions $f(x)$ on X: it allows us to compute for instance average values on subsets with bounded measure. Actually, both theories deal with quadratic mean values, and the natural framework of study is the Hilbert space $L^2(X, \mu)$ of all square-integrable functions $f(x)$ on the observation set X: $\int_X |f(x)|^2 \, \mu(dx) < \infty$. The function f is referred to as *finite-energy* signal in Signal Analysis and as quantum state in Quantum Mechanics. However, it is precisely at this stage that "quantum processing" of X differs from signal processing in at least three points:

1. not all square-integrable functions are eligible as quantum states,
2. a quantum state is defined up to a nonzero factor,
3. among the functions $f(x)$ those that are eligible as quantum states and that are of unit norm, $\int_X |f(x)|^2 \, \mu(dx) = 1$, give rise to a probabilistic interpretation: the correspondence $X \supset \Delta \rightarrow \int_\Delta |f(x)|^2 \mu(dx)$ is a probability measure which is interpreted in terms of localisation in the measurable set Δ and which allows to determine mean values of quantum observables, (essentially) self-adjoint operators defined in a domain that is included in the set of quantum states.

The first point lies at the heart of the *quantization* problem: what is the more or less canonical procedure allowing to select quantum states among simple signals? In other words, how to select the true (projective) Hilbert space of quantum states, denoted by $\mathcal{H}$, *i.e.* a closed subspace of $L^2(X, \mu)$, or equivalently the corresponding orthogonal projector $\mathbb{I}_{\mathcal{H}}$?

This problem can be solved if one finds a map from X to $\mathcal{H}$, $x \rightarrow |x\rangle \in \mathcal{H}$ (in Dirac notation), defining a family of states $\{|x\rangle\}_{x\in X}$ obeying the following two conditions:

- **Normalisation**

$$\langle\, x \,|x\rangle = 1, \tag{14}$$

- **Resolution of the unity in $\mathcal{H}$**

$$\int_X |x\rangle\langle x|\, \nu(dx) = \mathbb{I}_{\mathcal{H}}, \tag{15}$$

where $\nu(dx)$ is another measure on X, usually absolutely continuous with respect to $\mu(dx)$: this means that there exists a positive measurable function $h(x)$ such that $\nu(dx) = h(x)\mu(dx)$.

The quantization of a *classical* observable, that is to say of a function $f(x)$ on X having specific properties with respect to some supplementary structure allocated to X, (like topology, geometry or something else), simply consists of associating to the function $f(x)$ the operator defined by

$$F = \int_X f(x)|x\rangle\langle x|\, \nu(dx). \tag{16}$$

In this context, the function $f(x)$ is called upper symbol of the operator F by Lieb [10] (or contravariant by Berezin [5]), whereas the mean value $\langle x|F|x\rangle$ is said lower (or covariant) symbol of F [5]. One can say that, according to this approach, that a quantization of the observation set is in one-to-one correspondence with the choice of a frame in the sense of (14) and (15). To a certain extent, a quantization scheme consists of adopting a certain point of view in dealing with X. This frame can be discrete or continuous, depending on the topology furthermore allocated to the set X, and it can be overcomplete, of course. The validity of a precise frame choice is determined by comparing spectral characteristics of quantum observables F with experimental data. Of course, such a particular quantization scheme, associated to a specific frame, is intrinsically limited to all those classical observables for which the expansion (16) is mathematically justified within the theory of operators in Hilbert space (*e.g.* weak convergence). However, it is well known that limitations hold for *any* quantization scheme.

The explicit construction of a frame as well as its physical relevance are clearly crucial. It is remarkable that signal and quantum formalisms meet again on this level, since the frame is called, in a wide sense, *wavelet* family [11] or *coherent state* family [12] according to the practitioner's filiation. Two methods for constructing such families are generally in use. The first one rests upon group representation theory: a specific state or *probe*, say $|x_0\rangle$, is transported along the orbit $\{|g \cdot x_0 \equiv x\rangle\}_{g\in G}$ by the action of a group G for which X is a homogeneous space. Irreducibility (Schur's Lemma) and unitarity conditions, combined with square integrability of the representation in some restricted sense, automatically lead to properties (14) and (15). Various examples of such group-theoretical constructions are given in [13, 12]. The second method has a wave packet flavor in the sense that the state $|x\rangle$ is obtained from some superposition of elements in a fixed family of states $\{|\lambda\rangle\}_{\lambda\in\Lambda}$ which is total in $\mathcal{H}$:

$$|x\rangle = \int_\Lambda |\lambda\rangle\, \sigma(x, d\lambda). \tag{17}$$

Here, the complex-valued x-dependent measure σ has its support Λ contained in the support of the spectral resolution $E(d\lambda)$ of a certain self-adjoint operator A, and the $|\lambda\,\rangle$'s are precisely eigenstates of A: $A|\lambda\,\rangle = \lambda|\lambda\,\rangle$. More precisely, they can be eigenstates in a distributional sense in so as to put into the game of the construction portions belonging to the possible continuous part of the spectrum of A. Examples of such wave-packet constructions are given in [14, 15, 16, 17, 18], and we shall follow a similar procedure in the present paper.

For pedagogical purposes, we now suppose that A has a only discrete spectrum, say $\{a_n, 0 \leq n \leq L\}$, with L finite or infinite. Normalised eigenstates are denoted by $|n\rangle$ (Fock notation) and they form an orthonormal basis of $\mathcal{H}$. We also introduce the number operator N so that, for any "reasonable" function $f(N)$ of it, we have $f(N)|n\rangle = f(n)|n\rangle$. Now, suppose that the basis $\{|n\rangle\}_{n\in\mathbb{N}}$ is in one-to-one correspondence with an orthonormal set $\{\phi_n(x)\}_{n\in\mathbb{N}}$ of elements of $L^2(X, \mu)$. Furthermore, and this a decisive step in the wave packet construction, we assume that

$$\mathcal{N}(x) \equiv \sum_n |\phi_n(x)|^2 < \infty \text{ almost everywhere on } X. \tag{18}$$

Then, the states

$$|x\rangle \equiv \frac{1}{\sqrt{\mathcal{N}(x)}} \sum_n \phi_n^*(x)|n\rangle, \tag{19}$$

satisfy both of our requirements (14) and (15). Indeed, the normalisation is automatically ensured because of the orthonormality of the set $\{|n\rangle\}$ and the presence of the normalisation factor (18). The resolution of the unity in $\mathcal{H}$ holds by virtue of the orthonormality of the set $\{\phi_n(x)\}$ if $\nu(dx)$ is related to $\mu(dx)$ by

$$\nu(dx) = \mathcal{N}(x)\mu(dx). \tag{20}$$

4 Less standard example: motion of a particle on the circle

Quantization of the motion of a particle on the circle (like the quantization of polar coordinates in the plane) is an old question with so far unsatisfying answers. A large literature exists concerning this subject, more specifically devoted to the problem of angular localisation and related Heisenberg inequalities, see for instance [19].

Let us apply our scheme of coherent state quantization to this particular problem. We just have to follow the steps listed below:

4.1 De Bièvre-Gonzalez coherent states

- The observation set X is the cylinder $S^1 \times \mathbb{R} = \{x \equiv (\beta, J), \mid 0 \leq \beta < 2\pi,\ J \in \mathbb{R}\}$, *i.e.* the phase space of a particle moving on the circle.
- The real J and β are canonically conjugate variables and $dJ\,d\beta$ is the measure invariant with respect to canonical transformations.
- The measure on X is partly Gaussian, $\mu(dx) = \sqrt{\frac{\epsilon}{\pi}}\,\frac{1}{2\pi}e^{-\epsilon J^2}\,dJ\,d\beta$ where $\epsilon > 0$ can be arbitrarily small. This parameter could be viewed as the analogue of the Planck constant. Actually it represents a regularization.
- The functions $\phi_n(x)$ forming the orthonormal system needed to construct coherent states are suitably weighted Fourier exponentials:

$$\phi_n(x) = e^{(-\epsilon n^2/2)}\, e^{n(\epsilon J + i\beta)},\ n \in \mathbb{Z} \tag{21}$$

- The normalisation factor

$$\mathcal{N}(x) \equiv \mathcal{N}(J) = \sum_{n\in\mathbb{Z}} e^{(-\epsilon n^2)}\, e^{2n\epsilon J} < \infty \tag{22}$$

  is proportional to an elliptic Theta function.
- Coherent states read

$$|J, \beta\rangle = \frac{1}{\sqrt{\mathcal{N}(J)}} \sum_{n\in\mathbb{Z}} e^{(-\epsilon n^2/2)}\, e^{n(\epsilon J - i\beta)}|n\rangle, \tag{23}$$

  where the states $|n\rangle$'s, in one-to-one correspondence with the ϕ_n's, form an othonormal basis of some separable Hilbert space $\mathcal{H}$. For instance, they can be considered as Fourier exponentials $e^{in\beta}$ forming the orthonormal basis of the Hilbert space $L^2(S^1) \simeq \mathcal{H}$. They are the *spatial modes* in this representation.

The coherent states (23) have been proposed by De Bièvre-González (1992-93) [20], González-Del Olmo (1998) [21], Kowalski-Rembieliński-Papaloucas (1996) [22].

The quantization of the observation set is hence achieved by selecting in the original Hilbert space $L^2(S^1 \times \mathbb{R}, \sqrt{\frac{\epsilon}{\pi}}\,\frac{1}{2\pi}e^{-\epsilon J^2}\,dJ\,d\beta)$ *all Laurent series in the complex variable* $z = e^{\epsilon J - i\beta}$*, and this is the choice of polarization [23] leading to our quantization.*

4.2 Quantization of classical observables

- By virtue of (16) and (20), the quantum operator (acting on $\mathcal{H}$) associated to the classical observable $f(x)$ is obtained by

$$A_f := \int_X f(x)|x\rangle\langle x|\,\mathcal{N}(x)\mu(dx). \tag{24}$$

- For the most basic one, associated to the classical observable J, this yields

$$A_J = \int_X \mu(dx)\mathcal{N}(J)\, J\,|J,\beta\rangle\langle J,\beta| = \sum_{n\in\mathbb{Z}} n\,|n\rangle\langle n|, \tag{25}$$

  and this is nothing but the angular momentum operator, which reads in angular position representation (Fourier series): $A_J = -i\frac{\partial}{\partial\beta}$.
- For an arbitrary function $f(\beta)$, we have

$$A_{f(\beta)} = \int_X \mu(dx)\mathcal{N}(J)f(\beta)\,|J,\beta\rangle\langle J,\beta| \tag{26}$$

$$= \sum_{n,n'\in\mathbb{Z}} e^{-\frac{\epsilon}{4}(n-n')^2}\, c_{n-n'}(f)|n\rangle\langle n'|, \tag{27}$$

  where $c_n(f)$ is the nth Fourier coefficient of f. In particular, we have for the
- operator "angle" :

$$A_\beta = \pi\mathbb{I}_{\mathcal{H}} + \sum_{n\neq n'} i\frac{e^{-\frac{\epsilon}{4}(n-n')^2}}{n-n'}\,|n\rangle\langle n'|, \tag{28}$$

- operator "Fourier fundamental harmonic" :

$$A_{e^{i\beta}} = e^{-\frac{\epsilon}{4}}\sum_n |n+1\rangle\langle n|. \tag{29}$$

- In the isomorphic realisation of $\mathcal{H}$ in which the kets $|n\rangle$ are the Fourier exponentials $e^{i\,n\beta}$: $A_{e^{i\beta}}$ is multiplication operator by $e^{i\beta}$ up to the factor $e^{-\frac{\epsilon}{4}}$ (which is arbitrarily close to 1).

4.3 Did you say *canonical* ?

- The "canonical" commutation rule

$$[A_J, A_{e^{i\beta}}] = A_{e^{i\beta}}$$

  is canonical in the sense that it is in exact correspondence with the classical Poisson bracket

$$\left\{J, e^{i\beta}\right\} = ie^{i\beta}$$

  It is actually the only non trivial commutator having this exact correspondence.
- There could be interpretational difficulties with commutators of the type :

$$[A_J, A_{f(\beta)}] = \sum_{n,n'} \frac{e^{-\frac{\epsilon}{4}(n-n')^2}}{n-n'}\, c_{n-n'}(f)\,|n\rangle\langle n'|,$$

- in particular for the angle operator:

$$[A_J, A_\beta] = i\sum_{n\neq n'} e^{-\frac{\epsilon}{4}(n-n')^2}\,|n\rangle\langle n'|, \tag{30}$$

  to be compared with the classical $\{J,\beta\} = 1$!

Actually, these difficulties are only apparent ones and are due to the discontinuity of the 2π-periodic function $B(\beta)$ which is equal to β on $[0, 2\pi)$. They can be circumvented if we examine, for instance the behaviour of the corresponding lower symbols at the limit $\epsilon \to 0$. For the angle operator,

$$\langle J_0, \beta_0 | A_\beta | J_0, \beta_0 \rangle = \pi + \left(1 + e^{\epsilon(J_0 - \frac{1}{4})} \frac{\mathcal{N}(J_0 - \frac{1}{2})}{\mathcal{N}(J_0)} \right) \sum_{n \neq 0} \frac{e^{-\frac{\epsilon}{2} n^2 + in\beta_0}}{n}$$

$$\underset{\epsilon \to 0}{\sim} \pi + \sum_{n \neq 0} \frac{e^{in\beta_0}}{n}, \tag{31}$$

where we recognize at the limit the Fourier series of $B(\beta_0)$. For the commutator,

$$\langle J_0, \beta_0 | [A_J, A_\beta] | J_0, \beta_0 \rangle = -i \left(1 + e^{\epsilon(J_0 - \frac{1}{4})} \frac{\mathcal{N}(J_0 - \frac{1}{2})}{\mathcal{N}(J_0)} \right) + \sum_{n \in \mathbb{Z}} e^{-\frac{\epsilon}{2} n^2 + in\beta_0}$$

$$\underset{\epsilon \to 0}{\sim} -i + \sum_n \delta(\beta_0 - 2\pi n). \tag{32}$$

So we (almost) recover the canonical commutation rule except for the singularity at the origin $\bmod 2\pi$.

5 From the motion of the circle to the motion on 1+1-de Sitter space-time

The material of the previous section is now used to describe the quantum motion of a massive particle on a 1+1-de Sitter background, which means a one-sheeted hyperboloid embedded in a 2+1-Minkowski space. Here, we just summarize the content of Reference [24]. The phase space X is also a one-sheeted hyperboloid:

$$J_1^2 + J_2^2 - J_0^2 = \kappa^2 > 0, \tag{33}$$

with (local) canonical coordinates (J, β), as for the motion on the circle. Phase space coordinates are now viewed as basic classical observables,

$$J_0 = J, \; J_1 = J \cos\beta - \kappa \sin\beta, \; J_2 = J \sin\beta + \kappa \cos\beta, \tag{34}$$

and obey the Poisson bracket relations

$$\{J_0, J_2\} = -J_1, \; \{J_0, J_1\} = -J_1, \; \{J_1, J_2\} = J_0. \tag{35}$$

They are, as expected, the commutation relations of $so(1,2) \simeq sl(2, \mathbb{R})$, which is the kinematical symmetry algebra of the system. Applying the coherent states quantization (24) at $\epsilon \neq 0$ produces the basic quantum observables:

$$A_{J_0} = \sum_n n |n\rangle\langle n|, \tag{36}$$

$$A^\epsilon_{J_1} = \frac{1}{2} e^{-\frac{\epsilon}{4}} \sum_n (n + \frac{1}{2} + i\kappa) |n+1\rangle\langle n| + \text{ cc}, \tag{37}$$

$$A^\epsilon_{J_2} = \frac{1}{2i} e^{-\frac{\epsilon}{4}} \sum_n (n + \frac{1}{2} + i\kappa) |n+1\rangle\langle n| - \text{ cc}. \tag{38}$$

The quantization is asymptotically exact for these basic observables since

$$[A_{J_0}, A^\epsilon_{J_1}] = iA^\epsilon_{J_2}, \; [A_{J_0}, A^\epsilon_{J_2}] = -iA^\epsilon_{J_1}, \; [A^\epsilon_{J_1}, A^\epsilon_{J_2}] = -ie^{-\frac{\epsilon}{4}} A_{J_0}. \tag{39}$$

Moreover, the quadratic operator

$$C^\epsilon = (A^\epsilon_{J_1})^2 + (A^\epsilon_{J_2})^2 - (A_{J_0})^2 = \sum_n (e^{-\frac{\epsilon}{4}}(n^2 + \kappa^2 + \frac{1}{4}) - n^2)|n\rangle\langle n|, \tag{40}$$

admits the limit $C^\epsilon \underset{\epsilon\to 0}{\sim} (\kappa^2 + \frac{1}{4})\mathbb{I}$. Hence we have produced a coherent states quantization which leads asymptotically to the principal series of $SO_0(1,2)$.

6 Quantization with Daoud-Kibler k-fermionic coherent states

As a conclusion to this paper, let us sketch an application of our quantization procedure to a more elaborate mathematical structure, namely a Grassmann algebra for which coherent states have recently been proposed by Daoud and Kibler [25]. The construction of these coherent states fit the scheme described in Section 3:

- The observation set X is the Grassmann algebra Σ_k. The latter is defined as the linear span of $\{1, \theta, \cdots, \theta^{k-1}\}$ and their respective conjugates $\bar{\theta}^i$: here θ is a Grassmann variable satisfying $\theta^k = 0$.
- The measure on X is

$$\mu(d\theta d\bar{\theta}) = d\theta\, w(\theta, \bar{\theta})\; d\bar{\theta}. \tag{41}$$

  Here, the integral over $d\theta$ and $d\bar{\theta}$ should be understood in the sense of Berezin-Majid-Rodríguez-Plaza integrals:

$$\int d\theta\, \theta^n = 0 = \int d\bar{\theta}\, \bar{\theta}^n, \text{ for } n = 0, 1, \cdots, k-2$$

$$\int d\theta\, \theta^{k-1} = 1 = \int d\bar{\theta}\, \bar{\theta}^{k-1}.$$

  The "weight" $w(\theta, \bar{\theta})$ is given by the q-deformed polynomial

$$w(\theta, \bar{\theta}) = \sum_{n=0}^{k-1} (([n]_q!([n]_{\bar{q}}!)^{\frac{1}{2}}\, \theta^{k-1-n}\bar{\theta}^{k-1-n},$$

  where

$$[n]_q! = [1]_q[2]_q \cdots [n]_q, \text{ with } [x]_q := \frac{1-q^x}{1-q},$$

  and $q = e^{\frac{2\pi i}{k}}$ is a root of unity.
- Introduce an orthonormal basis $\{|n\rangle\}$ of the Hilbert space $\mathbb{C}^k$.
- The (nonnormalized) DK coherent states should be understood as elements of $\mathbb{C}^k \otimes \Sigma_k$. They read as

$$|\theta) = \sum_{n=0}^{k-1} \frac{\theta^n}{([n]_q!)^{\frac{1}{2}}}\, |n\rangle. \tag{42}$$

- The Daoud-Kibler quantization of the "spinorial" Grassman algebra rests upon the resolution of the unity $\mathbb{I}$ in $\mathbb{C}^k$:

$$\int\int d\theta|\theta)\, w(\theta, \bar{\theta})\, (\theta|\, d\bar{\theta} = \mathbb{I}$$

- The quantization of a Grassmann-valued function $f(\theta, \bar{\theta})$ maps f to the linear operator A_f on $\mathbb{C}^k$

$$A_f = \int\int d\theta|\theta)\, f(\theta, \bar{\theta}) w(\theta, \bar{\theta})\, (\theta|\, d\bar{\theta}.$$

Here we actually recover the $k \times k$-matrix realization of the so-called *k-fermionic algebra* F_k [25]. For instance, we have for the simplest functions:

$$A_\theta = \sum_{n=0}^{k-1} ([n+1]_q)^{\frac{1}{2}} |n\rangle\langle n+1|, \; A_{\bar\theta} = \sum_{n=0}^{k-1} ([n+1]_{\bar q})^{\frac{1}{2}} |n+1\rangle\langle n| = A_\theta^\dagger. \tag{43}$$

Their (anti-)commutator reads as

$$[A_\theta, A_{\bar\theta}] = \sum_{n=0}^{k-1} \frac{\cos \pi \frac{2n+1}{2k}}{\cos \frac{\pi}{2k}} |n\rangle\langle n|, \; \{A_\theta, A_{\bar\theta}\} = \sum_{n=0}^{k-1} \frac{\sin \pi \frac{2n+1}{2k}}{\sin \frac{\pi}{2k}} |n\rangle\langle n|. \tag{44}$$

In the purely fermionic case, $k = 2$, we recover the canonical anticommutation rule $\{A_\theta, A_{\bar\theta}\} = \mathbb{I}_2$. More details will be given in a forthcoming paper [26].

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References

[1] Gazeau, J-P : Il y a différentes manières de prendre position, *L'espace physique, entre mathématiques et philosophie*, Cargese, 2001, Ed. M. Lachièze-Rey, to appear (Éditions de Physique) 2004.

[2] Klauder, J. R. and Skagerstam B.-S. (eds.): Coherent states. Applications in physics and mathematical physics. World Scientific Publishing Co., Singapore, 1985.

[3] Klauder, J. R. : Continuous-Representation Theory I. Postulates of continuous-representation theory, *J. Math. Phys.* **4** (1963), 1055–1058; Continuous-Representation Theory II. Generalized relation between quantum and classical dynamics, *J. Math. Phys.* **4** (1963), 1058–1073.

[4] Schrödinger, E. : Der stetige Übergang von der Mikro- zur Makromechanik. *Naturwiss.***14** (1926), 664-666.

[5] Berezin, F. A. : General concept of quantization, *Comm. Math. Phys.* **40** (1975), 153–174.

[6] Ali S. T. and Englis M. : Quantization methods: a guide for physicists and analysts, *to appear*

[7] Van Hove, L. : Sur le problème des relations entre les transformations unitaires de la Mécanique quantique et les transformations canoniques de la Mécanique classique, *Bull. Acad. Roy. Belg., cl. des Sci.* **37** (1961), 610–620.

[8] Garidi T., Gazeau, J-P., Huguet E., Lachièze Rey M., and Renaud J. : Examples of Berezin-Toeplitz Quantization: Finite sets and Unit Interval *Symmetry in Physics. In memory of Robert T. Sharp 2002* eds. P. Winternitz et al (Montréal: CRM Proceedings and Lecture Notes) 2004.

[9] Garidi T., Gazeau, J-P., Huguet E., Lachièze Rey M., and Renaud J. : Quantization of the sphere with coherent states *Classical, Stochastic and Quantum gravity, String and Brane Cosmology, Peyresq 2002* Int. J. Theor. Phys. **42** (2003), 1301–1310.(http://arXiv.org/abs/math-ph/0302056).

[10] Feng, D. H., Klauder, J. R., and Strayer, M. (eds.) : *Coherent States: Past, Present and Future (Proc. Oak Ridge 1993)*, World Scientific, Singapore, 1994.

[11] Daubechies, I. : *Ten lectures on wavelets*, SIAM-CBMS, 1992.

[12] Ali, S. T., Antoine, J.-P., and Gazeau, J.-P. : *Coherent states, wavelets and their generalizations*, Graduate Texts in Contemporary Physics, Springer-Verlag, New York, 2000.

[13] Perelomov, A. M. : *Generalized Coherent States and their Applications*, Springer-Verlag, Berlin, 1986.

[14] Ali, S. T., Antoine, J.-P., and Gazeau, J.-P. : Continuous Frames in Hilbert Spaces, *Ann. of Phys.* **222** (1993), 1–37.

[15] Klauder, J. R. : Quantization without Quantization, *Ann. of Phys.* **237** (1995), 147–160.

[16] Klauder, J. R. : Coherent states for the hydrogen atom, *J. Phys. A: Math. Gen.* **29** (1996), L293–298.

[17] Gazeau, J-P., and Klauder, J. R. : Coherent states for systems with discrete and continuous spectrum, *J. Phys. A: Math. Gen.* **32** (1999), 123–132 .

[18] Gazeau, J-P., and Monceau, P. : Generalized coherent states for arbitrary quantum systems, in *Conférence Moshé Flato 1999 – Quantization, Deformations, and Symmetries*, edited by G. Dito and D. Sternheimer, Vol. II, pp. 131–144, Kluwer, Dordrecht, 2000.

[19] Lévy-Leblond, J. M. : Who is afraid of Nonhermitian Operators? A Quantum Description of Angle and Phase, *Ann. Phys.* **101** (1976), 319–341.

[20] De Bièvre, S., and González, J. A. : Semiclassical behaviour of coherent states on the circle *Quantization and Coherent States Methods in Physics* (Singapore: World Scientific) 1993.

[21] González, J. A., and del Olmo, M. A. : Coherent states on the circle, *J. Phys. A: Math. Gen.* **31** (1998), 8841–8857.

[22] Kowalski, K., Rembieliński, J., and Papaloucas, L. C. : Coherent states for a quantum particle on a circle, *J. Phys. A: Math. Gen.* **29** (1996), 4149–4167.

[23] Woodhouse, N. M. J. *Geometric Quantization*, (Oxford: Clarendon Press) 1992.

[24] Gazeau, J-P., Piechocki, W. : Coherent state quantization of a particle in de Sitter space *submitted.*

[25] Daoud M. and Kibler M. : A fractional supersymmetric oscillator and its coherent states *Proceedings of the International Wigner Symposium, Istanbul, Aout 1999*, Eds. M. Arik et al, Bogazici University Press (Istanbul) (2002).

[26] Gazeau, J-P., Gieres, F., and Kibler M. : Matrix realization of generalized Grassmann algebras via coherent state quantization, *in preparation*