

# Several Complex Variables

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## Abstract

This is a mere sketch of the course being given in the fall of 2004. I shall give the course in French, but if some student so wishes, I shall write on the blackboard in English. In the same eventuality, these lecture notes are in English. Prerequisites for the course are functions of one complex variable, functions of several real variables and topology, all at the undergraduate level.

## 1 Introduction

If the coordinates of  $z \in \mathbb{C}^n$  are given by  $z = (z_1, \dots, z_n)$ , and we write  $z_j = x_j + iy_j$ , where  $x_j = \Re z_j$  and  $y_j = \Im z_j$ , then we denote:

$$x = (x_1, \dots, x_n) = \Re z, \quad y = (y_1, \dots, y_n) = \Im z,$$

$$|z| = \sqrt{\sum |z_j|^2} = \sqrt{\sum (|x_j|^2 + |y_j|^2)} = \sqrt{|x|^2 + |y|^2},$$

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} + i \frac{\partial}{\partial y_j} \right).$$

When speaking, we call these the derivative with respect to  $z_j$  and the derivative with respect to  $\bar{z}_j$  respectively, however, they are not necessarily derivatives. That is, for a  $C^1$  function  $f$ , the expressions  $\frac{\partial f}{\partial z_j}$  and  $\frac{\partial f}{\partial \bar{z}_j}$  are well defined above, but cannot always be expressed as the limit of some differential quotient.

Recall that a function  $f$  defined in an open subset  $\Omega$  of  $\mathbb{C}$  is said to be holomorphic if  $f$  has a derivative at each point of  $\Omega$ . If  $f$  is holomorphic in

an open set  $\Omega$  of  $\mathbb{C}$ , then  $f$  satisfies the Cauchy-Riemann equation  $\partial f/\partial \bar{z} = 0$  in  $\Omega$ . The converse is false. For example, the function  $f$  defined to be 0 at 0 and  $e^{-1/z^4}$  elsewhere satisfies the Cauchy-Riemann equation at all points of  $\mathbb{C}$  but is not holomorphic at 0. However, if  $f \in C^1(\Omega)$ , then  $f$  is holomorphic in  $\Omega$  if and only if it satisfies the Cauchy-Riemann equation.

Let  $\Omega$  be an open subset of  $\mathbb{C}^n$ . A function  $f \in C^1(\Omega)$  is said to be *holomorphic* in  $\Omega$  if it is holomorphic in each variable, thus, if and only if  $f$  satisfies the system of (homogeneous) Cauchy-Riemann equations

$$\frac{\partial f}{\partial \bar{z}_j} = 0, \quad j = 1, \dots, n.$$

It is a deep result of Hartogs, that the condition that  $f$  be in  $C^1(\Omega)$  is superfluous. A function is said to be holomorphic on a subset  $E$  of  $\mathbb{C}^n$  if it is holomorphic in an open neighborhood of  $E$ . In complex analysis, the inhomogeneous system of Cauchy-Riemann equations

$$\frac{\partial f}{\partial \bar{z}_j} = u_j, \quad j = 1, \dots, n,$$

is also important. Loosely speaking, we say that a system of differential equations is *integrable* if the system has a solution. Of course, in order for a solution to exist to the above inhomogeneous system, the functions  $u_j$  must satisfy the following integrability (or compatibility) conditions.

$$\frac{\partial u_j}{\partial \bar{z}_k} = \frac{\partial u_k}{\partial \bar{z}_j} \quad j, k = 1, \dots, n.$$

A function defined in an open subset of  $\mathbb{R}^n$  (respectively  $\mathbb{C}^n$ ) is said to be real (respectively complex) *analytic* if it is locally representable by power series.

**Theorem 1.** *A function is holomorphic iff it is complex analytic.*

**Theorem 2.** *A function is complex analytic iff it is complex analytic in each variable.*

**Problem 1.** *Show Theorem 1 implies Theorem 2.*

**Problem 2.** *Show that the 'real' analog of Theorem 2 is false. This is a big difference between real analysis and complex analysis.*

Let  $f_j$  be holomorphic in a domain  $D_j, j = 1, 2$  and suppose If  $f_1 = f_2$  in some non-empty component  $G$  of  $D_1 \cap D_2$ , then  $f_2$  is said to be a direct holomorphic *continuation* of  $f_1$  through  $G$ . In shorthand, we also say  $(f_2, D_2)$  is a direct holomorphic continuation of  $(f_1, D_1)$ .

Let  $f$  be holomorphic in a domain  $D$  and let  $p \in \partial D$ . We say that  $f$  has a direct holomorphic continuation to  $p$  if there is a holomorphic function  $f_p$  in a neighborhood  $D_p$  of  $p$  such that  $(f_p, D_p)$  is a direct holomorphic continuation of  $(f, D)$  through some component  $G$  of  $D \cap D_p$  with  $p \in \partial G$ .

**Problem 3.** In  $\mathbb{C}$  give an example of a function  $f$  holomorphic in a domain  $D$  and a boundary point  $p$  such that  $f$  has a direct holomorphic continuation to  $p$ . Also, give an example where  $f$  has no direct holomorphic continuation to  $p$ .

A domain  $D$  is a *domain of holomorphy* if it is the 'natural' domain for some holomorphic function. That is, if there is a function  $f$  holomorphic in  $D$  which cannot be directly holomorphically continued to any boundary point of  $D$ . In particular,  $f$  cannot be directly holomorphically continued to any domain which contains  $D$ .

**Problem 4.** Give an example of a domain of holomorphy in  $\mathbb{C}^1$ .

**Problem 5.** Show that each domain in  $\mathbb{C}^1$  is a domain of holomorphy.

**Problem 6.** Give an example of a domain of holomorphy in  $\mathbb{C}^2$ .

The following theorem and its corollary show an important difference between complex analysis in one variable and in several variables.

**Theorem 3** (Hartogs phenomenon). *Let  $\Omega$  be a bounded domain in  $\mathbb{C}^n, n > 1$ . Then, any function holomorphic in a neighborhood of  $\partial\Omega$  has a direct holomorphic continuation to  $\Omega$ .*

**Corollary 4.** *In  $\mathbb{C}^n, n > 1$ , not every domain is a domain of holomorphy.*

**Problem 7.** *Show that the corollary follows from the theorem.*

**Corollary 5.** *Holomorphic functions of more than one variable have no isolated nonremovable singularities.*

**Problem 8.** *Show that the corollary follows from the theorem.*

**Corollary 6.** *Holomorphic functions of more than one variable have no isolated zeros.*

**Problem 9.** *Show that the corollary follows from the previous corollary.*

In  $\mathbb{C}$  there are two domains of particular interest,  $\mathbb{C}$  and the unit disc  $\mathbb{D}$ . The Riemann mapping theorem asserts that each simply connected domain in  $\mathbb{C}$  is equivalent, in the sense of complex analysis, to one of these two domains.

In  $\mathbb{C}^n$ , the analog of the Riemann mapping theorem fails. First of all, there are two natural generalizations of the unit disc, the unit ball  $\mathbb{B}^n = \{z : \|z\| < 1\}$  and the unit polydisc  $\mathbb{D}^n = \{z : |z_j| < 1, j = 1, \dots, n\}$ . Both of these domains are simply connected, but they are *not* equivalent in the sense of complex analysis. Let us be more precise.

A mapping from a domain of  $\mathbb{C}^n$  into  $\mathbb{C}^m$  is said to be *holomorphic* if each of its components is holomorphic. A holomorphic mapping from one domain to another is said to be *biholomorphic* if it is bijective and if the inverse mapping is also holomorphic. The two domains are then said to be *biholomorphically equivalent*. Poincaré has shown that, for  $n > 1$ , the unit polydisc  $\mathbb{D}^n$  and the unit ball  $\mathbb{B}^n$  are not biholomorphic!

The Hartogs phenomenon and the failure of the Riemann mapping theorem, for  $n > 1$ , are two major differences between complex analysis in one variable and in several variables.

## 2 Cauchy Integral Formula

Often, we shall restrict our attention to functions of two complex variables for simplicity.

**Theorem 7.** *Let  $f$  be holomorphic on the closed polydisc  $\overline{\mathbb{D}}^2$ . Then,*

$$f(z_1, z_2) = \frac{1}{(2\pi i)^2} \int_{|\zeta_1|=1} \int_{|\zeta_2|=1} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)} d\zeta_1 d\zeta_2$$

for each  $z \in \mathbb{D}^2$ .

*Proof.* For each fixed  $z_2$  in the unit disc,  $f(z_1, z_2)$  is holomorphic in  $z_1$  for  $z_1$  in the closed unit disc. Hence, for  $|z_1| < 1$ ,

$$f(z_1, z_2) = \frac{1}{2\pi i} \int_{|\zeta_1|=1} \frac{f(\zeta_1, z_2)}{\zeta_1 - z_1} d\zeta_1$$

by the usual Cauchy formula. For each fixed  $\zeta_1$  on the unit circle,  $f(\zeta_1, z_2)$  is holomorphic in  $z_2$  for  $z_2$  in the closed unit disc. Hence, for  $|z_2| < 1$ ,

$$f(\zeta_1, z_2) = \frac{1}{2\pi i} \int_{|\zeta_2|=1} \frac{f(\zeta_1, \zeta_2)}{\zeta_2 - z_2} d\zeta_2.$$

Combining the last two expressions, we obtain the theorem.  $\square$

**Problem 10.** *Show that each function holomorphic in the polydisc is the uniform limit on compact subsets of rational functions.*

**Problem 11.** *Show that each entire function (function holomorphic in  $\mathbb{C}^n$ ) is the uniform limit on compact subsets of rational functions.*

We have stated the Cauchy formula in the polydisc in  $\mathbb{C}^2$  for simplicity. As in one variable, there is also a Cauchy integral formula for derivatives. To state the formula in  $\mathbb{C}^n$ , we introduce multi-index notation.  $\alpha = (\alpha_1, \dots, \alpha_n)$ , where each  $\alpha_j$  is a non-negative integer and, by abuse of notation, we write  $1 = (1, \dots, 1)$  and  $0 = (0, \dots, 0)$ . If  $a \in \mathbb{C}^n$  and  $a_j \neq 0, j = 1, \dots, n$ , we write

$$\frac{z}{a} = \frac{z^1}{a^1} = \frac{z_1 \cdots z_n}{a_1 \cdots a_n}. \quad (1)$$

Set  $|\alpha| = \alpha_1 + \dots + \alpha_n$ ,  $\alpha! = \alpha_1! \cdots \alpha_n!$  and  $z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}$ . We denote derivatives with respect to real variables by

$$\frac{\partial^{|\beta|+|\gamma|} f}{\partial x^\beta \partial y^\gamma} = \frac{\partial^{|\beta|+|\gamma|} f}{\partial x_1^{\beta_1} \cdots \partial x_n^{\beta_n} \partial y_1^{\gamma_1} \cdots \partial y_n^{\gamma_n}}$$

and with respect to complex variables by

$$f^{(\alpha)} = \frac{\partial^{|\alpha|} f}{\partial z^\alpha} = \frac{\partial^{|\alpha|} f}{\partial z_1^{\alpha_1} \cdots \partial z_n^{\alpha_n}}.$$

The following lemma of Leibniz taken from Bartle allows one to differentiate under the integral sign.

**Lemma 8** (Leibniz). *Let  $\mu$  be a measure on a locally compact Hausdorff space  $Y$  with countable base and let  $I$  be an open interval. Consider a function  $f : I \times Y \rightarrow \mathbb{R}$ , with  $f(x, \cdot)$  (Borel) measurable, for each  $x \in I$ . Suppose*

there exists a point  $x_0$  such that  $f(x_0, \cdot)$  is  $\mu$ -integrable,  $\partial f/\partial x$  exists on  $I$  and there is a  $\mu$ -integrable function  $g$  on  $Y$  such that

$$\left| \frac{\partial f}{\partial x}(x, y) \right| \leq g(y), \quad \forall (x, y).$$

Then

$$\frac{\partial}{\partial x} \int f(x, \cdot) d\mu = \int \frac{\partial f}{\partial x}(x, \cdot) d\mu.$$

Let  $b\mathbb{D}^n = \{z : |z_j| = 1, j = 1, \dots, n\}$  denote the distinguished boundary of the polydisc and  $d\zeta = d\zeta_1 \cdots d\zeta_n$ .

**Theorem 9.** Let  $f$  be holomorphic on the closed polydisc  $\overline{\mathbb{D}^n}$ . Then,  $f \in C^\infty(\mathbb{D}^n)$ , and for each  $z \in \mathbb{D}^n$

$$\frac{\partial^{|\beta|+|\gamma|} f}{\partial x^\beta \partial y^\gamma}(z) = \frac{1}{(2\pi i)^n} \int_{b\mathbb{D}^n} f(\zeta) \frac{\partial^{|\beta|+|\gamma|}}{\partial x^\beta \partial y^\gamma} \left( \frac{1}{\zeta - z} \right) d\zeta.$$

All of these partial derivatives are holomorphic and, in particular,

$$f^{(\alpha)}(z) = \frac{\alpha!}{(2\pi i)^n} \int_{b\mathbb{D}^n} \frac{f(\zeta)}{(\zeta - z)^{\alpha+1}} d\zeta.$$

*Proof.* We already have the Cauchy integral formula for  $f$  itself, that is, for the multi-index  $\alpha = 0$ . In order to obtain the Cauchy formula for the first order partial derivatives of  $f$ , we apply the Leibniz theorem to differentiate the Cauchy formula for  $f$  by differentiating under the integral sign. Repetition of this process gives the general formula. We note that from this general Cauchy integral formula, it follows that all of the partial derivatives are continuous. Since all partial derivatives of the Cauchy kernel are holomorphic, the Leibniz formula yields that all partial derivatives of  $f$  are also holomorphic. The second formula is then a particular case of the first, since

$$\frac{\partial}{\partial z_j} = \frac{1}{2} \left( \frac{\partial}{\partial x_j} - i \frac{\partial}{\partial y_j} \right).$$

□

**Problem 12.** If  $f$  is holomorphic in an open set  $\Omega$  of  $\mathbb{C}^n$ , then  $f \in C^\infty(\Omega)$ .

**Problem 13.** If  $\mathcal{F}$  is a uniformly bounded family of holomorphic functions on an open set  $\Omega \subset \mathbb{C}^n$ , then, for each compact subset  $K \subset \Omega$  and each pair of multi-indices  $\alpha, \beta$ , there is a  $0 < M_{\alpha\beta}^K < +\infty$ , such that, for each  $f \in \mathcal{F}$ ,

$$\left| \frac{\partial^{|\alpha|+|\beta|} f}{\partial x^\alpha \partial y^\beta}(z) \right| \leq M_{\alpha\beta}^K, \text{ for all } z \in K.$$

**Problem 14.** If  $f$  is holomorphic in an open set  $\Omega$  of  $\mathbb{C}^n$ , then all partial derivatives of  $f$  are also holomorphic in  $\Omega$ .

**Theorem 10.** Let  $\varphi$  be continuous on the distinguished boundary of a polydisc  $\mathbb{D}$  and define  $F$  as the Cauchy integral of  $\varphi$ :

$$F(z) = \frac{1}{(2\pi i)^n} \int_{b\mathbb{D}} \frac{\varphi(\zeta)}{\zeta - z} d\zeta,$$

for  $z \in \mathbb{D}$ . Then,  $F$  is holomorphic in  $\mathbb{D}$ .

*Proof.* The proof is the same as that of the Cauchy integral formula. Using the Leibniz formula, we show that  $F$  is smooth and satisfies the Cauchy-Riemann equations.  $\square$

### 3 Sequences of holomorphic functions

**Theorem 11.** On an open set, the uniform limit of holomorphic functions is holomorphic.

*Proof.* Let  $f_n$  be holomorphic on an open set  $\Omega$  and suppose  $f_n \rightarrow f$  uniformly. It is sufficient to show that  $f$  is holomorphic in a neighborhood of each point of  $\Omega$ . It is sufficient to show that  $f$  is holomorphic in each polydisc whose closure is contained in  $\Omega$ . Let  $\mathbb{D}$  be such a polydisc. From the uniform convergence, we have, for  $z \in \mathbb{D}$ ,

$$f(z) = \lim f_j(z) = \lim \frac{1}{(2\pi i)^n} \int_{b\mathbb{D}} \frac{f_j(\zeta)}{\zeta - z} d\zeta = \frac{1}{(2\pi i)^n} \int_{b\mathbb{D}} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

Thus,  $f$  is a Cauchy integral in  $\mathbb{D}$  and hence  $f$  is holomorphic in  $\mathbb{D}$ .  $\square$

One of the most fundamental facts concerning numerical sequences is the Bolzano-Weierstrass theorem. Recall that a sequence of numbers  $\{z_j\}$  is *bounded* if there is a number  $M > 0$  such that  $|z_j| \leq M$ , for all  $j$ .

**Theorem 12** (Bolzano-Weierstrass). *Any bounded sequence of numbers has a convergent subsequence.*

A sequence of functions  $\{f_j\}$  is (uniformly) *bounded* on a set  $E$  if there is a number  $M > 0$  such that  $|f_j| \leq M$ , for all  $j$ . For sequences of functions, we have the following analog of the Bolzano-Weierstrass theorem, known as Montel's theorem.

**Theorem 13** (Montel). *Let  $\mathcal{F}$  be a bounded family of holomorphic functions on an open set  $\Omega \subset \mathbb{C}^n$ . Then, each sequence of functions in  $\mathcal{F}$  has a subsequence which converges uniformly on compact subsets.*

In order to prove Montel's theorem, we gather a certain amount of material which is, in any case, interesting in itself.

Recall that a family  $\mathcal{F}$  of complex-valued functions, defined on a metric space  $(X, d)$  is *equicontinuous* if for each  $\epsilon > 0$ , there is a  $\delta > 0$  such that, for all  $f \in \mathcal{F}$  and for all  $p, q \in X$ ,

$$d(p, q) < \delta \quad \text{implies} \quad |f(p) - f(q)| < \epsilon.$$

**Theorem 14** (Arzelà-Ascoli). *If  $K$  be a compact metric space and  $\{f_j\}$  is a sequence of complex-valued functions which is pointwise bounded and equicontinuous on  $K$ , then*

- (a)  $\{f_j\}$  is uniformly bounded on  $K$ ;
- (b)  $\{f_j\}$  has a uniformly convergent subsequence.

**Problem 15.** *Use Problem 13 to show that if  $\mathcal{F}$  is a bounded family of holomorphic functions on an open subset  $\Omega \subset \mathbb{C}^n$ , then the family  $\nabla\mathcal{F} = \{\nabla f : f \in \mathcal{F}\}$  is bounded on compact subsets of  $\Omega$ .*

**Problem 16.** *Let  $f$  be a smooth function defined in an open convex subset  $B$  of  $\mathbb{R}^n$ . If  $|\nabla f| \leq M$  in  $B$ , then  $|f(p) - f(q)| \leq M|p - q|$ , for each  $p, q \in B$ .*

Let  $X$  be a topological space. An exhaustion of  $X$  by compact sets is a sequence  $\{K_j\}$  of nested compact subsets,  $K_j \subset K_{j+1}^0$ , whose union is  $X$ .

**Problem 17.** *Show that each open subset of  $\mathbb{R}^n$  admits an exhaustion by compact sets.*



*Proof.* (of Montel theorem) Let  $\{f_j\}$  be a bounded sequence of holomorphic functions on an open set  $\Omega \subset \mathbb{C}^n$ . Now, let  $K$  be a compact subset of  $\Omega$ . Let  $d$  be the distance of  $K$  from  $\partial\Omega$  and choose  $2r < d$ . We may cover  $K$  by finitely many balls  $B(a_1, r), \dots, B(a_m, r)$  whose centers are in  $K$ . From Problem 13, it follows that the sequence  $\{\nabla f_j\}$  of gradients of the sequence  $\{f_j\}$  is uniformly bounded on the union of the closed balls  $\overline{B}(a_1, 2r), \dots, \overline{B}(a_m, 2r)$  by some  $M < +\infty$ . If  $z, \zeta \in K$  and  $|z - \zeta| < r$ , then, since  $z$  lies in some  $B(a_k, r)$ , both  $z$  and  $\zeta$  lie in  $B(a_k, 2r)$ . By Problem 16, it follows that  $|f_j(z) - f_j(\zeta)| \leq M|z - \zeta|, j = 1, 2, \dots$ . Thus, the sequence  $\{f_j\}$  is equicontinuous on  $K$ . Since the sequence is also by hypothesis bounded on  $K$ , it follows from the Arzelá-Ascoli theorem that the sequence  $\{f_j\}$  has a subsequence, which converges uniformly on  $K$ .

By Problem 17, the open set  $\Omega$  has an exhaustion by compact sets:

$$K_1 \subset K_2^0 \subset K_2 \subset \dots \subset K_k^0 \subset K_{k+1} \subset \dots$$

From the previous paragraph,  $\{f_j\}$  has a subsequence which converges uniformly on  $K_1$ . Applying the same argument to this subsequence, we see that the subsequence has itself a subsequence which converges uniformly on  $K_2$ . Continuing in this manner, we construct an infinite matrix  $\{f_{kj}\}$  of functions. The first row is the sequence  $\{f_j\}$ ; each row is a subsequence of the previous row and, for each  $k = 1, 2, \dots$ , the  $k$ -th row converges uniformly on  $K_k$ . The diagonal sequence  $\{f_{kk}\}$  is thus a subsequence of  $\{f_j\}$  which converges uniformly on each  $K_m, m = 1, 2, \dots$ .

Now, let  $K$  be an arbitrary compact subset of  $\Omega$ . Since  $\{K_m^0\}$  is a nested open cover of  $\Omega$ , it follows from compactness that  $K$  is contained in some  $K_m$ . Since  $\{f_{kk}\}$  converges uniformly on  $K_m$  it also converges uniformly on  $K$ .  $\square$

Let  $\Omega$  be an open set in  $\mathbb{R}^n$ . Denote by  $C(\Omega)$  the family of continuous complex-valued functions on  $\Omega$ . Fix an exhaustion  $\{K_j\}$  of  $\Omega$  and for  $f, g \in C(\Omega)$ , denote

$$d_j(f, g) = \sup_{z \in K_j} |f(z) - g(z)|$$

and

$$d(f, g) = \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{d_j(f, g)}{1 + d_j(f, g)}.$$

**Problem 18.** Let  $\Omega$  be an open subset of  $\mathbb{R}^n$ . Show that  $d$  is a distance function on  $C(\Omega)$ , that the induced metric space is complete and separable and that a sequence of functions in  $C(\Omega)$  converges with respect to this distance iff it converges uniformly on compact subsets of  $\Omega$ . The induced topology on  $C(\Omega)$  is called the topology of uniform convergence on compacta. Show that the space  $C(\Omega)$  is a topological algebra.

If  $\Omega$  is an open subset of  $\mathbb{C}^n$ , denote by  $\mathcal{O}(\Omega)$  the family of holomorphic functions on  $\Omega$ .

**Problem 19.** Let  $\Omega$  be an open subset of  $\mathbb{C}^n$ . Show that  $\mathcal{O}(\Omega)$  is a closed subalgebra of  $C(\Omega)$ .

## 4 Series

In the introduction, we asserted that holomorphic functions are the same as (complex) analytic functions. In order to discuss analytic functions of several variables, we must first discuss multiple series. We follow the presentation in Range. As with ordinary series, by abuse of notation, the expression

$$\sum_{\alpha \in \mathbb{N}^n} b_\alpha, \quad b_\alpha \in \mathbb{C}.$$

will have two meanings depending on the context. The first meaning is that this is simply a formal expression which we call a multiple series. The second meaning will be the sum of this multiple series, when it exists. Of course we now have to define what we mean by the sum of a multiple series. If  $n > 1$ , the index set  $\mathbb{N}^n$  does not carry any natural ordering, so that there is no canonical way to consider  $\sum b_\alpha$  as a sequence of (finite) partial sums as in the case  $n = 1$ . The ambiguity is avoided if one considers absolutely convergent series as follows. The multiple series  $\sum_{\alpha \in \mathbb{N}^n} b_\alpha$  is called absolutely convergent if

$$\sum_{\alpha \in \mathbb{N}^n} |b_\alpha| = \sup \left\{ \sum_{\alpha \in \Lambda} |b_\alpha| : \Lambda \text{ finite} \right\} < \infty.$$

In fact, absolutely convergent series are precisely the elements in  $L^1(\mathbb{N}^n, \mu)$ , where  $\mu$  is counting measure.

Cauchy's theorem on multiple series asserts that the absolute convergence of  $\sum b_\alpha$  is necessary and sufficient for the following to hold.

Any arrangement of  $\sum b_\alpha$  into an ordinary series

$$\sum_{j=0}^{\infty} b_{\sigma(j)},$$

where  $\sigma : \mathbb{N} \rightarrow \mathbb{N}^n$  is a bijection, converges in the usual sense to a limit  $L \in \mathbb{C}$  which is independent of  $\sigma$ . This number  $L$  is called the limit (or sum) of the multiple series, and one writes

$$\sum_{\alpha \in \mathbb{N}^n} b_\alpha = L.$$

In particular, if  $\sum b_\alpha$  converges absolutely, its limit can be computed from the *homogeneous* expansion

$$L = \sum_{k=0}^{\infty} \left( \sum_{|\alpha|=k} b_\alpha \right).$$

Furthermore, for any permutation  $\tau$  of  $\{1, \dots, n\}$ , the iterated series

$$\sum_{\alpha_{\tau(n)}=0}^{\infty} \left( \cdots \left( \sum_{\alpha_{\tau(1)}=0}^{\infty} b_{\alpha_1 \dots \alpha_n} \right) \cdots \right)$$

converges to  $L$  as well. Here, as in any mathematical expression, we first perform the operation in the innermost parentheses and work our way out. Conversely, if  $b_\alpha \geq 0$ , the convergence of any one of the iterated series implies the convergence of  $\sum b_\alpha$ .

The Cauchy theorem on multiple series can be viewed as a special case of the Fubini-Tonelli theorem in integration theory, but we shall finesse integration theory and prove the Cauchy theorem for the case  $n = 2$ ; that is, for double series.

Suppose a double series  $\sum b_{jk}$  converges absolutely. Then, any arrangement of  $\sum b_{jk}$  into a simple series converges absolutely. We know that if a simple series converges absolutely then it converges and any rearrangement converges to the same sum. Since any two arrangements of  $\sum b_{jk}$  into simple

series are rearrangements of each other, it follows that all arrangements of  $\sum b_{jk}$  into simple series converge and to the same sum  $L$ . This proves the first part of Cauchy's double series theorem.

Now, let  $\mathcal{P}^1, \mathcal{P}^2, \dots$  be any partition of the set  $\mathbb{N} \times \mathbb{N}$  of indices of the double series  $\sum b_{jk}$ . Cauchy's double series theorem further asserts that

$$L = \sum_{\nu} \left( \sum_{\mathcal{P}^{\nu}} b_{jk} \right).$$

We may consider each  $\sum_{\mathcal{P}^{\nu}} b_{jk}$  as a double series obtained from the double series  $\sum b_{jk}$  by possibly setting some of the terms equal to zero. Since the double series  $\sum b_{jk}$  converges absolutely, it follows that the double series  $\sum_{\mathcal{P}^{\nu}} b_{jk}$  also converges absolutely. Hence it converges. Denote the sum of  $\sum_{\mathcal{P}^{\nu}} b_{jk}$  by  $L_{\mathcal{P}^{\nu}}$ . We must show that

$$L = \sum_{\nu} L_{\mathcal{P}^{\nu}}.$$

Fix  $\epsilon > 0$ . Let  $\sum_{i=1}^{\infty} b_{\sigma(i)}$  be any arrangement of  $\sum b_{jk}$  and choose  $n_1$  so large that

$$\sum_{i=n_1}^{\infty} |b_{\sigma(i)}| < \epsilon.$$

Now choose  $n_2$  so large that each of the terms  $b_{\sigma(i)}, i < n_1$  are in one of the  $\mathcal{P}^{\nu}, \nu < n_2$ . Set  $n(\epsilon) = \max\{n_1, n_2\}$ . For  $n > n(\epsilon)$  we have

$$\begin{aligned} |L - \sum_{\nu=1}^n L_{\mathcal{P}^{\nu}}| &= \left| \sum_{i=1}^{\infty} b_{\sigma(i)} - \sum_{\nu=1}^n L_{\mathcal{P}^{\nu}} \right| < \\ & \left| \sum_{i=n_1}^{\infty} b_{\sigma(i)} - \sum_{\nu=1}^n L'_{\mathcal{P}^{\nu}} \right| < \epsilon + \sum_{\nu=1}^n |L'_{\mathcal{P}^{\nu}}|, \end{aligned}$$

where  $L'_{\mathcal{P}^{\nu}}$  is the sum  $L_{\mathcal{P}^{\nu}}$  from which those  $b_{\sigma(i)}$  for which  $i < n_1$  (if there are any such) have been removed. We note that

$$\sum_{\nu=1}^n |L'_{\mathcal{P}^{\nu}}| \leq \lim_{m \rightarrow \infty} \sum_{\nu=1}^n \sum_{i=1}^m \{ |b_{\sigma(i)}| : \sigma(i) \in \mathcal{P}^{\nu}, n_1 \leq i < m \} \leq \sum_{i=n_1}^{\infty} |b_{\sigma(i)}| < \epsilon.$$

Combining the above estimates, we have that, for  $n > n(\epsilon)$ ,

$$|L - \sum_{\nu=1}^n L_{\mathcal{P}^\nu}| < 2\epsilon,$$

which concludes the proof of Cauchy's theorem for double series.

We recall the following from undergraduate analysis.

**Theorem 15** (Weierstrass M-test). *Let  $f_n$  be sequence of functions defined on a set  $E$  and  $M_n$  a sequence of constants. If  $|f_n| \leq M_n$  and  $\sum M_n$  converges, then  $\sum f_n$  converges absolutely and uniformly.*

**Problem 20.** *For  $\zeta \in \mathbb{D}^n$ , and recalling the abusive notation  $1 = (1, \dots, 1)$  as well as the notation given by (1) show that:*

$$\frac{1}{1 - \zeta} = \sum_{\alpha \geq 0} \zeta^\alpha;$$

*the series converges absolutely and any arrangement converges uniformly on compact subsets of  $\mathbb{D}^n$ .*

The next theorem asserts that holomorphic functions are analytic. The converse will come later.

**Theorem 16.** *Let  $f$  be holomorphic in a domain  $\Omega \subset \mathbb{C}^n$  and let  $a \in \Omega$ . Then,  $f$  can be expanded in an absolutely convergent power series:*

$$f(z) = \sum_{\alpha \geq 0} c_\alpha (z - a)^\alpha,$$

*in a neighborhood of  $a$ . The series is the Taylor series of  $f$ ; that is,*

$$c_\alpha = \frac{f^{(\alpha)}(a)}{\alpha!}.$$

*The representation of  $f$  as the sum of its Taylor series is valid in any polydisc centered at  $a$ .*

*Proof.* Consider a polydisc

$$\mathbb{D}^n(a, r) = \{z : |z_j - a_j| < r, j = 1, \dots, n\},$$

which for simplicity we denote by  $\mathbb{D}$ , whose closure is contained in  $\Omega$ . By the Cauchy integral formula,

$$f(z) = \frac{1}{(2\pi i)^n} \int_{b\mathbb{D}} \frac{f(\zeta)}{\zeta - z} d\zeta.$$

By an earlier problem, we may write

$$\frac{f(\zeta)}{\zeta - z} = \frac{f(\zeta)}{(\zeta - a) - (z - a)} = \frac{f(\zeta)}{\zeta - a} \cdot \frac{1}{1 - \frac{z-a}{\zeta-a}} = \frac{f(\zeta)}{\zeta - a} \sum_{\alpha \geq 0} \left( \frac{z - a}{\zeta - a} \right)^\alpha$$

and the convergence is uniform on  $b\mathbb{D} \times K$  for any compact subset  $K$  of  $\mathbb{D}$ . Integrating term by term, we have

$$f(z) = \sum_{\alpha \geq 0} \left( \frac{1}{(2\pi i)^n} \int_{b\mathbb{D}} \frac{f(\zeta)}{(\zeta - a)^{\alpha+1}} d\zeta \right) (z - a)^\alpha = \sum_{\alpha \geq 0} c_\alpha (z - a)^\alpha$$

and the convergence is uniform on compact subsets of  $\mathbb{D}$ . By the Cauchy formula for derivatives,  $c_\alpha = f^{(\alpha)}(a)/\alpha!$ .

We have assumed that the closure of the polydisc is contained in  $\Omega$ , but any polydisc whose closure is contained in  $\Omega$  can be written as the union of an increasing sequence of polydiscs with the same center whose closures are contained in  $\Omega$ . The function  $f$  is represented by its Taylor series about  $a$  for each of the polydiscs in this sequence and hence the representation is valid on the union of these polydiscs.  $\square$

We have now established that holomorphic functions are analytic. In the proof we did not require the property that holomorphic functions are  $C^1$ . We merely required uniform convergence to allow us to integrate term by term, and for this it is sufficient that holomorphic functions be locally bounded.

To prove conversely that analytic functions are holomorphic, we need a little more familiarity with multiple power series.

**Theorem 17 (Abel).** *If the power series  $\sum c_\alpha z^\alpha$  converges at the point  $a$  for some arrangement (as a simple series) and if  $a_j \neq 0, j = 1, \dots, n$ , then the series converges absolutely and uniformly on each compact subset of the polydisc*

$$\{z : |z_j| < |a_j|, j = 1, \dots, n\}.$$

*Proof.* Since some arrangement of the series converges, it follows that the terms are bounded. Thus  $|c_\alpha a^\alpha| < M$  for all  $\alpha$ . Fix  $0 < r_j < |a_j|, j = 1, \dots, n$  and suppose  $|z_j| \leq r_j, j = 1, \dots, n$ . Then,

$$|c_\alpha z^\alpha| = |c_\alpha z_1^{\alpha_1} \cdots z_n^{\alpha_n}| = |c_\alpha a_1^{\alpha_1} \cdots a_n^{\alpha_n}| \cdot \left| \left( \frac{z_1}{a_1} \right)^{\alpha_1} \cdots \left( \frac{z_n}{a_n} \right)^{\alpha_n} \right| \leq$$

$$M \left| \frac{r_1}{a_1} \right|^{\alpha_1} \cdots \left| \frac{r_n}{a_n} \right|^{\alpha_n} = M \rho^\alpha,$$

where  $\rho_j < 1, j = 1, \dots, n$ . Since  $\sum \rho^\alpha$  converges, the power series converges absolutely and uniformly on the closed polydisc  $|z_j| \leq r_j, j = 1, \dots, n$ , by the Weierstrass  $M$ -test. Since any compact subset of the open polydisc  $\{z : |z_j| < |a_j|, j = 1, \dots, n\}$  is contained in such a closed polydisc, the proof is complete.  $\square$

Let  $\sigma_\alpha$  be a multiple power series. Abel's theorem asserts that  $\sigma$  converges absolutely and uniformly on compact subsets of the polydisc  $|z_j| < |a_j|, j = 1, \dots, n$ , if some arrangement of  $\sigma_\alpha$  converges at the point  $a$ . Suppose we write  $a = (b, c)$  and  $\sigma_\alpha$  as the iteration  $\sigma_\beta \sigma_\gamma$  of two power series, which in some sense converges at  $(b, c)$ . Can we hope for the same conclusion that  $\sigma_\alpha$  converges absolutely and uniformly on compact subsets of the polydisc  $|z_j| < |a_j|, j = 1, \dots, n$ ? Our meaning will be made clear by the following example which illustrates the futility of such a hope.

*Example.* Write  $z = (\zeta, w)$  and consider the double power series

$$\sum_{\alpha} z^\alpha = \sum_{j,k} c_{j,k} \zeta^j w^k,$$

where  $c_{j,j} = 4^j, c_{j,j+1} = -4^j$  and  $c_{j,k} = 0$  if  $k$  is different from  $j$  or  $j + 1$ . Then, for  $z = (\zeta, w) = (1, 1)$ ,

$$\sum_j c_{j,k} 1^j = 4^j - 4^j = 0, \quad k = 0, 1, \dots,$$

and consequently,

$$\sum_k \left( \sum_j c_{j,k} 1^j \right) 1^k = \sum_k 0 \cdot 1^k = 0.$$

It is certainly not true that the double power series converges absolutely on the polydisc  $|\zeta| < 1, |w| < 1$ . This would imply that for any such point  $(\zeta, w)$ , the terms  $c_{j,k}\zeta^j w^k$  would tend to zero. However, for the point  $(1/2, 1/2)$  the 'diagonal' terms are

$$c_{j,j} \left(\frac{1}{2}\right)^j \left(\frac{1}{2}\right)^j = 4^j \left(\frac{1}{2}\right)^{2j} = 1.$$

**Theorem 18.** *On an open set  $\Omega$  in  $\mathbb{C}^n$ , a function is holomorphic iff it is analytic.*

*Proof.* We have shown earlier that every holomorphic function is analytic. Conversely, suppose  $f$  is analytic on  $\Omega$ . It is sufficient to show that  $f$  is holomorphic in a polydisc about each point of  $\Omega$ . Fix  $a \in \Omega$  and let  $\mathbb{D}$  be a polydisc containing  $a$  and contained in  $\Omega$ , such that  $f$  can be represented as a power series in  $\mathbb{D}$ . We have seen that the power series converges uniformly on compact subsets of  $\mathbb{D}$ . In particular, let  $Q$  be a polydisc containing  $a$  and whose closure is compact in  $\mathbb{D}$ . Then the power series converges uniformly in  $Q$  and, since the terms are polynomials, they are holomorphic. Thus,  $f$  is the uniform limit of holomorphic functions on  $Q$ . Hence,  $f$  is holomorphic on  $Q$ . We have shown that  $f$  is holomorphic in a neighborhood of each point of  $\Omega$  and so  $f$  is holomorphic in  $\Omega$ .  $\square$

**Theorem 19** (uniqueness). *Let  $f$  be holomorphic in a domain  $\Omega$  and suppose  $f = 0$  on a (non empty) open subset of  $\Omega$ . Then  $f = 0$  on  $\Omega$ .*

**Problem 21.** *Prove the theorem.*

**Corollary 20** (uniqueness). *Let  $f$  and  $g$  be holomorphic in a domain  $\Omega$  and suppose  $f = g$  on an open subset of  $\Omega$ . Then  $f = g$  on  $\Omega$ .*

## 5 Holomorphic mappings

**Problem 22** (chain rule). *Suppose  $\zeta \rightarrow z$  is a smooth mapping from an open set  $D \subset \mathbb{C}$  into an open set  $\Omega \subset \mathbb{C}^n$  and  $z \rightarrow w$  is a smooth function from  $\Omega$  into  $\mathbb{C}$ , then*

$$\frac{\partial w}{\partial \zeta} = \sum_{j=1}^n \left( \frac{\partial w}{\partial z_j} \frac{\partial z_j}{\partial \zeta} + \frac{\partial w}{\partial \bar{z}_j} \frac{\partial \bar{z}_j}{\partial \zeta} \right)$$



and

$$\frac{\partial w}{\partial \bar{\zeta}} = \sum_{j=1}^n \left( \frac{\partial w}{\partial z_j} \frac{\partial z_j}{\partial \bar{\zeta}} + \frac{\partial w}{\partial \bar{z}_j} \frac{\partial \bar{z}_j}{\partial \bar{\zeta}} \right).$$

A mapping  $f : \Omega \rightarrow \mathbb{C}^m$ , defined on an open subset  $\Omega$  of  $\mathbb{C}^n$ , is said to be *holomorphic* if each of the components  $f_1, \dots, f_m$  of  $f$  are holomorphic.

**Problem 23.** *The composition of holomorphic mappings is holomorphic. That is, if  $D$  is an open subset of  $\mathbb{C}^k$ ,  $\Omega$  is an open subset of  $\mathbb{C}^n$ ,  $g : D \rightarrow \mathbb{C}^n$  and  $f : \Omega \rightarrow \mathbb{C}^m$  are holomorphic mappings, and  $g(D) \subset \Omega$ , then the mapping  $f \circ g : D \rightarrow \mathbb{C}^m$  is holomorphic.*

Let  $f : \Omega \rightarrow \mathbb{C}^m$  be a holomorphic mapping defined in an open set  $\Omega \subset \mathbb{C}^n$ . To each  $z \in \Omega$ , we associate a unique linear transformation  $f'(z) : \mathbb{C}^n \rightarrow \mathbb{C}^m$ , called the *derivative of  $f$  at  $z$* , such that

$$f(z+h) = f(z) + f'(z)h + r(h),$$

where  $r(h) = O(\|h\|^2)$  as  $h \rightarrow 0$ .

**Problem 24.** *Prove the uniqueness of the derivative.*

With respect to the standard coordinates in  $\mathbb{C}^n$  and  $\mathbb{C}^m$ , the linear transformation  $f'(a)$  at a point  $a$  is represented by the (complex) Jacobian matrix

$$J(f)(a) = \left( \frac{\partial f_j}{\partial z_k}(a) \right), \quad j = 1, \dots, m; \quad k = 1, \dots, n.$$

Of course this matrix represents a linear transformation, so we need only verify that it has the required approximation property. Since the vector  $r(h)$  is small iff each of its components is small, it is sufficient to check the claim for each component  $f_j$  of  $f$ . Thus, it is enough to suppose that  $f$  itself is a function rather than a mapping. From the Taylor formula,

$$f(z+h) = f(z) + J(f)(z)h + \sum_{|\alpha| \geq 2} \frac{f^{(\alpha)}(z)}{\alpha!} h^\alpha.$$

Write  $h = t\zeta$ , with  $t$  positive. Then,

$$r(h) = \sum_{k=2}^{\infty} \left( \sum_{|\alpha|=k} \frac{f^{(\alpha)}(z)}{\alpha!} \zeta^\alpha \right) t^k = t^2 \sum_{k=0}^{\infty} \left( \sum_{|\alpha|=k+2} \frac{f^{(\alpha)}(z)}{\alpha!} \zeta^\alpha \right) t^k.$$

Since the original power series in  $h$  converges absolutely for small  $h$ , the power series in  $\zeta$  and  $t$  converges for some positive  $t$  and some  $\zeta$  none of whose coordinates are zero. It follows that the series

$$\sum_{k=0}^{\infty} \left( \sum_{|\alpha|=k+2} \frac{f^{(\alpha)}(z)}{\alpha!} \zeta^{\alpha} \right) t^k$$

in  $\zeta$  and  $t$  converges for all small  $\zeta$  and  $t$ . Thus, for some  $t_0 > 0$  and  $\rho > 0$ , this sum is bounded, by say  $M$ , for  $|t| \leq t_0$  and  $|\zeta| \leq \rho$ . If  $\|h\| \leq t_0\rho$ , we may write

$$h = \left( \frac{\|h\|}{\rho} \right) \left( \frac{h}{\|h\|} \rho \right) = t\zeta,$$

with  $|t| \leq t_0$  and  $|\zeta| \leq \rho$ . Thus, for  $\|h\| \leq t_0\rho$ , we have

$$|r(h)| \leq t^2 M = \left( \frac{\|h\|}{\rho} \right)^2 M = O(\|h\|)^2,$$

which concludes the proof that  $f'$  is represented by the Jacobian matrix  $J(f)$ .

**Problem 25.** *If  $f$  and  $g$  are holomorphic mappings such that  $f \circ g$  is defined, then  $(f \circ g)' = (f'(g))g'$  and  $J(f \circ g) = J(f)J(g)$ . More precisely, if  $w = g(z)$ , then  $(f \circ g)'(z) = f'(w)g'(z)$  and  $J(f \circ g)(z) = [J(f)(w)][J(g)(z)]$ .*

Let  $\Omega$  be open in  $\mathbb{C}^n$  and  $f : \Omega \rightarrow \mathbb{C}^m$  be a mapping, which we may write  $f(z) = w$ , with  $z \in \Omega$  and  $w \in \mathbb{C}^m$ . Let  $x$  and  $y$  be the real and imaginary parts of  $z$  and let  $u$  and  $v$  be the real and imaginary parts of  $w$ . We may think of  $\Omega$  as an open subset of  $\mathbb{R}^{2n}$  and we may view the complex mapping  $z \mapsto w$  as a real mapping  $(x, y) \mapsto (u, v)$  of the open subset  $\Omega$  of  $\mathbb{R}^{2n}$  into  $\mathbb{R}^{2m}$ . If the complex mapping  $f$  is smooth, let  $J_R(f)$  denote the (real) Jacobian matrix of the associated (real) mapping  $(x, y) \mapsto (u, v)$ . If  $f$  is an equidimensional smooth complex mapping, then the complex and real Jacobian matrices  $J(f)$  and  $J_R(f)$  are square.  $\det J(f)$  is called the (complex) Jacobian determinant of  $f$  and  $\det J_R(f)$  is called the real Jacobian determinant of  $f$ .

**Theorem 21.** *If  $f$  is an equidimensional holomorphic mapping, then*

$$\det J_R(f) = |\det J(f)|^2.$$

**Problem 26.** *Verify this for  $n = 1$ .*

*Proof.* In this proof we shall sometimes denote the determinant of a square matrix  $A$  by  $|A|$ . We shall write matrices as block matrices, where for example  $\partial u/\partial x$  represents the matrix  $(\partial u_j/\partial x_k)$ . Since an even number of permutations of rows and columns does not change the determinant, we may write

$$\det J_R(f) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}.$$

Adding a constant multiple of a row to another row does not change the determinant, so we may add  $i$  times the lower blocks to the upper blocks and use the Cauchy-Riemann equations to obtain

$$\det J_R(f) = \begin{vmatrix} \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} & i\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial u}{\partial x} \end{vmatrix}.$$

Now, if we subtract  $i$  times the left blocks from the right blocks, we have

$$\det J_R(f) = \begin{vmatrix} \frac{\partial f}{\partial x} & 0 \\ \dots & \frac{\partial \bar{f}}{\partial x} \end{vmatrix}.$$

We have  $\partial \bar{f}/\partial x = \overline{\partial f/\partial x}$  and, since  $f$  is holomorphic  $\partial f/\partial x = \partial f/\partial z$ . Thus,

$$\det J_R(f) = \begin{vmatrix} \frac{\partial f}{\partial z} & 0 \\ \dots & \frac{\partial \bar{f}}{\partial z} \end{vmatrix} = \left| \frac{\partial f}{\partial z} \right| \left| \frac{\partial \bar{f}}{\partial z} \right| = \left| \frac{\partial f}{\partial z} \right| \overline{\left| \frac{\partial f}{\partial z} \right|} = \det J(f) \cdot \overline{\det J(f)}.$$

Hence,

$$\det J_R(f) = |\det J(f)|^2.$$

□

**Theorem 22** (inverse mapping). *Let  $f$  be a holomorphic mapping defined in a neighborhood of a point  $a$ . If  $f'(a)$  is invertible, then  $f$  is invertible in a neighborhood of  $a$  and the inverse mapping is also holomorphic.*

*Proof.* Since  $f$  is holomorphic it is smooth. Since  $f'(a)$  is invertible it is equidimensional and so the Jacobian matrix  $J(f)(a)$  is square and invertible. Thus,  $\det J(f)(a) \neq 0$ . By the previous theorem,  $\det J_R(f)(a) \neq 0$ . Thus, we may invoke the real inverse mapping theorem to conclude that  $w = f(z)$  considered as a real mapping is locally invertible at  $a$ . Let  $g$  denote the local

inverse mapping defined in a neighborhood of  $b = f(a)$ . Then,  $g$  is smooth and, since  $z = (g \circ f)(z)$ , we have for  $j = 1, \dots, n$  and  $k = 1, \dots, n$ :

$$0 = \frac{\partial z_j}{\partial \bar{z}_k} = \sum_{\nu} \frac{\partial g_j}{\partial w_{\nu}} \frac{\partial f_{\nu}}{\partial \bar{z}_k} + \sum_{\nu} \frac{\partial g_j}{\partial \bar{w}_{\nu}} \frac{\partial \bar{f}_{\nu}}{\partial \bar{z}_k} = \sum_{\nu} \frac{\partial g_j}{\partial \bar{w}_{\nu}} \frac{\partial \bar{f}_{\nu}}{\partial \bar{z}_k}.$$

Since

$$\frac{\partial \bar{f}_{\nu}}{\partial \bar{z}_k} = \overline{\frac{\partial f_{\nu}}{\partial z_k}},$$

we have the matrix equation:

$$(0) = \left( \frac{\partial g}{\partial \bar{w}} \right) \overline{\left( \frac{\partial f}{\partial z} \right)} = \left( \frac{\partial g}{\partial \bar{w}} \right) \overline{J(f)}. \quad (2)$$

Now, since  $f'(a)$  is invertible,  $\det J(f)(a) \neq 0$  and so  $\overline{\det J(f)(z)} \neq 0$  for  $z$  in a neighborhood of  $a$ . Thus,  $J(f)$  and consequently  $\overline{J(f)}$  also is invertible for  $z$  in a neighborhood of  $a$ . Multiplying equation (2) on the right by the inverse matrix of  $\overline{J(f)}$ , we have

$$(0) = \left( \frac{\partial g}{\partial \bar{w}} \right).$$

That is, the components of  $g$  satisfy the Cauchy-Riemann equations in a neighborhood of  $b$ . Therefore, the inverse mapping  $g$  is also holomorphic in a neighborhood of  $b = f(a)$ .  $\square$

Next, we shall present the implicit mapping theorem. But first, we shall try to motivate the formulation by an informal heuristic discussion, which the student should not take too seriously. Suppose  $f$  is a holomorphic mapping from an open set  $W$  in  $\mathbb{C}^{n+m}$  to  $\mathbb{C}^k$  and we would like the level set  $f^{-1}(0)$  near a point  $(a, b)$  in  $\mathbb{C}^{n+m}$  where  $f(a, b) = 0$  to look like a graph in  $\mathbb{C}^n \times \mathbb{C}^m$  of a function  $w = g(z)$  defined in a neighborhood of  $a$  and taking its values in  $\mathbb{C}^m$ . First of all, we had better have  $k = m$ . Secondly, if we want the level set to be a graph over a neighborhood of  $a$  we would not wish the level set to be 'vertical' at  $(a, b)$ . In the real case with  $n = m = 1$ , we preclude this by the condition  $\partial f / \partial y \neq 0$  at  $(a, b)$ . In the multi-variable situation, we preclude this strongly by asking that the matrix  $\partial f / \partial y$  be invertible at  $(a, b)$ .

**Theorem 23** (implicit mapping). *Let  $f(z, w)$  be a holomorphic mapping from a neighborhood of a point  $(a, b)$  in  $\mathbb{C}^{n+m}$  to  $\mathbb{C}^m$  and suppose  $f(a, b) = 0$ . If*

$$\det \frac{\partial f}{\partial w}(a, b) \neq 0, \quad (3)$$

*then, there are neighborhoods  $U$  and  $V$  of  $a$  and  $b$  respectively and a holomorphic mapping  $g : U \rightarrow V$  such that  $f(z, w) = 0$  in  $U \times V$  iff  $w = g(z)$ .*

*Proof.* As in the proof of the inverse mapping theorem, we obtain all of the conclusions from the real implicit mapping theorem except the holomorphy of  $g$ . For  $z \in U$ , we have  $f(z, g(z)) = 0$  and hence, for  $j = 1, \dots, m; k = 1, \dots, n$ :

$$0 = \frac{\partial f_j}{\partial \bar{z}_k} = \sum_{\nu} \frac{\partial f_j}{\partial z_{\nu}} \frac{\partial z_{\nu}}{\partial \bar{z}_k} + \sum_{\nu} \frac{\partial f_j}{\partial w_{\nu}} \frac{\partial g_{\nu}}{\partial \bar{z}_k} = \sum_{\nu} \frac{\partial f_j}{\partial w_{\nu}} \frac{\partial g_{\nu}}{\partial \bar{z}_k}.$$

Fix  $z \in U$  and define  $f_z(w) = f(z, w)$  for  $w \in V$ . Then, the preceding equations can be written as the matrix equation

$$(0) = \left( \frac{\partial f_z}{\partial w} \right) \left( \frac{\partial g}{\partial \bar{z}} \right). \quad (4)$$

By continuity, we may assume that (4) holds not only at  $(a, b)$ , but also at all points  $(z, w) \in U \times V$ . Thus, for all  $z \in U$ , the Jacobian matrix  $\partial f_z / \partial w$  is invertible at all points  $w \in V$ . Now, if we multiply both members of (4) on the left by the inverse of the matrix  $\partial f_z / \partial w$ , we obtain that the matrix  $\partial g / \partial \bar{z}$  is the zero matrix. Thus,  $g$  satisfies the Cauchy-Riemann equations and is therefore holomorphic.  $\square$

The following Rank Theorem is taken from Kaup's book. In the Rank Theorem, we assume we are given a holomorphic mapping  $f$  such that the rank of  $f'$  is a constant  $r$  near a point  $a$ . The simplest example would be when  $f$  is a linear transformation of rank  $r$ , for then  $f'(x) = f$ , for each  $x$ . The simplest example of a linear transformation of rank  $r$  from  $\mathbb{C}^n$  to  $\mathbb{C}^m$  is the mapping  $(z_1, \dots, z_n) \mapsto (z_1, \dots, z_r, 0, \dots, 0)$ . The Rank Theorem asserts that near  $a$  the mapping  $f$  can be put in this form by a biholomorphic change of coordinates.

**Theorem 24 (Rank).** *Let  $f$  be a holomorphic mapping from a neighborhood of a point  $a$  in  $\mathbb{C}^n$  to  $\mathbb{C}^m$  and suppose  $f'$  has constant rank  $r$  near  $a$ . Then, there are neighborhoods  $U$  and  $V$  of  $a$  and  $b = f(a)$ , polydiscs  $D^n \subset \mathbb{C}^n$  and  $D^m \subset \mathbb{C}^m$ , each centered at 0, and biholomorphic mappings  $\varphi : D^n \rightarrow U$  and  $\psi : V \rightarrow D^m$  with  $\varphi(0) = a$  and  $\psi(b) = 0$  such that, with  $\chi(z_1, \dots, z_n) = (z_1, \dots, z_r, 0, \dots, 0)$ , we have  $\chi = \psi \circ f \circ \varphi$ .*

*Proof.* Without loss of generality, set  $a = b = 0$ ; moreover, let the coordinates of  $\mathbb{C}^n$  and  $\mathbb{C}^m$  be chosen in such a manner that  $f'(0)$  has the matrix representation

$$f'(0) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

Then, for the mapping

$$g(z) = (f_1(z), \dots, f_r(z), z_{r+1}, \dots, z_n),$$

we obviously have that  $g'(0) = I_n$ . By the inverse mapping theorem, there exists an open neighborhood  $U$  of 0 in  $\mathbb{C}^n$  that is mapped biholomorphically by  $g$  onto a polydisc  $D^n$ ; set  $\varphi := (g|_U)^{-1}$ . For  $w \in D^n$  and  $z := \varphi(w)$ , we have, for  $j = 1, \dots, r$ ,

$$(f \circ \varphi)_j(w) = f_j(\varphi(w)) = f_j(z) = w_j,$$

and hence,

$$(f_1, \dots, f_m)(z) = f(z) = (f \circ \varphi)(w) =: (w_1, \dots, w_r, h_{r+1}(w), \dots, h_m(w)),$$

where, in addition to  $f$  and  $\varphi$ , every  $h_j$  is holomorphic. The mapping  $f \circ \varphi$  satisfies  $\text{rank } (f \circ \varphi)' \geq r$  on  $D^n$ . We may assume that  $\text{rank } f' = r$  on  $D^n$ ; thus, by the chain rule,  $\text{rank } (f \circ \varphi)' = r$ , so

$$\frac{\partial h_j}{\partial w_k} = 0, \quad \text{for all } j, k \geq r + 1.$$

Hence, the  $h_j$ 's do not depend on the variables  $w_{r+1}, \dots, w_n$ ; their restrictions to the first  $r$  components determine a mapping  $h : D^r \rightarrow \mathbb{C}^{m-r}$ . For the bijective mapping

$$\begin{aligned} \gamma : D^r \times \mathbb{C}^{m-r} &\rightarrow D^r \times \mathbb{C}^{m-r} \\ (u, v) &\mapsto (u, v - h(u)), \end{aligned}$$

the derivative  $\gamma'$  has the following matrix

$$\begin{pmatrix} I_r & 0 \\ * & I_{m-r} \end{pmatrix};$$

by the inverse mapping theorem,  $\gamma$  is biholomorphic. Now choose a sufficiently large polydisc  $D^{m-r}$  in  $\mathbb{C}^{m-r}$  such that

$$(\gamma \circ f \circ \varphi)(D^n) \subset D^r \times D^{m-r} =: D^m;$$

for  $V := \gamma^{-1}(D^m)$  and  $\psi := \gamma|_V$ , we conclude that

$$\begin{aligned} (\psi \circ f \circ \varphi)(w) &= \gamma(w_1, \dots, w_r, h_{r+1}(w), \dots, h_m(w)) = \\ &= (w_1, \dots, w_r, 0, \dots, 0) = \chi(w). \end{aligned}$$

□

The rank theorem has a real version for smooth mappings (see [4], Theorem 9.32 and comments following), which we shall call the real rank theorem and we shall refer to the holomorphic version which we have just proved as the complex rank theorem.

## 6 Pluriharmonic functions

**Problem 27.** *If  $f$  is a holomorphic function of several complex variables, then the real part of  $f$  is harmonic.*

Recall that in one complex variable, there is a sort of converse. If  $u(x, y)$  is a harmonic function of two real variables, then  $u$  is locally the real part of a holomorphic function  $f(z) = u(x, y) + iv(x, y)$ .

In several variables, there is no such converse. Consider the function  $u(x_1, y_1, x_2, y_2) = x_1^2 - x_2^2$ , where  $z_1 = x_1 + iy_1$  and  $z_2 = x_2 + iy_2$ . Then,  $u$  is harmonic, but suppose there were locally a holomorphic function  $f(z_1, z_2)$  such that  $f = u + iv$ . Then, for fixed  $z_2$ , the function  $f_1(z_1) = f(z_1, z_2)$  would be holomorphic and hence the real part  $u_1(x_1, y_1) = x_1^2 - x_2^2$  would be harmonic in  $(x_1, y_1)$  which it is not.

A *complex line* in  $\mathbb{C}^n$  is a set of the form  $\ell = \{z : z = a + \lambda b, \lambda \in \mathbb{C}\}$ , where  $a$  and  $b$  are fixed points in  $\mathbb{C}^n$ , with  $b \neq 0$ . Let us say that  $\ell$  is the complex line through  $a$  in the 'direction'  $b$ . Let  $e^1, \dots, e^n$  be the standard

basis of  $\mathbb{C}^n$ . Thus, the coordinates of  $e^j$  are given by the Kronecker delta  $\delta_k^j$ . The complex line through  $a$  in the direction of  $e^j$  is called the complex line through  $a$  in direction of the  $j$ -th coordinate.

If  $\Omega$  is an open set in  $\mathbb{C}^n$ , we defined  $f$  to be holomorphic in  $\Omega$  if  $f \in C^1(\Omega)$  and  $f$  is holomorphic in each variable. That is, if the restriction of  $f$  to  $\ell \cap \Omega$  is holomorphic for each complex line  $\ell$  in the direction of a coordinate. A much stronger result holds.

**Theorem 25.** *Let  $\Omega$  be open in  $\mathbb{C}^n$  and  $f \in C^1(\Omega)$ . Then,  $f \in \mathcal{O}(\Omega)$  if and only if the restriction of  $f$  to  $\ell \cap \Omega$  is holomorphic, for each complex line  $\ell$ .*

*Proof.* The restriction of  $f$  to a complex line  $\{z = a + \lambda b : \lambda \in \mathbb{C}\}$  is the function  $f(a + \lambda b)$ . It follows from Theorem 23 that if  $f \in \mathcal{O}(\Omega)$ , then the restriction of  $f$  to  $\ell \cap \Omega$  is holomorphic, for each complex line  $\ell$ .  $\square$

A real-valued function  $u$  defined in an open subset  $\Omega$  of  $\mathbb{C}^n$ , is said to be *pluriharmonic* in  $\Omega$  if  $u \in C^2(\Omega)$  and the restriction of  $u$  to  $\ell \cap \Omega$  is harmonic for each complex line  $\ell$ . Unlike the holomorphic situation, this is *not* equivalent to being harmonic in each *coordinate* direction.

Let  $\Omega$  be an open set in  $\mathbb{C}^n$ . For  $u \in C^2(\Omega)$ , the Hermitian matrix

$$L_u = \left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right)$$

is called the complex Hessian matrix of  $u$ . We use the letter  $L$  for the complex Hessian, because the letter  $H$  is already being used for the real Hessian and because the quadratic form associated to the complex Hessian is usually called the Levy form.

A direct calculation shows that a real function  $u \in C^2(\Omega)$  is pluriharmonic in  $\Omega$  if and only if its complex Hessian matrix vanishes identically,  $L_u = 0$ , that is if and only  $u$  satisfies the system of differential equations

$$\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z) = 0, \quad \forall z \in \Omega.$$

In real form this system of equations becomes

$$\frac{\partial^2 u}{\partial x_j \partial x_k} + \frac{\partial^2 u}{\partial y_j \partial y_k} = 0, \quad \frac{\partial^2 u}{\partial x_j \partial y_k} - \frac{\partial^2 u}{\partial x_k \partial y_j} = 0. \quad (5)$$

We may now characterize real parts of holomorphic functions.



**Theorem 26.** *The real part of any holomorphic function is pluriharmonic. Conversely, every pluriharmonic function is locally the real part of a holomorphic function.*

*Proof.* It is an immediate consequence of Theorem 24 that the real part of a holomorphic function is pluriharmonic.

To show the converse, it is sufficient to show that any function  $u$  pluriharmonic in a polydisc  $\mathbb{D}^n$  is the real part of a holomorphic function therein.

We shall use the Poincaré lemma which asserts that in a convex domain, every closed form is exact (see, for example [4, Theorem 10.39]).

We wish to show that there exists a function  $v$  such that  $f = u + iv$  is holomorphic. If  $u$  did have such a conjugate function  $v$ , we could write

$$v(z) - v(a) = \int_a^z dv.$$

Since conjugate functions are only determined up to additive imaginary constants, we could even assume that  $v(a) = 0$ . From the Cauchy-Riemann equations, we would have

$$dv = \sum_k \left( \frac{\partial v}{\partial x_k} dx_k + \frac{\partial v}{\partial y_k} dy_k \right) = \sum_k \left( -\frac{\partial u}{\partial y_k} dx_k + \frac{\partial u}{\partial x_k} dy_k \right) = *dv.$$

Now  $dv$  is undefined, since we are trying to prove the existence of  $v$ , but the conjugate differential  $*dv$  of  $u$  is well defined by the last equality. Set  $\omega = *dv$ . If we can show that  $\omega$  is an exact differential, that is, that there is in fact a  $C^1$ -function  $v$  such that  $dv = \omega$ , then  $u$  and  $v$  will satisfy the Cauchy-Riemann equations and so  $f = u + iv$  will indeed be holomorphic.

Since we are working in a polydisc, which is thus a convex domain, we need only check that the differential form  $\omega$  is closed. By the Poincaré lemma it will then be exact.

$$\begin{aligned} d\omega &= - \sum_{j,k} \frac{\partial^2 u}{\partial x_j \partial x_k} dx_j \wedge dx_k + \sum_{j,k} \frac{\partial^2 u}{\partial y_j \partial x_k} dy_j \wedge dy_k + \\ &\quad + \sum_{j,k} \left( \frac{\partial^2 u}{\partial x_j \partial x_k} + \frac{\partial^2 u}{\partial y_j \partial y_k} \right) dx_j \wedge dy_k. \end{aligned}$$

The first sum is zero because

$$\frac{\partial^2 u}{\partial x_j \partial x_k} = \frac{\partial^2 u}{\partial x_k \partial x_j} \quad \text{while} \quad dx_j \wedge dx_k = -dx_k \wedge dx_j.$$

The second sum is zero for a similar reason. The third sum is zero because by (5) the terms are zero. Thus  $d\omega = 0$  and the proof is complete.  $\square$

## 7 Plurisubharmonic functions

A function  $u$  defined in an open subset  $\Omega$  of  $\mathbb{C}^n$ , and taking values in  $[-\infty, +\infty)$  is said to be plurisubharmonic in  $\Omega$  if  $u$  is upper semi-continuous,  $u$  is not identically  $-\infty$  in any component of  $\Omega$  and, for each complex line  $\ell$ , the restriction of  $u$  to each component of  $\ell \cap \Omega$  is subharmonic or identically  $-\infty$ . For the definition of a subharmonic function, see for example [1].

Similarly, one can define plurisuperharmonic functions, and it is easy to see that a function  $u$  is plurisuperharmonic if and only if  $-u$  is plurisubharmonic.

**Problem 28.** *If  $f$  is holomorphic, then  $|f|$  is plurisubharmonic.*

Subharmonicity and plurisubharmonicity resemble convexity in some ways. We present an example of this resemblance by first characterizing convex  $C^2$ -functions and then giving a similar characterization for plurisubharmonic  $C^2$ -functions.

Our discussion of convex functions is taken from Fleming.

Recall that a real valued function  $u$  defined on a convex open set  $\Omega$  in  $\mathbb{R}^n$  is said to be *convex* if for each  $a, b \in \Omega$ ,

$$u\left(\frac{a+b}{2}\right) \leq \frac{u(a)+u(b)}{2}.$$

**Theorem 27.** *Let  $u$  be a real-valued function defined on a convex subset  $\Omega$  of  $\mathbb{R}^n$ . If  $u$  is continuous, then  $u$  is convex if and only if*

$$u(ta + (1-t)b) \leq tu(a) + (1-t)u(b), \tag{6}$$

for every  $x, y \in \Omega$  and for each  $t \in (0, 1)$ .

*Proof.* From the symmetry between  $a$  and  $b$  and between  $t$  and  $1-t$ , it is sufficient to prove (6) for  $t \in (0, 1/2)$ . Moreover, (6) is equivalent to

$$u(x + t(y-x)) \leq u(x) + t(u(y) - u(x)), \tag{7}$$

First of all, we show (7) for all  $t$  of the form

$$t = j2^{-n}, j = 0, \dots, 2^n. \quad (8)$$

We proceed by induction on  $n$ . For  $n = 0$ , this is trivial and for  $n = 1$  it is just the definition of convexity. Suppose then that we have established (7) for  $n$ . Let  $t = j2^{-(n+1)}$ . Then, setting  $w = 2^{-1}(x + y)$ , we have

$$\begin{aligned} u(x + j2^{-(n+1)}(y - x)) &= u\left(x + j2^{-n}\frac{y - x}{2}\right) \leq u(x + j2^{-n}(w - x)) \leq \\ &u(x) + j2^{-n}(u(w) - u(x)) = u(x) + j2^{-n}\left(u\left(\frac{x + y}{2}\right) - u(x)\right) \leq \\ &\leq u(x) + j2^{-(n+1)}(u(y) - u(x)), \end{aligned}$$

by the inductive hypothesis, which is legitimate provided  $j2^{-n} \leq 1$ . If  $j = 2^n + 1, \dots, 2^{n+1}$ , we set  $k = 2^{n+1} - j$ . Then,

$$x + j2^{n+1}(y - x) = y + k2^{n+1}(x - y)$$

and we are back in the justifiable situation. By induction, we have established (7) for all  $t$  of the form (8). Now fix  $x, y \in \Omega$  and set

$$\varphi(t) = u(x + t(y - x)) - u(x) - t(u(y) - u(x)),$$

for  $t \in (0, 1)$ . By hypothesis,  $\varphi$  is continuous and we have shown that  $\varphi(t) \leq 0$ , for the dense set of  $t$  of the form (8). It follows that  $\varphi(t) \leq 0$ , for all  $t \in (0, 1)$  which proves (7) and concludes the proof.  $\square$

**Corollary 28.** *Let  $u$  be a real-valued function defined on a convex subset  $\Omega$  of  $\mathbb{R}^n$ . If  $u$  is continuous, then  $u$  is convex if and only if*

$$u(p_1x_1 + \dots + p_mx_m) \leq p_1u(x_1) + \dots + p_mu(x_m), \quad (9)$$

whenever  $x_1, \dots, x_m \in \Omega$  and  $0 \leq p_j \leq 1$ , with  $p_1 + \dots + p_m = 1$ .

*Proof.* We proceed by induction. The assertion for  $m = 1$  is trivial and for  $m = 2$  is the theorem. Suppose the assertion is true for  $m$  and let  $x_j$  and  $p_j$  be as in the theorem with  $j = 1, \dots, m + 1$ . We may assume that  $0 < p_{m+1} < 1$ . Note that

$$p_1x_1 + \dots + p_mx_m = (1 - p_{m+1})y,$$

where

$$y = \frac{p_1x_1 + \cdots + p_mx_m}{1 - p_{m+1}} = \sum_{j=1}^m \frac{p_j}{1 - p_{m+1}} x_j,$$

and

$$\sum_{j=1}^m \frac{p_j}{1 - p_{m+1}} = 1.$$

Thus, since  $\Omega$  is convex,  $y \in \Omega$ . From the theorem,

$$\begin{aligned} & u(p_1x_1 + \cdots + p_mx_m) = \\ & u((1 - p_{m+1})y + p_{m+1}x_{m+1}) \leq (1 - p_{m+1})u(y) + p_{m+1}u(x_{m+1}) = \\ & (1 - p_{m+1})u\left(\sum_{j=1}^m \frac{p_j}{1 - p_{m+1}} x_j\right) + p_{m+1}u(x_{m+1}) \leq \\ & (1 - p_{m+1}) \sum_{j=1}^m \frac{p_j}{1 - p_{m+1}} u(x_j) = p_1u(x_1) + \cdots + p_mu(x_m), \end{aligned}$$

where the inequality preceding the last equality is by the induction hypothesis.  $\square$

Having characterized continuous convex functions, we now characterize differentiable convex functions.

**Theorem 29.** *Let  $u$  be a real-valued function defined on a convex open subset  $\Omega$  of  $\mathbb{R}^n$ . If  $u$  is differentiable, then  $u$  is convex if and only if*

$$u(y) \geq u(x) + \nabla u(x) \cdot (y - x), \quad (10)$$

for every  $x, y \in \Omega$ .

*Proof.* The condition in the theorem certainly corresponds to the intuitive notion of a function being convex if its graph  $\{(y, u(y)) : y \in \Omega\}$  in  $\mathbb{R}^{n+1}$  is concave, for the condition says that the graph lies above the tangent space to the graph at  $(x, u(x))$ , for each  $x \in \Omega$ .

Suppose  $u$  is convex in  $\Omega$  and let  $x, y \in \Omega$ . Let  $h = y - x$  and  $t \in (0, 1)$ . By the convexity of  $u$ ,

$$u(x + th) \leq tu(x + h) + (1 - t)u(x).$$

This inequality may be rewritten as

$$u(x + th) - u(x) \leq t[u(x + h) - u(x)].$$

Subtracting  $t\nabla u(x) \cdot h$  from both sides and dividing by  $t$ ,

$$\frac{u(x + th) - u(x) - t\nabla u(x) \cdot h}{t} \leq u(x + h) - u(x) - \nabla u(x) \cdot h.$$

Since  $u$  is differentiable, the left-hand side tends to 0 as  $t \rightarrow 0^+$ . Thus we have the inequality (10).

Conversely, assume that (10) holds for every  $x, y \in \Omega$ . Let  $x_1, x_2 \in \Omega, x_1 \neq x_2$ . Let

$$x = \frac{x_1 + x_2}{2}, \quad h = x_1 - x.$$

Then  $x_2 = x - h$ . By (10) we have

$$u(x_1) \geq u(x) + \nabla u(x) \cdot h,$$

$$u(x_2) \geq u(x) + \nabla u(x) \cdot (-h).$$

Adding the inequalities, we get

$$u(x_1) + u(x_2) \geq 2u(x) \quad \text{or} \quad \frac{u(x_1) + u(x_2)}{2} \geq u\left(\frac{x_1 + x_2}{2}\right).$$

Thus,  $u$  is convex. □

For a real-valued  $C^2$ -function  $u$ , we denote the Hessian matrix by  $H_u$ . We write  $H_u \geq 0$  to mean that the associated quadratic form is positive semi-definite. Having characterized differentiable convex functions, we now characterize  $C^2$ -convex functions.

**Theorem 30.** *Let  $u$  be a real-valued function defined on a convex open subset  $\Omega$  of  $\mathbb{R}^n$ . If  $u \in C^2(\Omega)$ , then  $u$  is convex if and only if  $H_u \geq 0$ , that is,*

$$\left(\frac{\partial^2 u}{\partial x_j \partial x_k}\right) \geq 0. \tag{11}$$

*Proof.* We must prove that  $u$  is convex if and only if

$$\sum_{j,k=1}^n \frac{\partial^2 u}{\partial x_j \partial x_k}(x) h_j h_k \geq 0, \quad \text{for all } h \in \mathbb{R}^n, x \in \Omega.$$

Since  $\Omega$  is convex, we may use Taylor's formula for any pair of points  $x, y \in \Omega$ :

$$u(y) = u(x) + \nabla u(x) \cdot h + \sum_{|\alpha|=2} \frac{1}{\alpha!} \frac{\partial^2 u}{\partial x^\alpha}(x + sh) h^\alpha = \quad (12)$$

$$\begin{aligned} & u(x) + \nabla u(x) \cdot h + \sum_{j=k} \frac{1}{2} \frac{\partial^2 u}{\partial x_j^2}(x + sh) h_j^2 + \sum_{j < k} \frac{\partial^2 u}{\partial x_j \partial x_k}(x + sh) h_j h_k = \\ & u(x) + \nabla u(x) \cdot h + \frac{1}{2} \sum_{j,k=1}^n \frac{\partial^2 u}{\partial x_j \partial x_k}(x + sh) h_j h_k, \end{aligned}$$

where  $s \in (0, 1)$  and  $h = y - x$ . Thus, if  $Q_u(x, h)$  is the quadratic form associated to the Hessian  $H_u(x)$ , we have

$$u(y) = u(x) + \nabla u(x) \cdot h + \frac{1}{2} Q_u(x + sh, h). \quad (13)$$

To prove the theorem, suppose we have (11) for each  $z \in \Omega$ . Then, in particular, for  $z = x + sh$ , we conclude from (13) that

$$u(y) \geq u(x) + \nabla u(x) \cdot h.$$

Thus,  $u$  satisfies (6) and so  $u$  is convex.

On the other hand, if it is not true that (11) holds at every point  $x \in \Omega$ , then  $Q_u(x_0, h_0) < 0$  for some  $x_0 \in \Omega$  and some  $h_0 \neq 0$ . Since  $u \in C^2(\Omega)$ , the function  $Q_u(\cdot, h_0)$  is continuous in  $\Omega$ . Hence, there exists a  $\delta > 0$  such that  $Q_u(y, h_0) < 0$  for every  $y$  in the  $\delta$ -neighborhood of  $x_0$ . Let  $h = ch_0$ , where  $c$  is small enough that  $|h| < \delta$ , and set  $x = x_0 + h$ . Fix any  $s \in (0, 1)$ . Since  $Q(x_0 + sh, \cdot)$  is quadratic,

$$Q(x_0 + sh, h) = c^2 Q(x_0 + sh, h_0) < 0.$$

From (12)

$$u(x) < u(x_0) + \nabla u(x_0) \cdot h.$$

By (6),  $u$  is not convex in  $\Omega$ . □

We now state an analogous characterization of plurisubharmonic functions.

**Theorem 31.** *Let  $\Omega$  be an open set in  $\mathbb{C}^n$ . A real valued function  $u \in C^2(\Omega)$  is plurisubharmonic in  $\Omega$  if and only if  $L_u \geq 0$ , that is,*

$$\left( \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \right) \geq 0.$$

Since a real function in  $C^2(\Omega)$  is pluriharmonic if and only if  $L_u = 0$  on  $\Omega$ , it follows that a real  $C^2$ -function is pluriharmonic if and only if it is both plurisubharmonic and pluri-superharmonic.

## 8 The Dirichlet problem

The classical Dirichlet problem is the following. Given a bounded open subset  $\Omega$  of  $\mathbb{R}^n$  and a continuous function  $\varphi$  on the boundary  $\partial\Omega$ , find a harmonic function  $u$  in  $\Omega$  having boundary values  $\varphi$ . That is, find a function  $u$  continuous on  $\bar{\Omega}$  such that  $\Delta u = 0$  in  $\Omega$  and  $u = \varphi$  on  $\partial\Omega$ . One way of attacking the Dirichlet problem is via the method of Perron using subharmonic functions.

Harmonic functions in  $\mathbb{C}^n$  have the serious drawback that harmonicity is not preserved by biholomorphic change of coordinates. That is, if  $u$  is harmonic and  $L$  is a linear change of coordinates in  $\mathbb{C}^n$ , then  $u \circ L$  need not be harmonic. For the purposes of complex analysis in several variables, it would seem more appropriate to find a solution to the Dirichlet problem which is pluriharmonic. The class of pluriharmonic functions is a more restricted class than the class of harmonic functions. For the Dirichlet problem, the class of pluriharmonic functions is in fact too restricted. There exist continuous functions  $\varphi$  on the boundary of such smooth domains as the ball, for which there is no solution to the Dirichlet problem in the class of pluriharmonic functions. We seek to enlarge the class of pluriharmonic functions sufficiently to solve the Dirichlet problem, while retaining the property that this larger class will be preserved by complex change of coordinates. A solution is provided in terms of the complex Monge-Ampère equation.

The complex Monge-Ampère equation is the non-linear partial differential equation

$$\det H(u) = 0.$$

Since  $H(u) = 0$  for pluriharmonic functions, it is trivial that the class of solutions to the complex Monge-Ampère equation contains the pluriharmonic functions.

**Problem 29.** *The class of solutions to the complex Monge-Ampère equation is preserved by linear change of coordinates.*

Just as the Perron method uses subharmonic functions to find a harmonic solution  $h$  to the Dirichlet problem, it is possible in the ball to use the Perron method with plurisubharmonic functions to find a solution  $u$  to the Dirichlet problem which is plurisubharmonic and satisfies the complex Monge-Ampère equation.

**Theorem 32.** *The Dirichlet problem for the complex Monge-Ampère equation has a solution in the ball.*

*Proof.* Let  $B$  be a ball in  $\mathbb{C}^n$  and  $\varphi \in \partial B$ . Denote by  $\mathcal{U}$  the family of all plurisubharmonic functions  $v$  in  $B$  which are dominated by  $\varphi$  at the boundary  $\partial B$ . That is,

$$\limsup v(z)_{z \rightarrow \zeta} \leq \varphi(\zeta) \quad \forall \zeta \in \partial B.$$

Now set

$$\omega(z) = \sup_{v \in \mathcal{U}} v(z).$$

The regularization  $\omega^*$  of  $\omega$  is defined as follows:

$$\omega^*(z) = \limsup_{w \rightarrow z} \omega(w) \quad \forall z \in B.$$

One can show that  $\omega^*$  is plurisubharmonic in  $B$ , continuous in  $\overline{B}$  and satisfies

$$\det H(\omega^*) = 0 \quad \text{and} \quad \omega^*|_{\partial B} = \varphi.$$

□

This Monge-Ampère solution is smaller than the harmonic solution, since both solutions are obtained by taking suprema over classes of functions and the harmonic solution is the supremum over a larger class of functions. Similarly, one can use the Perron method with plurisuperharmonic rather than plurisubharmonic functions to obtain a solution which is plurisuperharmonic and satisfies the Monge-Ampère equation. This solution is greater than the harmonic solution. The Perron method thus yields a plurisubharmonic solution  $u$  and a plurisuperharmonic solution  $v$  both satisfying the Monge-Ampère equation such that  $u \leq h \leq v$ , where  $h$  is the harmonic solution.



## 9 Complex manifolds

Complex manifolds are higher dimensional analogs of Riemann surfaces. A manifold is, loosely speaking, a topological space which is locally Euclidean.

Let  $M$  be a connected, Hausdorff, space having a countable base of open sets. Suppose we are given a covering  $\mathcal{U} = \{U_\alpha\}$  of  $M$  by open sets and homeomorphisms  $\varphi_\alpha : U_\alpha \rightarrow V_\alpha$ , where each  $V_\alpha$  is an open set in real Euclidean space  $\mathbb{R}^n$ . A pair  $(U_\alpha, \varphi_\alpha)$  is called a *chart* and the family of charts  $\mathcal{A} = \{(U_\alpha, \varphi_\alpha)\}_\alpha$  is called an *atlas*. The open sets  $U_\alpha$  are called *coordinate neighborhoods* and the variable  $x^\alpha = \varphi_\alpha(p)$ , where  $p \in U_\alpha$ , is called a local coordinate corresponding to  $U_\alpha$ .

If  $U_\alpha \cap U_\beta \neq \emptyset$ , then we have a homeomorphism

$$\varphi_{\alpha\beta} = \varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \rightarrow \varphi_\beta(U_\alpha \cap U_\beta).$$

Such a homeomorphism is called a *change of coordinates* for the atlas  $\mathcal{A}$ . We say that an atlas  $\mathcal{A}$  is of smoothness  $k$  if each change of coordinates  $\varphi_{\alpha\beta}$  is of smoothness  $k$ .

Two atlases  $\mathcal{A}$  and  $\mathcal{A}'$  of smoothness  $k$ , corresponding respectively to coverings  $\mathcal{U}$  and  $\mathcal{U}'$ , are said to be *equivalent* if their union  $\mathcal{A} \cup \mathcal{A}'$  is again an atlas of smoothness  $k$ . A *real manifold* of smoothness  $k$  is a topological space  $M$  as above, together with an equivalence class of atlases of smoothness  $k$ . The (real) *dimension* of  $M$  is the dimension  $n$  of the open sets  $V_\alpha$  to which the coordinate neighborhoods  $U_\alpha$  are homeomorphic. If  $k = 0$ , we say that  $M$  is a *topological manifold*.

Loosely speaking, a real manifold is a topological space which is locally real Euclidean. We shall now introduce *complex manifolds*, which are, loosely speaking, locally complex Euclidean. Indeed, to define a complex manifold of (complex) dimension  $n$ , we copy the definition of a real manifold of (real) dimension  $n$ . The only difference is that, instead of requiring that the  $V_\alpha$  be open sets in real Euclidean space  $\mathbb{R}^n$ , we require that they be open sets in complex Euclidean space  $\mathbb{C}^n$ . We may speak of *complex coordinates*, charts, atlases etc. Thus, a complex manifold of complex dimension  $n$  can be considered as a real manifold of real dimension  $2n$ . Thus, it would seem that the study of complex manifolds is merely the study of real manifolds in even real dimensions. If we consider only *topological manifolds*, this point of view is plausible. However, when considering complex manifolds, we usually require a very high level of smoothness. A complex atlas  $\mathcal{A}$  is said to be a *holomorphic atlas* if the changes of coordinates  $\varphi_{\alpha\beta}$  are biholomorphic. A

*holomorphic structure* on  $M$  is an equivalence class of holomorphic atlases on  $M$ . Often, we shall, as elsewhere in mathematics, merely give a holomorphic atlas  $\mathcal{U}$  for a manifold and think of it as the equivalence class of all structures which are (biholomorphically) compatible with it. Of course we shall associate the same holomorphic structure to two holomorphic atlases  $\mathcal{U}$  and  $\mathcal{V}$  if and only if the two atlases are compatible. Since the union of compatible holomorphic atlases is a holomorphic atlas, for any holomorphic atlas  $\mathcal{A}$ , there is a maximal holomorphic atlas compatible with  $\mathcal{A}$ . This is merely the union of all holomorphic atlases compatible with  $\mathcal{A}$ . Thus, we may think of a holomorphic structure on  $M$  as a maximal holomorphic atlas. It seems we have now defined a holomorphic structure on  $M$  in three ways: as an equivalence class of holomorphic atlases, as a holomorphic atlas which is maximal with respect to equivalence or simply as a holomorphic atlas  $\mathcal{U}$ , meaning the equivalence class of  $\mathcal{U}$  or the maximal holomorphic atlas equivalent with  $\mathcal{U}$ . All that matters at this point is to be able to tell whether two holomorphic structures on  $M$  are the same or not. No matter which definition we use, we shall always come up with the same answer. That is two structures will be considered different with respect to one of the definitions if and only if they are considered different with respect to the other definitions.

A *complex holomorphic manifold* is a topological space  $M$  as above, together with a holomorphic structure. Since complex manifolds of dimension  $n$  of smoothness less than holomorphic are merely real manifolds of dimension  $2n$ , we shall consider only *holomorphic* complex manifolds. Thus, for brevity, when we speak of a complex structure, we shall mean a holomorphic structure and when we speak of a complex manifold, we shall always mean a manifold endowed with a complex (holomorphic) structure. A *Riemann surface* is a complex manifold of dimension one. Thus, complex manifolds are higher dimensional analogs of Riemann surfaces.

**Example.** Let  $M = \mathbb{R}^2 = \{(s, t) : s, t \in \mathbb{R}\}$ . We shall consider two atlases  $\mathcal{U}$  and  $\mathcal{V}$  on  $M$ . Each of these atlases will consist of a single chart.

$$\mathcal{U} = \{(\mathbb{R}^2, \varphi)\}, \varphi : \mathbb{R}^2 \rightarrow \mathbb{C},$$

where  $\varphi(s, t) = z = x + iy$ , with  $x = s, y = t$  and

$$\mathcal{V} = \{(\mathbb{R}^2, \psi)\}, \psi : \mathbb{R}^2 \rightarrow \mathbb{C},$$

where  $\psi(s, t) = w = u + iv$ , with  $u = s, v = -t$ . Since the change of charts  $z \mapsto w$  is given by  $w = \bar{z}$ , these two charts are not compatible. Hence the

two atlases  $\mathcal{U}$  and  $\mathcal{V}$  are not compatible, that is, are not equivalent. Thus, the atlases  $\mathcal{U}$  and  $\mathcal{V}$  define two *different* complex structures on  $\mathbb{R}^2$ .

The preceding example is an instance of the following fact which the student should verify. Let  $\mathcal{U}$  and  $\mathcal{V}$  be two holomorphic atlases on the same topological manifold  $M$ . The atlases  $\mathcal{U}$  and  $\mathcal{V}$  are compatible if and only if the identity mapping  $p \mapsto p$  from the complex manifold  $(M, \mathcal{U})$  to the complex manifold  $(M, \mathcal{V})$  is biholomorphic. In other words, two complex structures on the same topological manifold are the same if and only if the identity mapping is biholomorphic with respect to these two complex structures.

Since holomorphy, pluriharmonicity and plurisubharmonicity of functions are invariant under biholomorphic mappings, these notions may be well defined on complex manifolds. Namely, we define a function  $f$  on a complex manifold  $M$  to be holomorphic, pluriharmonic or plurisubharmonic if it is so in each coordinate. More precisely,  $f$  is said to be holomorphic, pluriharmonic or plurisubharmonic on an open set  $U \subset M$ , if for each coordinate neighborhood  $U_\alpha$  which meets  $U$ , the composition  $f \circ \varphi_\alpha^{-1}$  is respectively holomorphic, pluriharmonic or plurisubharmonic on  $\varphi_\alpha(U \cap U_\alpha)$ . Similarly, we define a mapping between manifolds to be holomorphic if it is holomorphic in the coordinates. More precisely, a mapping  $f : U \rightarrow M$  from an open subset  $U$  of a complex manifold  $M$  of dimension  $m$  to a complex manifold  $N$  of dimension  $n$  is said to be *holomorphic* if, for each chart  $(U_\alpha, \varphi_\alpha)$  for which  $U_\alpha$  meets  $U$  and each chart  $(V_\beta, \psi_\beta)$  for which  $V_\beta$  meets  $f(U \cap U_\alpha)$ , the composition  $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$  is a holomorphic mapping from the open subset  $\varphi_\alpha(f^{-1}(V_\beta) \cap U_\alpha)$  of  $\mathbb{C}^m$  into  $\mathbb{C}^n$ . It is easily verified that holomorphy, pluriharmonicity and plurisubharmonicity are preserved by holomorphic mappings between manifolds. That is, if  $g$  is a holomorphic mapping from an open subset  $U$  of a complex manifold  $M$  to a complex manifold  $N$  and if  $f$  is a function defined in a neighborhood of  $g(U)$ , which is holomorphic, pluriharmonic or plurisubharmonic, then the composition  $f \circ g$  is respectively holomorphic, pluriharmonic or plurisubharmonic on  $U$ . It also follows that the composition of holomorphic mappings between manifolds is holomorphic.

## 10 Examples of manifolds

In this section we give several examples of complex manifolds.

## 10.1 Domains

**Problem 30.** *Let  $M$  be a complex manifold and  $\Omega$  be a domain in  $M$ , that is, an open connected subset. Then, the complex structure of  $M$  induces a complex structure on  $\Omega$  making  $\Omega$  a complex manifold. The holomorphic and plurisubharmonic functions on  $\Omega$  considered as a complex manifold are precisely the holomorphic and plurisubharmonic functions on  $\Omega$  considered as an open subset of  $M$ .*

In particular, if  $\Omega$  is a domain in  $\mathbb{C}^n$ , then the holomorphic and plurisubharmonic functions on  $\Omega$  considered as a complex manifold are precisely the holomorphic and plurisubharmonic functions on  $\Omega$  considered as an open subset of  $\mathbb{C}^n$ . This shows that complex analysis on manifolds is a generalization of complex analysis on domains in  $\mathbb{C}^n$ .

## 10.2 Submanifolds

A connected subset  $M$  of  $\mathbb{R}^n$  is said to be a *submanifold* of  $\mathbb{R}^n$  of smoothness  $C^\ell$  if for each  $p \in M$  there is an open neighborhood  $U_p$  of  $p$ , a number  $k \in \{0, \dots, n\}$  and a  $C^\ell$ -diffeomorphism  $f = (f_1, \dots, f_n)$  of  $U_p$  onto an open neighborhood  $V_0$  of the origin such that

$$M \cap U_p = \{t \in U_p : f_{k+1}(t) = \dots = f_n(t) = 0\}.$$

If we write  $s = f(t)$  and  $N = f(M \cap U_p)$ , then in the local coordinates  $s_1, \dots, s_n$ ,

$$N \cap V_0 = \{s \in V_0 : s_{k+1} = \dots = s_n = 0\}.$$

The number  $k$  is called the *dimension* of the submanifold  $M$  at the point  $p$ .

We shall say that a subset  $M$  of  $\mathbb{C}^n$  is a real submanifold of  $\mathbb{C}^n$  of dimension  $k$  if  $M$  is a submanifold of dimension  $k$  of the space  $\mathbb{R}^{2n}$  underlying  $\mathbb{C}^n$ .

Analogously, a connected subset  $M$  of  $\mathbb{C}^n$  is said to be a complex *submanifold* of  $\mathbb{C}^n$  if for each  $p \in M$  there is an open neighborhood  $U_p$  of  $p$ , a number  $k \in \{0, \dots, n\}$  and  $f = (f_1, \dots, f_n)$  mapping  $U_p$  biholomorphically onto an open neighborhood  $V_0$  of the origin such that

$$M \cap U_p = \{z \in U_p : f_{k+1}(z) = \dots = f_n(z) = 0\}.$$

If we write  $\zeta = f(z)$  and  $N = f(M \cap U_p)$ , then in the holomorphic coordinates  $\zeta_1, \dots, \zeta_n$ ,

$$N \cap V_0 = \{\zeta \in V_0 : \zeta_{k+1} = \dots = \zeta_n = 0\}.$$

The number  $k$  is called the *dimension* of the complex submanifold  $M$  at the point  $p$ . Obviously, every submanifold of  $\mathbb{C}^n$  of dimension  $k$  can be thought of as a real submanifold of dimension  $2k$  (but not conversely).

**Problem 31.** *Show that a complex submanifold  $M$  of  $\mathbb{C}^n$  is indeed a complex manifold and if a function  $u$  is holomorphic, pluriharmonic or plurisubharmonic on the subset  $M$ , then,  $u$  is respectively holomorphic, pluriharmonic or plurisubharmonic on the submanifold  $M$ .*

Manifolds can be thought of as higher dimensional analogs of curves. The level curves of a function  $f(x, y)$  are familiar examples of curves in  $\mathbb{R}^2$  and the level surfaces of a function  $f(x, y, z)$  are familiar examples of surfaces in  $\mathbb{R}^3$ . In fact, we shall characterize submanifolds as level sets of mappings or, equivalently, as zero sets of mappings, or equivalently, as the common zero set of finitely many functions.

We have defined submanifolds of  $\mathbb{R}^n$  and  $\mathbb{C}^n$ . Let  $D$  be a domain in  $\mathbb{R}^n$  or  $\mathbb{C}^n$ . It is obvious how to define a submanifold of  $D$ . A submanifold  $M$  of a domain  $D$  is said to be a *closed* submanifold of  $D$  if  $M$  is a closed subset of  $D$ . For example the open intervals  $(0, 1)$  and  $(-1, +1)$  are both submanifolds of the open unit disc  $D$ . The first is not a closed submanifold of  $D$ , whereas the second is. Henceforth, when we speak of a submanifold of a domain  $D$ , we shall mean a closed submanifold. By a smooth manifold, we shall mean one such that the changes of coordinates are smooth mappings, by which we mean  $C^1$ -mappings.

**Theorem 33.** *Let  $M$  be a closed subset of a domain  $D$  in  $\mathbb{R}^n$ . Then  $M$  is a smooth submanifold of  $D$  if and only if, for each  $a \in M$ , there exists a neighborhood  $U \subset D$  and a smooth mapping  $f : U \rightarrow \mathbb{R}^m$  such that*

$$U \cap M = \{t \in U : f(t) = 0\}$$

and

$$\text{rank} \left( \frac{\partial f}{\partial t} \right) = \text{constant}, \quad \text{on } U.$$

*Proof.* Suppose  $M$  is a smooth submanifold of  $D$  and let  $k$  be the dimension of  $M$ . Fix  $a \in M$ . From the definition of submanifold, there is a diffeomorphism  $g$  of a neighborhood  $U$  of  $a$  onto an open neighborhood  $V$  of the origin in  $\mathbb{R}^n$ , such that

$$U \cap M = \{t \in U : g_{k+1}(t) = \cdots = g_n(t) = 0\}.$$

Let  $\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$  be the projection  $(s_1, \dots, s_n) \mapsto (s_{k+1}, \dots, s_n)$ . Set  $f = \pi \circ f$ . Then,  $f$  has the properties required of the theorem.

Suppose conversely that  $M$  is a closed subset of  $D$  and  $f$  a mapping having the properties stated in the theorem. Then, by the real rank theorem, there are neighborhoods  $U$  and  $V$  of  $a$  and of  $0 = f(a)$ , polydiscs  $D^n$  and  $D^m$  in  $\mathbb{R}^n$  and  $\mathbb{R}^m$  containing the origin, and diffeomorphisms  $\varphi : U \rightarrow D^n$  and  $\psi : V \rightarrow D^m$ , such that the mapping  $\psi \circ f \circ \varphi^{-1}$  has the form  $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_r, 0, \dots, 0)$ . We may assume that  $U \cap M = \{t \in U : f(t) = 0\}$ . Thus, if  $N = \varphi(M \cap U)$ , we have

$$N = \{x \in D^n : \psi \circ f \circ \varphi^{-1}(x) = 0\} = \{x \in D^n : x_1 = \dots = x_r = 0\}.$$

Thus, the pair  $(U, \varphi)$  is a smooth chart for  $M$  at  $a$  and  $M$  is a smooth submanifold of  $D$ .  $\square$

As an application of the above theorem, let  $f : D \rightarrow \mathbb{R}$  be a smooth function which is nonsingular, that is  $\nabla f(t) \neq 0$ , for each  $t \in D$ . Then, for each  $c \in \mathbb{R}$ , each component of the level set  $f(t) = c$  is a smooth submanifold of  $D$ . For example, the unit sphere

$$S^{n-1} = \{t \in \mathbb{R}^n : t_1^2 + \dots + t_n^2 = 1\}$$

is a smooth compact submanifold of dimension  $n - 1$  in  $\mathbb{R}^n$ .

The preceding results on smooth submanifolds of domains in  $\mathbb{R}^n$  have analogs for (complex) submanifolds of domains in  $\mathbb{C}^n$ .

**Theorem 34.** *Let  $M$  be a closed subset of a domain  $D$  in  $\mathbb{C}^n$ . Then  $M$  is a complex submanifold of  $D$  if and only if, for each  $a \in M$ , there exists a neighborhood  $U \subset D$  and a holomorphic mapping  $f : U \rightarrow \mathbb{C}^m$  such that*

$$U \cap M = \{z \in U : f(z) = 0\}$$

and

$$\text{rank} \left( \frac{\partial f}{\partial z} \right) = \text{constant}, \quad \text{on } U.$$

*Proof.* The proof is the same as for the real version using the complex rank theorem rather than the real rank theorem.  $\square$

As an application of the above theorem, let  $f : D \rightarrow \mathbb{C}$  be a holomorphic function which is nonsingular, that is  $(\partial f / \partial z)(z) \neq 0$ , for each  $z \in D$ . Then,

for each  $c \in \mathbb{C}$ , each component of the level set  $f(z) = c$  is a complex submanifold of  $D$ . For example, each component of the complex sphere

$$\{z \in \mathbb{C}^n : z_1^2 + \cdots + z_n^2 = 1\}$$

is a complex submanifold of dimension  $n - 1$  in  $\mathbb{C}^n$ . Note that the complex sphere is unbounded!

Fermat's last theorem (Wiles' Theorem), asserts that the equation

$$x^n + y^n = z^n, \quad n > 2,$$

has no integer solutions with  $xyz \neq 0$ . Note that, by the above theorem, each component of the intersection of the set

$$\{(x, y, z) \in \mathbb{C}^3 : x^n + y^n - z^n = 0\}$$

with the open set  $xyz \neq 0$  is a complex submanifold of the open set  $xyz \neq 0$ . Weils' theorem asserts that this submanifold does not intersect any points with integer coordinates.

### 10.3 Projective space

Before introducing projective spaces, we first recall the notion of a quotient topological space.

Let  $X$  be a topological space,  $Y$  a set and  $f : X \rightarrow Y$ . The quotient topology induced by  $f$  is the largest topology on  $Y$  such that  $f$  is continuous. The open sets in  $Y$  for the quotient topology are precisely those sets  $V \subset Y$  such that  $f^{-1}(V)$  is open.

To each equivalence relation on a set  $X$ , we associate the partition of  $X$  consisting of equivalence classes. Conversely, to each partition of  $X$ , we may associate the equivalence relation defined by saying that two elements of  $X$  are equivalent if they belong to the same member of the partition. This gives a one-to-one correspondence between equivalence relations  $\sim$  on  $X$  and partitions  $\mathcal{P}$  of  $X$ . A quotient set of  $X$  is defined as a set  $X/\sim$  of equivalence classes with respect to an equivalence relation  $\sim$  on  $X$ . There is a natural projection of  $X$  onto a quotient set  $X/\sim$  denoted  $p : X \rightarrow X/\sim$  defined by sending a point  $x$  to its equivalence class  $[x]$ . Let us say that a function  $f$  on  $X$  is  $\sim$ -invariant if  $f(x) = f(y)$ , whenever  $x \sim y$ . The projection induces a natural bijection between  $\sim$ -invariant functions on  $X$  and functions on  $X/\sim$ .

Let  $X$  be a topological space and  $\sim$  an equivalence relation on  $X$ . The quotient topological space induced by an equivalence relation  $\sim$  on  $X$  is the quotient set  $X/\sim$  endowed with the quotient topology induced by the natural projection  $p : X \rightarrow X/\sim$ . We sometimes speak of the quotient topology induced by an equivalence relation (partition) as the identification topology, since we identify points in the same equivalence class (member of the partition).

As an example, let  $X$  be the closed unit interval  $[0, 1]$  with the usual topology and let  $\mathcal{P}$  be the partition which identifies 0 and 1. That is, the members of the partition are the set  $\{0, 1\}$  and the singletons  $\{t\}, 0 < t < 1$ . The quotient space  $[0, 1]/\mathcal{P}$  is then the circle with its usual topology.

If a topological space  $X$  is connected and has a countable base for its topology, then any quotient space of  $X$  is also connected and has a countable base for its topology. However, a quotient space of a Hausdorff space need not be Hausdorff.

**Lemma 35.** *Let  $X/\sim$  be a quotient space of a Hausdorff space  $X$  with respect to an equivalence relation  $\sim$  on  $X$ . Then,  $X/\sim$  is also Hausdorff if and only if for each  $[x] \neq [y]$  in  $X/\sim$ , there are disjoint open subsets  $U$  and  $V$  of  $X$ , both of which are unions of equivalence classes, such that  $[x] \subset U$  and  $[y] \subset V$ .*

There is a general notion of a quotient manifold, which we shall not define in this section. We do present, however, the most important example, complex projective space  $\mathbb{P}^n$  of dimension  $n$ , which we think of as a compactification of the complex Euclidean space  $\mathbb{C}^n$  obtained by adding 'points at infinity' to  $\mathbb{C}^n$ . For  $n = 1$  we obtain the Riemann sphere  $\mathbb{C}^1 = \overline{\mathbb{C}}$ . Projective space  $\mathbb{C}^n$  is the most fundamental space for algebraic geometry.

We define projective space  $\mathbb{P}^n$  as the set of all complex lines in  $\mathbb{C}^{n+1}$  which pass through the origin. Let us denote a point  $\omega \neq 0$  in  $\mathbb{C}^{n+1}$  by  $(\omega_0, \dots, \omega_n)$ . Two points  $\omega$  and  $\omega'$  both different from zero lie on the same line through the origin if and only if  $\omega = \lambda\omega'$  for some  $\lambda \in \mathbb{C}$ . This is an equivalence relation,  $\sim$  on  $\mathbb{C}^{n+1} \setminus \{0\}$ . Projective space is the quotient space

$$\mathbb{C}^{n+1} \setminus \{0\} / \sim .$$

Since  $\mathbb{C}^{n+1} \setminus \{0\}$  is connected and has a countable base for its topology, the same is true of the quotient space  $\mathbb{P}^n$ .



Let us show that  $\mathbb{P}^n$  is Hausdorff. We invoke Lemma 35. Suppose then that  $[\omega]$  and  $[\omega']$  represent two distinct complex lines in  $\mathbb{C}^{n+1}$  passing through the origin. For any subset  $E$  of the unit sphere  $S$  in  $\mathbb{C}^{n+1}$ , we denote

$$C(E) = \{e^{i\theta}w : w \in E, \theta \in [0, 2\pi]\}.$$

We may assume that the points  $\omega$  and  $\omega'$  lie on the sphere  $S$ . Let  $u_j$  and  $v_j$  be sequences of open subsets of  $S$  decreasing respectively to  $\omega$  and  $\omega'$ . Suppose, for  $j = 1, \dots$ , there is a point  $p_j \in C(u_j) \cap C(v_j)$ . Since this sequence lies on the sphere  $S$  which is compact, we may assume that the sequence converges. The limit point must lie on both of the circles  $C(\omega)$  and  $C(\omega')$ , which however are disjoint. This contradiction shows that there exist open neighborhoods  $u$  and  $v$  of  $\omega$  and  $\omega'$  respectively in  $S$  such that  $C(u) \cap C(v) = \emptyset$ . Set

$$U = \{[w] : w \in C(u)\}, \quad V = \{[w] : w \in C(v)\}.$$

Then,  $U$  and  $V$  are disjoint open subsets of  $\mathbb{C}^{n+1} \setminus \{0\}$  which contain  $[\omega]$  and  $[\omega']$  respectively and which are both unions of equivalence classes. By Lemma 35, the quotient space  $\mathbb{P}^n$  is Hausdorff.

Let us denote the equivalence class (the line passing through  $\omega$ ) by the 'homogeneous coordinates'  $[\omega] = [\omega_0, \dots, \omega_n]$ . Let

$$U_j = \{[\omega_0, \dots, \omega_n] : \omega_j \neq 0\}, \quad j = 0, \dots, n,$$

and define a mapping  $\varphi_j : U_j \rightarrow \mathbb{C}^n$  by

$$\varphi_j([\omega_0, \dots, \omega_n]) = \left( \frac{\omega_0}{\omega_j}, \dots, \frac{\omega_{j-1}}{\omega_j}, \frac{\omega_{j+1}}{\omega_j}, \dots, \frac{\omega_n}{\omega_j} \right).$$

The family  $U_j, j = 0, \dots, n$  is a finite cover of  $\mathbb{P}^n$  by open sets and the mappings  $\varphi_j$  are homeomorphisms from  $U_j$  onto  $\mathbb{C}^n$ . Thus,  $\mathbb{P}^n$  has a countable base, since it has a finite cover by open sets each of which has a countable base. We have shown that  $\mathbb{P}^n$  is a connected Hausdorff space whose topology has a countable base and we have exhibited a topological atlas  $\mathcal{A} = \{(U_j, \varphi_j) : j = 0, \dots, n\}$ . Thus,  $\mathbb{P}^n$  is a topological manifold of complex dimension  $n$ .

**Problem 32.** *The atlas  $\mathcal{A} = \{(U_j, \varphi_j) : j = 0, \dots, n\}$  is a holomorphic atlas giving projective space  $\mathbb{P}^n$  the structure of a complex manifold.*

We may express projective space  $\mathbb{P}^n$  as the disjoint union of  $U_0$  which is biholomorphic to  $\mathbb{C}^n$  and the set  $\{[\omega] = [0, \omega_1, \dots, \omega_n] : \omega \neq 0\}$  which is in one-to-one correspondence with the points of  $\mathbb{P}^{n-1}$  in homogeneous coordinates. Thus,

$$\mathbb{P}^n = \mathbb{C}^n \cup \mathbb{P}^{n-1}.$$

Thus, we may think of projective space as a compactification of Euclidean space obtained by adding 'points at infinity'.

In view of our definition of projective space, it is natural to define  $\mathbb{P}^0$  to be the space of complex lines through the origin in  $\mathbb{C}$ . Thus,  $\mathbb{P}^0$  is a singleton which we may think of as a zero-dimensional complex manifold. Let us denote this ideal point by  $\infty$ . The preceding formula in this case becomes

$$\mathbb{P} = \mathbb{C} \cup \{\infty\},$$

the one-point compactification of  $\mathbb{C}$ . The complex projective space of dimension one is therefore the Riemann sphere.

## 10.4 Tori

In this section, we shall present, as an example of a complex manifold, the complex  $n$ -torus. But first we present the real  $n$ -torus.

In  $\mathbb{R}^n$ , let  $\omega_1, \dots, \omega_n$  be linearly independent. Let  $L$  be the lattice generated by these vectors:

$$L = \{k_1\omega_1 + \dots + k_n\omega_n : k_j \in \mathbb{Z}\} = \mathbb{Z}\omega_1 + \dots + \mathbb{Z}\omega_n.$$

Two points  $x$  and  $y$  in  $\mathbb{R}^n$  are said to be equivalent mod  $L$  if and only if  $y = x + \omega$ , for some  $\omega \in L$ . The real  $n$ -torus induced by  $L$  is the quotient space with respect to this equivalence relation and we denote it by  $\mathbb{R}^n/L$ . If  $L'$  is the lattice on  $\mathbb{R}^n$  by another set of independent vectors  $\omega'_1, \dots, \omega'_n$ , let  $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$  be a linear change of basis in  $\mathbb{R}^n$  mapping the basis  $\omega_1, \dots, \omega_n$  to the basis  $\omega'_1, \dots, \omega'_n$ . This is a homeomorphism which maps the lattice  $L$  to the lattice  $L'$  and two vectors are equivalent mod  $L$  if and only if their images are equivalent mod  $L'$ . Thus,  $f$  induces a homeomorphism of the quotient spaces  $\mathbb{R}^n/L$  and  $\mathbb{R}^n/L'$ . All real  $n$ -tori are thus homeomorphic to the standard real  $n$ -torus arising from the standard basis  $e_1, \dots, e_n$  of  $\mathbb{R}^n$ .

It can be shown that any real  $n$ -torus is homeomorphic to  $(S^1)^n$ , the  $n$ -fold product of the circle.

**Problem 33.** *Verify this for the case  $n = 1$ .*

Clearly,  $(S^1)^n$  is a compact real manifold of dimension  $n$  and so the same is true of any real  $n$ -torus.

We shall now construct complex tori. In  $\mathbb{C}^n$ , let  $\omega_1, \dots, \omega_{2n}$  be  $\mathbb{R}$ -independent and let  $L$  be the associated lattice:

$$L = \{k_1\omega_1 + \dots + k_{2n}\omega_{2n} : k_j \in \mathbb{Z}\} = \mathbb{Z}\omega_1 + \dots + \mathbb{Z}\omega_{2n}.$$

Two points  $z$  and  $\zeta$  in  $\mathbb{C}^n$  are said to be equivalent mod  $L$  if and only if  $\zeta = z + \omega$ , for some  $\omega \in L$ . The complex  $n$ -torus induced by  $L$  is the quotient space with respect to this equivalence relation and we denote it by  $\mathbb{C}^n/L$ . If we think of the real Euclidean space  $\mathbb{R}^{2n}$  underlying the complex Euclidean space  $\mathbb{C}^n$ , then we see that the complex  $n$ -torus  $\mathbb{C}^n/L$  can be (topologically) identified with the real  $2n$ -torus  $\mathbb{R}^{2n}/L$ . Thus, the complex  $n$ -torus is a compact real manifold of (real) dimension  $2n$  and hence with  $(S^1)^{2n}$ . We shall endow the complex  $n$ -torus with a complex structure with respect to which it is a complex manifold of (complex) dimension  $n$ .

For  $z \in \mathbb{C}^n$ , let  $B(z, r)$  be the ball of center  $z$  and radius  $r$  and set

$$[B(z, r)] = \bigcup_{\zeta \sim z} B(\zeta, r) = \bigcup_{w \in B(z, r)} [w]$$

$$U([z], r) = \{[w] : w \in B(z, r)\}.$$

By abuse of notation,  $[w]$  denotes the equivalence class of  $w$  as subset of  $\mathbb{C}^n$  in the first equation and denotes the corresponding point of  $\mathbb{C}^n/L$  in the second equation. If  $p$  is the natural projection from  $\mathbb{C}^n$  onto  $\mathbb{C}^n/L$ , then  $p([B(z, r)]) = U([z], r)$ . Thus,  $U([z], r)$  is an open neighborhood of the point  $[z]$  in the complex  $n$ -torus  $\mathbb{C}^n/L$ .

We claim that  $|\omega|$  is bounded below for  $\omega \in L$ . Consider first the lattice  $L_0$  generated by the standard basis  $e_1, \dots, e_{2n}$  of the underlying real vector space  $\mathbb{R}^{2n}$ . If  $\omega \in L_0$ , then  $\omega = k_1e_1 + \dots + k_{2n}e_{2n}$ . Thus,

$$\min\{|\omega| : \omega \in L_0, \omega \neq 0\} = 1. \tag{14}$$

Now the  $\mathbb{R}$ -linearly independent vectors  $\omega_1, \dots, \omega_{2n}$  generating the lattice  $L$  can be obtained from the standard basis  $e_1, \dots, e_{2n}$  by isomorphism of  $\mathbb{R}^{2n}$  and this isomorphism also maps the lattice  $L_0$  to the lattice  $L$ . Since this isomorphism is bilipschitz, it follows from (6) that for some  $r_L > 0$ ,

$$\inf\{|\omega| : \omega \in L, \omega \neq 0\} \geq r_L. \tag{15}$$

From (7) we see that if  $|z - \zeta| > r_L$ , then  $z$  and  $\zeta$  are not equivalent. From (7) we also see that  $[z]$  is a discrete set, since  $|a - b| \geq r_L$ , for any two distinct points in  $[z]$ . Thus, if  $\zeta$  is not equivalent to  $z$ , it follows that  $\zeta$  is at a positive distance from  $[z]$ , since  $\zeta$  is not in the discrete set  $[z]$ . Further, we claim that if  $[z] \neq [\zeta]$ , then these two sets are at a positive distance from one another. Suppose not. Then, there are  $z_j \sim z$  and  $\zeta_j \sim \zeta$  with  $|z_j - \zeta_j| \rightarrow 0$ . We have  $z_j = z + \alpha_j$  for some  $\alpha_j \in L$ . Thus,  $\zeta_j - \alpha_j$  is a sequence of points in  $[\zeta]$  which converges to  $z$ . This contradicts the fact that  $[\zeta]$  is at a positive distance from  $z$ . We have established that if  $[z] \neq [\zeta]$ , then  $[z]$  and  $[\zeta]$  are at a positive distance  $2r > 0$  from each other. Thus, the open sets  $[B(z, r)]$  and  $[B(\zeta, r)]$  in  $\mathbb{C}^n$  are disjoint. Let us now show that the complex  $n$ -torus  $\mathbb{C}^n/L$  carries a natural structure of a complex manifold of dimension  $n$  which is induced by the projection. Notice that the distance between any two points in the same equivalence class  $[z]$  is bounded below by  $r_L$ . Choose  $r < r_L/2$  and for each point  $[z] \in \mathbb{C}^n/L$ , denote by  $V_z$  the open ball  $B(z, r)$  in  $\mathbb{C}^n$ . Set

$$U_{[z]} = p(V_z) \quad \text{and} \quad \varphi_{[z]} = (p|_{V_z})^{-1}.$$

Then  $U_{[z]}$  is an open neighborhood of the point  $[z]$  in the complex  $n$ -torus  $\mathbb{C}^n/L$ . The family  $\{(U_{[z]}, \varphi_{[z]})\}$  is an atlas for the complex  $n$ -torus  $\mathbb{C}^n/L$ . Indeed, the projection is both open and continuous from the definition of the quotient topology. Moreover, no two points in  $V_z = B(z, r)$  are equivalent, since  $2r < r_L$ . Thus,  $p$  restricted to  $V_z$  is a homeomorphism of  $V_z$  onto  $U_{[z]}$ . The changes of coordinates are holomorphic since  $\varphi_{[z]} \circ \varphi_{[\zeta]}^{-1}$  is the identity if  $V_z \cap V_\zeta \neq \emptyset$ . Thus, the complex  $n$ -torus  $\mathbb{C}^n/L$  is a compact complex manifold of dimension  $n$ .

## 11 Automorphic functions and forms

### 11.1 The quotient manifold with respect to an automorphism group

Let  $M$  be a complex manifold. We denote by  $Aut(M)$  the group of biholomorphic mappings of  $M$  onto itself.  $Aut(M)$  is called the automorphism group of  $M$ . Let  $G$  be a subgroup of  $Aut(M)$ . We say that  $G$  acts on  $M$ . If  $p \in M$ , the  $G$ -orbit of  $p$  is  $[p] = gp : g \in G$ . Let  $M/G$  be the space of

$G$  – orbits with the quotient topology induced by the projection

$$\begin{aligned} M &\rightarrow M/G \\ p &\mapsto [p] = Gp. \end{aligned}$$

A group  $G$  of automorphisms of  $M$  acts *properly discontinuously* if for each compact  $K \subset M$ ,

$$g(K) \cap K \neq \emptyset,$$

for at most finitely many  $g \in G$ .

**Lemma 36.** *If  $G$  acts properly discontinuously on  $M$ , then  $M/G$  is Hausdorff.*

*Proof.* Let  $p_1 \neq p_2$  in  $M$  and let  $(\varphi_1, U_1)$  and  $(\varphi_2, U_2)$  be disjoint charts for  $p_1$  and  $p_2$ . We may suppose that  $\varphi_j(U_j) \supset \{|z| < 1\}$ . Set  $K_j = \varphi_j^{-1}(|z| \leq 1)$  and

$$A_n = \{p \in U_1 : |\varphi_1(p)| < 1/n\}, \quad B_n = \{p \in U_2 : |\varphi_2(p)| < 1/n\}.$$

Denote by  $\pi$  the projection of  $M$  onto  $M/G$ . Suppose  $\pi(A_n) \cap \pi(B_n) \neq \emptyset$ , for each  $n$ . Then, there is a  $b_n \in \pi(B_n)$  such that

$$b_n \in \pi^{-1}\pi A_n = \bigcup_{g \in G} g(A_n).$$

Thus, there is a  $g_n \in G$  and a point  $a_n \in A_n$  such that  $g_n(a_n) = b_n$ . Thus,  $g_n(K) \cap K \neq \emptyset$ , where  $K = K_1 \cup K_2$ . Since  $G$  is properly discontinuous,  $\{g_1, g_2, \dots\}$  is finite. For a subsequence, which we continue to denote  $\{g_n\}$ , we have  $g_n = g, n = 1, 2, \dots$

$$g(p_1) = \lim g(a_n) = \lim g_n(a_n) = p_2.$$

Thus,  $p_1$  and  $p_2$  are  $G$ -equivalent.

To show that  $M/G$  is Hausdorff, suppose  $[p_1]$  and  $[p_2]$  are distinct. Thus, by the previous paragraph, there is an  $n$  such that  $\pi(A_n) \cap \pi(B_n) = \emptyset$ . Since,  $\pi$  is open,  $\pi(A_n)$  and  $\pi(B_n)$  are disjoint open neighborhoods of  $[p_1]$  and  $[p_2]$ . Thus,  $M/G$  is Hausdorff.  $\square$

A group  $G \subset \text{Aut}(M)$  is said to act *freely* on  $M$  if the only  $g \in G$  having a fixed point is  $id_M$ .

**Theorem 37.** *Let  $M$  be a complex manifold and  $G$  a subgroup of  $\text{Aut}(M)$ . If  $G$  acts freely and properly discontinuously on  $M$ , then the projection  $\pi : M \rightarrow M/G$  induces on  $M/G$  the structure of a complex manifold and the projection is holomorphic.*

*Proof.* The quotient space  $M/G$  is connected and second countable and from the lemma it is Hausdorff. We need only show that  $\pi$  induces a holomorphic atlas.

Fix  $p \in M$ ; let  $(\varphi_p, U)$  be a chart at  $p$  such that  $\varphi_p(U) \supset \{|z| < 1\}$  and let  $K = \varphi_p^{-1}(|z| \leq 1)$  and  $A_n$  be defined as in the proof of the lemma. We show that  $\pi|_{A_n}$  is injective for sufficiently large  $n$ . Suppose not. Then, there are points  $a_n \neq b_n$  in  $A_n$  with  $\pi(a_n) = \pi(b_n)$ . Hence, for some  $g_n \in G$ , we have  $g_n(a_n) = b_n$ . Since  $a_n \neq b_n$ , no  $g_n \neq id_M$ . We have  $b_n \in K \cap g_n(K)$ . Since  $G$  is properly discontinuous, it follows that  $\{g_1, g_2, \dots\}$  is finite. Some subsequence, which we continue to denote  $\{g_n\}$  is constant,  $g_n = g$ . We continue to denote the corresponding subsequences of  $\{a_n\}$  and  $\{b_n\}$  as  $\{a_n\}$  and  $\{b_n\}$ . Then  $g(a_n) = b_n$ . Now  $a_n, b_n \rightarrow p$  and so by continuity,  $g(p) = p$ . Since  $G$  acts freely and  $g \neq id_M$ , this is a contradiction. Hence, for each  $p$  there is a  $A_n$  which we denote by  $A_p$  on which  $\pi$  is a homeomorphism. Thus,  $M/G$  is a manifold if we take as charts the family  $(\varphi_p \circ \pi|_{A_p}^{-1}, \pi(A_p)), p \in M$ . It is easy to check that the change of charts is biholomorphic so that  $M/G$  is in fact a complex manifold.  $\square$

The complex  $n$ -torus  $\mathbb{C}^n/L$  is an example of a quotient manifold of this type. If  $L$  is a lattice

$$L = \{k_1\omega_1 + \dots + k_{2n}\omega_{2n} : k_j \in \mathbb{Z}\} = \mathbb{Z}\omega_1 + \dots + \mathbb{Z}\omega_{2n}.$$

Write  $k = (k_1, \dots, k_{2n})$  and let  $g_k$  be the automorphism of  $\mathbb{C}^n$  given by

$$z \mapsto g_k(z) = k_1\omega_1 + \dots + k_{2n}\omega_{2n}.$$

We may identify the lattice  $L$  with the group of all such automorphisms  $g_k, k \in \mathbb{Z}^{2n}$ . The group  $L$  acts freely and properly discontinuously on  $\mathbb{C}^n$ .

## 11.2 Automorphic functions

Let  $M$  be a complex manifold and  $\Gamma \subset \text{Aut}(M)$  a subgroup of the group of automorphisms of  $M$ . A family of (zero free) holomorphic functions on  $M$ ,

$$\{j_\gamma(p) : \gamma \in \Gamma\},$$

is a *factor of automorphy* if

$$j_{\gamma\gamma'}(p) = j_\gamma(\gamma'(p))j_{\gamma'}(p), \quad \forall \gamma, \gamma' \in \Gamma, p \in M.$$

A function  $f$  meromorphic on  $M$  is called an automorphic function with respect to the factor of automorphy  $\{j_\gamma\}_{\gamma \in \Gamma}$  if

$$f \circ \gamma = fj_\gamma, \quad \forall \gamma \in \Gamma.$$

If all of the  $j_\gamma$  are 1, that is, when  $f$  is  $\Gamma$ -invariant, we simply say that  $f$  is *automorphic with respect to*  $\Gamma$ :

$$f \circ \gamma = f, \quad \forall \gamma \in \Gamma.$$

**Example**  $M = \mathbb{C}$ ,  $\Gamma = \{\gamma_n : n \in \mathbb{Z}\}$  the group of translations  $\gamma_n(z) = z + n$ . Then, the automorphic functions with respect to  $\Gamma$  are the periodic meromorphic functions with period 1:

$$f(z + 1) = f(z).$$

For example,  $f(z) = \sin(2\pi z)$  is such an automorphic function. So is  $f(z) = \tan(2\pi z)$ .

**Example** In  $\mathbb{C}^n$ , let  $\omega_1, \dots, \omega_{2n}$  be  $\mathbb{R}$ -independent; let

$$L = \{k_1\omega_1 + \dots + k_{2n}\omega_{2n} : k_j \in \mathbb{Z}\}$$

be the associated lattice and consider  $L$  as a group of translations  $L \subset \text{Aut}(\mathbb{C}^n)$ . The  $L$ -automorphic functions on  $\mathbb{C}^n$  then correspond to the meromorphic functions on the complex torus  $\mathbb{C}^n/L$ .

If  $n = 1$ , the  $L$ -automorphic functions are the doubly periodic meromorphic functions (elliptic functions) with periods  $\omega_1$  and  $\omega_2$ .

**Warning.** The student may have noticed that in defining automorphic functions we have slipped in meromorphic functions for the first time. In one variable, a meromorphic function  $f$  is merely the quotient  $f = g/h$  of holomorphic functions  $g$  and  $h$ , with  $h \neq 0$ . In several variables, meromorphic functions are rather more complicated to define and we shall avoid doing so for the present. The student could simply restrict his or her attention to automorphic functions which are holomorphic, but this would exclude one of the most interesting examples, elliptic functions. We therefore invite the student to accept that there are 'things' called meromorphic functions of several complex variables. We shall define meromorphic functions later.

**Example.** If  $A$  is a ring, we denote the general linear group and the special linear group, of degree 2 over  $A$  respectively

$$GL(2, A) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in A, ad - bc \neq 0 \right\}$$

$$SL(2, A) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in A, ad - bc = 1 \right\}.$$

To any

$$\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL(2, \mathbb{C}),$$

we associate an automorphism of the Riemann sphere  $\overline{\mathbb{C}}$

$$z \mapsto \gamma z = \frac{az + b}{cz + d}.$$

Every automorphism of  $\overline{\mathbb{C}}$  is of this form and if we multiply all four of the coefficients by the same constant, we obtain the same automorphism. Thus we may assume that  $ad - bc = 1$ . Moreover, if we replace  $a, b, c, d$  by  $-a, -b, -c, -d$  we still obtain the same automorphism. The representation is now unique. Thus,

$$Aut(\overline{\mathbb{C}}) = SL(2, \mathbb{C}) / \{I, -I\},$$

where  $\{I, -I\}$  is the subgroup of  $GL(2, \mathbb{C})$  consisting of the identity  $I$  and  $-I$ .

Let  $\mathbb{H}$  be the half-plane

$$\mathbb{H} = \{z \in \mathbb{C} : \Im z > 0\}.$$

Then

$$Aut(\mathbb{H}) = SL(2, \mathbb{R}) / \{I, -I\}.$$

The special linear group of degree 2 over  $\mathbb{Z}$ ,

$$SL(2, \mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\},$$

is the famous *modular group*. It acts properly discontinuously and freely on  $\mathbb{H}$ . An (elliptic) modular function is a function automorphic with respect to the modular group.



### 11.3 Automorphic forms

Let  $\Gamma \subset \text{Aut}(X)$ , and denote  $J_\gamma(p)$  the Jacobian determinant of  $\gamma$ , where  $X \subset \mathbb{C}^n$  is open. Then  $\{J_\gamma^{-m}\}$  is a factor of automorphy, for

$$J_{\gamma\gamma'}(z) = J_\gamma(\gamma'(z))J_{\gamma'}(z)$$

and hence

$$J_{\gamma\gamma'}(z)^{-m} = J_\gamma(\gamma'(z))^{-m}J_{\gamma'}(z)^{-m}.$$

A function  $f$  which is automorphic with respect to  $\{J_\gamma^{-m}\}$  is called an *automorphic form* of weight  $m$ . Thus,

$$f(\gamma(z)) = f(z)J_\gamma(z)^{-m}.$$

Notice that if  $f$  and  $g$  are automorphic forms of weight  $m$ , then  $f/g$  is an automorphic function,

$$\frac{f(\gamma(z))}{g(\gamma(z))} = \frac{f(z)J_\gamma(z)^{-m}}{g(z)J_\gamma(z)^{-m}} = \frac{f(z)}{g(z)}.$$

The above considerations can be used to prove the existence of meromorphic functions on a Riemann surface  $S$ . There exists a simply connected covering  $\tilde{S} \rightarrow S$ , the universal covering of  $S$ . By the uniformization theorem,  $\tilde{S}$  is biholomorphic to the Riemann sphere  $\overline{\mathbb{C}}$ , the complex plane  $\mathbb{C}$ , or the unit disc  $\mathbb{D}$ . Thus,  $\text{Aut}(\tilde{S})$  can be considered as a group of Moebius transformations. Let  $\Gamma$  be the subgroup of  $\text{Aut}(\tilde{S})$  which preserves the 'fibers'  $\pi^{-1}(p)$ . Then,  $\tilde{S}/\Gamma \simeq S$  and so  $\text{Mer}(S) \simeq \text{Mer}(\tilde{S}/\Gamma)$ , which may be thought of as the set of  $\Gamma$ -automorphic functions on the domain  $\tilde{S}$  of  $\overline{\mathbb{C}}$ . Thus, to show the existence of a meromorphic function on  $S$ , it is sufficient to construct two automorphic forms of the same weight  $m$  on the unit disc, complex plane or Riemann sphere.

### 11.4 Modular forms

These are automorphic forms with respect to the modular group  $SL(2, \mathbb{Z})$ .

Modular forms are important for Wiles' theorem (Fermat's last theorem).

Modular forms are also important for the *theory of everything*, by which we mean the unification of quantum theory and general relativity. The key is string theory which has the following prerequisites: Riemann surfaces, modular forms, representations.

## 12 The Riemann manifold of a function

### 12.1 Holomorphic continuation

The notion of holomorphic continuation is familiar from the study of functions of a single complex variable and was defined in the introduction for functions of several complex variables. We repeat the remarks made in the introduction. Let  $f_j$  be holomorphic in a domain  $D_j, j = 1, 2$  and suppose  $f_1 = f_2$  in some component  $G$  of  $D_1 \cap D_2$ , then  $f_2$  is said to be a direct holomorphic *continuation* of  $f_1$  through  $G$ . In shorthand, we also say  $(f_2, D_2)$  is a direct holomorphic continuation of  $(f_1, D_1)$ .

In one variable, holomorphic continuation is usually done from one disc to another using power series. We can do the same with polydiscs. Recall that if a power series converges in a polydisc, we have shown that the sum is holomorphic in that polydisc.

**Problem 34.** *Let  $f_1$  be the sum of a power series converging in a polydisc  $\mathbb{D}_1$  centered at  $a_1$  and let  $a_2 \in \mathbb{D}_1$ . Let  $f_2$  be the sum of the Taylor series of  $f_1$  about  $a_2$ . Then,  $f_2$  converges in any polydisc centered at  $a_2$  and contained in  $\mathbb{D}_1$ . If  $\mathbb{D}_2$  is any polydisc centered at  $a_2$  in which  $f_2$  converges, then  $(f_2, \mathbb{D}_2)$  is a direct holomorphic continuation of  $(f_1, \mathbb{D}_1)$ .*

**Problem 35.** *Let  $f$  be holomorphic in a domain  $D$ , let  $\Omega$  be a domain which meets  $D$  and let  $G$  be a component of the intersection. Show that, if there is a direct holomorphic continuation of  $f$  to  $\Omega$  through  $G$ , then it is unique.*

Let  $f$  be holomorphic in a domain  $D$  and let  $p \in \partial D$ . We say that  $f$  has a direct holomorphic continuation to  $p$  if there is a holomorphic function  $f_p$  in a neighborhood  $D_p$  of  $p$  such that  $(f_p, D_p)$  is a direct holomorphic continuation of  $(f, D)$  through some component  $G$  of  $D \cap D_p$  with  $p \in \partial G$ .

A domain  $D$  is a *domain of holomorphy* if it is the 'natural' domain for some holomorphic function. That is, if there is a function  $f$  holomorphic in  $D$  which cannot be directly holomorphically continued to any boundary point of  $D$ . In particular,  $f$  cannot be directly holomorphically continued to any domain which contains  $D$ .

In the introduction, it was left as an exercise to show that in  $\mathbb{C}$ , each domain is a domain of holomorphy.

An important difference between complex analysis in one variable and in several variables is the existence of domains which are not domains of holomorphy in  $\mathbb{C}^n, n > 1$ .

The following fundamental example was discovered by Hartogs. In studying this example, the student should draw the absolute value diagram associated to this figure. This can be found in any book on several complex variables, in particular in Kaplan. This diagram will also be drawn in class.

**Theorem 38.** . In  $\mathbb{C}^2$ , consider the domain

$$H = \{z : |z_1| < 1/2, |z_2| < 1\} \cup \{z : |z_1| < 1, 1/2 < |z_2| < 1\}.$$

Every function holomorphic in the domain  $H$  extends to the unit polydisc  $\mathbb{D} = \{z : |z_j| < 1, j = 1, 2\}$ .

*Proof.* Let  $f$  be holomorphic in  $H$ . Fix  $1/2 < \delta < 1$ . Then,

$$F(z_1, z_2) = \frac{1}{2\pi i} \int_{|\zeta|=\delta} \frac{f(z_1, \zeta)}{\zeta - z_2} d\zeta \quad (16)$$

defines a holomorphic function in the polydisc

$$\mathbb{D}_\delta = \{z : |z_1| < 1, |z_2| < \delta\}.$$

The proof that  $F$  is holomorphic is the same as that of Theorem 12, using the Leibniz formula, noting that  $F(z) = \int K(z, \zeta) d\zeta$ , where the kernel  $K$  is continuous and holomorphic in  $z$  in the domain  $|z_1| < 1, |z_2| \neq \delta$ . Since for  $|z_1| < 1/2$  the function  $f(z_1, \cdot)$  is holomorphic on  $|z_2| < 1$ , formula (2) implies that  $F(z_1, z_2) = f(z_1, z_2)$  in the polydisc  $|z_1| < 1/2, |z_2| < \delta$ . The uniqueness property of holomorphic functions implies that  $F = f$  on the intersection of  $H$  and this polydisc. Thus,  $F$  is a direct holomorphic continuation of  $f$  from  $H$  to the polydisc  $\mathbb{D}_\delta$ . Since the union of  $H$  and this polydisc is the unit polydisc  $\mathbb{D}$ , this concludes the proof.  $\square$

Having considered *direct* holomorphic continuation, we now introduce *indirect* holomorphic continuation. By holomorphic continuation along a chain of elements, we understand a sequence  $(f_1, D_1), (f_2, D_2), \dots, (f_m, D_m)$  of holomorphic elements, with given components  $G_j$  of successive intersections, such that  $(f_{j+1}, D_{j+1})$  is a direct holomorphic continuation of  $(f_j, D_j)$  through  $G_j$ , for  $j = 1, \dots, m-1$ . We say that there is a holomorphic continuation along a chain of elements from an element  $(f, D)$  to an element  $(h, \Omega)$  if there is a holomorphic continuation along a chain whose first element is  $(f, D)$  and whose last element is  $(h, \Omega)$ . We say that a holomorphic element

$(h, \Omega)$  is a holomorphic continuation of a holomorphic element  $(f, D)$  if there is a holomorphic continuation along a chain of elements from the element  $(f, D)$  to the element  $(h, \Omega)$ . We say that  $(h, \Omega)$  is an *indirect* holomorphic continuation of  $(f, D)$  if  $(h, \Omega)$  is a holomorphic continuation of  $(f, D)$  but is not a direct holomorphic continuation of  $(f, D)$ .

**Problem 36.** *Give an example of an indirect holomorphic continuation.*

**Problem 37.** *Holomorphic continuation along a given chain of domains through a given sequence of components of the respective intersections is unique. That is, if  $(f_1, D_1), \dots, (f_m, D_m)$  and  $(g_1, D_1), \dots, (g_m, D_m)$  are holomorphic continuations along the same chain of domains  $D_1, \dots, D_m$  through the same components  $G_j$  of successive intersections with same initial functions  $f_1 = g_1$ , then the terminal functions are the same  $f_m = g_m$ .*

Later, we shall construct the *Riemann manifold* of a holomorphic element  $(f, D)$  by considering all holomorphic continuations along chains  $(f_j, D_j)$  through components  $G_j$  of successive intersections with initial element  $(f, D)$  equal to  $(f_1, D_1)$  and by 'gluing' successive domains along the sets  $G_j$ . In the case of one variable, this is the familiar construction of the Riemann surface of a holomorphic element. For functions of several variables, the procedure is the same. The term Riemannian domain of  $f$  is often used rather than Riemann manifold of  $f$ . The Riemann manifold of a function  $f$  is a concrete instance of a complex manifold. Before explaining this gluing process used to construct the Riemann manifold of a function, we shall in the next section introduce the general notion of a spread manifold.

**Problem 38.** *Describe the Riemann surface of the logarithm.*

## 12.2 Spread manifolds

In this subsection we introduce the notion of a manifold  $N$  *spread* (or *étalé*) over a manifold  $M$ .

Suppose  $X$  and  $Y$  are topological spaces and let  $\varphi : Y \rightarrow X$ . We shall say that  $Y$  is *spread over*  $X$  by  $\varphi$  if  $\varphi$  is a local homeomorphism, i.e. each  $y \in Y$  has an open neighborhood  $V_y$  so that the restriction of  $\varphi$  to  $V_y$  is a continuous bijection onto its image with a continuous inverse.

**Example 1.** *Take  $X$  to be the unit circle  $C$  in  $\mathbb{C}$  with the usual topology, let  $Y$  be an open interval  $-\infty \leq a < t < b \leq +\infty$ , and define  $\varphi : (a, b) \rightarrow C$  by  $\varphi(t) = e^{it}$ . Then,  $(a, b)$  is spread over  $C$  by the mapping  $\varphi$ .*

From this example, we see that if  $\varphi : Y \rightarrow X$  is a spread of  $Y$  over  $X$ , then  $\varphi$  need not be surjective. Moreover, for those who know what a covering space is, this example shows that a spread over  $X$  need not be a covering space, even if it is surjective. However, a covering space is always a spread.

**Problem 39.** *If  $Y$  is a connected Hausdorff space with countable base spread over a complex manifold  $X$ , then there is a unique complex structure on  $Y$  with respect to which  $Y$  is a complex manifold and the projection is holomorphic.*

Under these circumstances, we say that  $Y$  is a (complex) manifold spread over the (complex) manifold  $X$ . We also say that  $Y$  is a spread manifold over  $X$ . The quotient  $M/G$  of a manifold with respect to a properly discontinuous automorphism group  $G$  which acts freely is an example. That is,  $M$  is a manifold spread over  $M/G$  by the natural projection. In the next section, we give another example, the Riemann manifold of a holomorphic function.

### 12.3 Riemann manifolds

In this subsection we introduce the notion of the Riemann manifold of a holomorphic function of several complex variables. Our nomenclature is not standard. What we shall call a Riemann manifold is usually called a Riemann domain, but we have chosen to call it a Riemann manifold because it is the higher dimensional analog of the Riemann surface of a function of a single complex variable. A Riemann manifold is a special case of a spread manifold. It is natural to confuse the notion of a *Riemann* manifold with that of a *Riemannian* manifold, especially in translating from one language to another. However, these two notions are quite different. Riemannian manifolds belong to real analysis, not complex analysis, and we shall perhaps define them later. Our purpose in this subsection is to define Riemann manifolds.

Since we have already alluded to the dangers of translation and since this course is likely to be given in French, I should at this point introduce French terms for the preceding notions. The French term for manifold is variété. A Riemannian manifold is une variété riemannienne. The Riemann surface of a function of one complex variable is la surface de Riemann de la fonction. The Riemann manifold of a function of several complex variables is la variété de riemann de la fonction.

Let  $f \in \mathcal{O}(\Omega)$ , where  $\Omega$  is a domain in  $\mathbb{C}^n$ . Let  $\pi : M \rightarrow \mathbb{C}^n$  be a manifold spread over  $\mathbb{C}^n$ . We shall say that  $M$  contains  $\Omega$  if there is a domain  $\tilde{\Omega}$  in  $M$

such that  $\pi$  maps  $\tilde{\Omega}$  biholomorphically onto  $\Omega$ . By the Riemann manifold of  $f$  we mean the maximal complex manifold spread over  $\mathbb{C}^n$  containing  $\Omega$  to which  $f$  extends holomorphically. Let us denote the Riemann manifold of  $f$  by  $M_f$ . Thus,  $M_f$  is the natural domain of  $f$  (over  $\mathbb{C}^n$ ).

**Problem 40.** *Let  $\Omega$  is a domain of holomorphy. Then there exists  $f \in \mathcal{O}(\Omega)$  such that  $M_f = \Omega$ .*

At this point, it would be good for the student to reconsider the earlier problem of describing the Riemann surface of the logarithm. Fix a branch  $f$  of  $\ln z$  in some domain  $D$  of  $\mathbb{C}$ . Describe the Riemann surface  $M_f$ .

Let us now return to the task of describing the Riemann manifold  $M_f$  of a holomorphic function  $f \in \mathcal{O}(\Omega)$ , where  $\Omega$  is a domain in  $\mathbb{C}^n$ . Since  $M_f$  will be associated to all possible holomorphic continuations of  $f$ , the end result will be the same if we construct the Riemann manifold of the restriction of  $f$  to some ball  $B$  contained in  $\Omega$ , for the direct holomorphic continuation of  $f$  from  $B$  to  $\Omega$  is unique. We thus assume from the outset that we are given an (holomorphic) element  $(f, B)$ . By this we mean that  $B$  is a ball in  $\mathbb{C}^n$  and  $f \in \mathcal{O}(B)$ . We shall construct the Riemann manifold  $M_f$  of the element  $(f, B)$ . Let

$$\mathcal{F} = \bigsqcup \{(f, B) : B \subset \mathbb{C}^n, f \in \mathcal{O}(B)\}$$

be the disjoint union of all elements  $(f, B)$  for all balls  $B$  in  $\mathbb{C}^n$ . We consider  $\mathcal{F}$  as a topological space by putting the topology of the ball  $B$  on each  $(f, B)$ . It may help to think of  $f$  as a mere index on the ball  $B$ . Thus, the topological space  $\mathcal{F}$  is a disjoint union of balls  $(f, B)$ . The ball  $(f, B)$  can be considered to be over the ball  $B$ . We merely map the point  $(f, z)$  of  $(f, B)$  to the point  $z \in B$ . Thus, the topological space  $\mathcal{F}$  is spread over  $\mathbb{C}^n$ . Note that  $\mathcal{F}$  is not connected. All of the balls  $(f, B)$  are disjoint from each other and are in fact distinct components of  $\mathcal{F}$ .

Let us now introduce an equivalence relation on  $\mathcal{F}$ . Let  $(f_\alpha, z_\alpha) \in (f, B_\alpha)$  and  $(f_\beta, z_\beta) \in (f_\beta, B_\beta)$  be any two points in  $\mathcal{F}$ . We write  $(f_\alpha, z_\alpha) \sim (f_\beta, z_\beta)$  if  $z_\alpha = z_\beta$  and  $f_\alpha = f_\beta$  on  $B_\alpha \cap B_\beta$ .

**Problem 41.** *This is an equivalence relation on  $\mathcal{F}$ .*

Let us denote the quotient space  $\mathcal{O} = \mathcal{F} / \sim$ . We describe the preceding construction by saying that the space  $\mathcal{O}$  is obtained by gluing two balls  $(f_\alpha, B_\alpha)$  and  $(f_\beta, B_\beta)$  in  $\mathcal{F}$  together along the intersection  $B_\alpha \cap B_\beta$  if and only if  $f_\alpha = f_\beta$  on this intersection. Let us denote an element of  $\mathcal{O}$  by

$[f, z]$ , where  $[f, z]$  is the equivalence class of the element  $(f, z)$  in  $\mathcal{F}$ . An element  $[f, z]$  is called a germ of a holomorphic function at  $z$  and  $\mathcal{O}$  is called the space of germs of holomorphic functions. We claim that the space  $\mathcal{O}$  is Hausdorff. Let  $[f_\alpha, z_\alpha]$  and  $[f_\beta, z_\beta]$  be distinct points of  $\mathcal{O}$  representing respectively equivalence classes of points  $(f_\alpha, z_\alpha)$  and  $(f_\beta, z_\beta)$  in  $\mathcal{F}$ . Since  $[f_\alpha, z_\alpha] \neq [f_\beta, z_\beta]$ , either  $z_\alpha \neq z_\beta$  or  $z_\alpha = z_\beta$  but  $f_\alpha$  and  $f_\beta$  are distinct holomorphic functions in some neighborhood  $B$  of  $z_\alpha = z_\beta$ .

In the first case, we may choose disjoint small balls  $D_\alpha$  and  $D_\beta$  containing  $z_\alpha$  and  $z_\beta$  and contained in the domains  $B_\alpha$  and  $B_\beta$  of  $f_\alpha$  and  $f_\beta$  respectively. Since,  $D_\alpha$  and  $D_\beta$  are disjoint, the disjoint open sets  $(f_\alpha, D_\alpha)$  and  $(f_\beta, D_\beta)$  in  $\mathcal{F}$  are remain disjoint in  $\mathcal{O}$ . These yield disjoint neighborhoods of  $[f_\alpha, z_\alpha]$  and  $[f_\beta, z_\beta]$ .

In the second case, no points of  $(f_\alpha, B)$  and  $(f_\beta, B)$  are identified, for this would imply that  $f_\alpha = f_\beta$  in  $B$ , which is not the case. Thus,  $(f_\alpha, B)$  and  $(f_\beta, B)$  are disjoint neighborhoods of  $[f_\alpha, z_\alpha]$  and  $[f_\beta, z_\beta]$ . We have shown that  $\mathcal{O}$  is Hausdorff.

At last, we may define the Riemann manifold  $M_f$  of an arbitrary holomorphic element  $(f, \Omega)$ , that is, of an arbitrary holomorphic function defined on a domain  $\Omega$  in  $\mathbb{C}^n$ . Let  $B$  be a ball in  $\Omega$  and define the Riemann manifold  $M_f$  of  $f$  (more precisely, of  $(f, \Omega)$ ) to be the component of  $\mathcal{O}$  containing the element  $(f, B)$ .

It can be verified that the Riemann manifold  $M_f$  of a holomorphic element  $(f, B)$  is indeed a manifold. Since  $M_f$  is spread over  $\mathbb{C}^n$ , we have only to check that  $M_f$  is Hausdorff, connected and has a countable base. First of all  $M_f$  is connected by definition, since it is a component of  $\mathcal{O}$ . To check that  $M_f$  is Hausdorff, it is sufficient to note that  $\mathcal{O}$  is Hausdorff, since  $M_f$  is a subspace of  $\mathcal{O}$ , which we have shown to be Hausdorff.

To show that the Riemann manifold  $M_f$  of a holomorphic element  $(f, B)$  is second countable is not so simple. We shall merely sketch the proof. It will be sufficient to construct a second countable subset  $X_f$  of  $M_f$  which is both open and closed. Since  $M_f$  is connected it will follow that  $X_f = M_f$  so  $M_f$  second countable.

We shall define the Riemann manifold associated to holomorphic continuation along a chain as in one complex variable. Let  $(f_1, B_1), \dots, (f_m, B_m)$  be a holomorphic continuation along a chain of balls  $B_1, \dots, B_m$ . We construct the Riemann manifold associated to this holomorphic continuation by gluing two balls  $(f_j, B_j)$  and  $(f_k, B_k)$  along their intersection if and only if their intersection is non empty and  $f_j = f_k$  on this intersection. The resulting

space is connected since in this process any two successive balls are glued together. This yields a complex manifold spread over  $\mathbb{C}^n$ . Using the same gluing rule, we may construct a complex manifold from any two holomorphic continuations along balls  $(f_1, B_1), \dots, (f_m, B_m)$  and  $(g_1, K_1), \dots, (g_m, K_m)$  having the same initial element  $(f_1, B_1)$ . That is, the balls  $B_1$  and  $K_1$  are the same and the holomorphic functions  $f_1$  and  $g_1$  coincide thereon. We can do the same with any finite collection of holomorphic continuations having the same initial element  $(f, B)$ . The result will always be connected and hence a complex manifold spread over  $\mathbb{C}^n$ .

Now let us perform such holomorphic continuations in a more systematic manner. Fix an initial holomorphic element  $(f, B)$ . Let  $M_{j,k}$  be the complex manifold spread  $\mathbb{C}^n$  obtained by holomorphic continuation along chains of at most  $k$  balls of radius  $1/j$  whose centers are obtained, starting from the center of  $B$  by taking (at most  $k$ ) steps of length  $1/j$  in the directions of the coordinate axes. Now we glue to such manifolds together along the intersection of two of their balls according to the usual rule. If we do this simultaneously to the whole family  $M_{j,k}$ , where  $j, k = 1, 2, 3, \dots$ , the result is connected and yields a second countable Hausdorff space  $X_f$  spread over  $\mathbb{C}^n$ . Now, if  $M_f$  is the Riemann manifold of a holomorphic element  $(f, B)$  and  $X_f$  is the manifold associated to holomorphic continuation of the element  $(f, B)$  constructed in the manner we have just described, it is not difficult to see that  $X_f$  is an open and closed subset of  $M_f$ . Since  $M_f$  is connected,  $M_f = X_f$ . Since  $X_f$  is by construction second countable, we have that  $M_f$  is second countable. Thus, the Riemann manifold of a holomorphic element  $(f, B)$  is indeed a manifold.

Holomorphic continuation of a holomorphic element  $(f, B)$  usually leads to a multiple-valued 'function'. That is, if by continuation along a chain we return to a former point, the new function may differ from the former function at that point. The Riemann manifold  $M_f$  of a holomorphic element  $(f, B)$  is constructed to remove this ambiguity. That is, there is a holomorphic function  $\tilde{f}$  on  $M_f$  such that  $\tilde{f} = f$  on the initial  $B$ . The function  $\tilde{f}$  is defined on any  $(f_\alpha, B_\alpha)$  arising in the definition of  $M_f$ , by setting  $\tilde{f} = f_\alpha$ . The equivalence relation used in defining  $M_f$  is designed precisely to assure that  $\tilde{f}$  is well defined, that is,  $\tilde{f}$  is a (single-valued) function on  $M_f$ . We recapitulate by once again saying that the Riemann manifold  $M_f$  of the holomorphic element  $(f, B)$  is the natural domain of  $f$ , that is the maximal domain over  $\mathbb{C}^n$  to which  $f$  extends holomorphically.



## 13 The tangent space

Here is a short definition which we shall explain after. The tangent space  $T(X)$  of a real manifold  $X$  of dimension  $n$  is the set of formal expressions

$$T(X) = \left\{ a_1 \frac{\partial}{\partial x_1} + \cdots + a_n \frac{\partial}{\partial x_n} : a_j \in C^1(X) \right\},$$

which is the space of smooth vector fields on  $X$ . We shall define the tangent space  $T_p(X)$  of  $X$  at a point  $p \in X$  and we shall set

$$T(X) = \bigcup_{p \in X} T_p(X).$$

There remains to define  $T_p(X)$ .

Let  $X$  be a smooth (real) manifold. If  $U$  is an open subset of  $X$ , we denote by  $\mathcal{E}(U)$  the set of smooth functions on  $U$ . If  $p \in X$ , let us say that  $f$  is a smooth function at  $p$  if  $f \in \mathcal{E}(U)$  for some open neighborhood  $U$  of  $p$ . Two smooth functions  $f$  and  $g$  at  $p$  are said to be equivalent if  $f = g$  in some neighborhood of  $p$ . This is an equivalence relation and the equivalence classes are called germs of smooth functions at  $p$ . For simplicity, we shall denote the germ of a smooth function  $f$  at  $p$  also by  $f$ . Denote by  $\mathcal{E}_p$  the set of germs of smooth functions at  $p$ . The set  $\mathcal{E}_p$  is an  $\mathbb{R}$ -algebra.

A *derivation* of the algebra  $\mathcal{E}_p$  is a vector space homomorphism

$$D : \mathcal{E}_p \rightarrow \mathbb{R}$$

such that

$$D(fg) = D(f) \cdot g(p) + f(p) \cdot D(g),$$

where  $g(p)$  and  $f(p)$  are the evaluations at  $p$  of the germs  $g$  and  $f$  at  $p$ .

The tangent space of  $X$  at  $p$ , denoted  $T_p(X)$ , is the vector space of derivations of the algebra  $\mathcal{E}_p$ .

Since  $X$  is a smooth manifold, there is a diffeomorphism  $h$  of an open neighborhood  $U$  of  $p$  onto an open set  $U' \subset \mathbb{R}^n$ :

$$h : U \rightarrow U',$$

and if we set  $h^*f(x) = f \circ h(x)$ , then  $h$  has the property that, for open  $V \subset U'$ ,

$$h^* : \mathcal{E}(V) \rightarrow \mathcal{E}(h^{-1}(V))$$

is an algebra isomorphism. Thus  $h^*$  induces an algebra isomorphism on germs:

$$h^* : \mathcal{E}_{h(p)} \rightarrow \mathcal{E}_p,$$

and hence induces an isomorphism on derivations:

$$h_* T_p(X) \rightarrow T_{h(p)}(\mathbb{R}^n).$$

Indeed, if  $D \in T_p(X)$  we define  $h_*(D) \in T_{h(p)}(\mathbb{R}^n)$  as follows: if  $f \in \mathcal{E}_{h(p)}$ , we set  $h_*(D)f \equiv D(h^*f)$ .

**Problem 42.** Fix  $a \in \mathbb{R}^n$ . Then,

- i)  $\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}$  are derivations of  $\mathcal{E}_a(\mathbb{R}^n)$  and
- ii) form a basis of  $T_a(\mathbb{R}^n)$ .

Applying this to the point  $a = h(p)$ , we see that the vector space  $T_p(X)$  is of dimension  $n$ , for each  $p \in X$ . The derivations given in the previous problem are the directional derivatives evaluated at the point  $h(p)$ .

Let  $f : M \rightarrow N$  be a smooth mapping between smooth manifolds. Then, there is a natural mapping:

$$f^* : \mathcal{E}_{f(p)} \rightarrow \mathcal{E}_p,$$

which in turn induces a natural mapping

$$df_p : T_p(M) \rightarrow T_{f(p)}(N)$$

given by  $df_p(D_p) = D_p \circ f^*$ . The mapping  $df_p$  is linear.

In local coordinates  $x$  for  $p$  and  $y$  for  $q = f(p)$ , consider the case  $D_p = \frac{\partial}{\partial x_i}$ . Let  $g \in \mathcal{E}_q$ .

$$df_p\left(\frac{\partial}{\partial x_i}\right)(g) = \frac{\partial}{\partial x_i} \circ f^*(g) = \frac{\partial}{\partial x_i}(g \circ f) = \sum_{j=1}^m \frac{\partial g}{\partial y_j} \frac{\partial f_j}{\partial x_i} = \left( \sum_{j=1}^m \frac{\partial f_j}{\partial x_i} \frac{\partial}{\partial y_j} \right) g.$$

Thus

$$df_p\left(\frac{\partial}{\partial x_i}\right) = \sum_{j=1}^m \frac{\partial f_j}{\partial x_i} \frac{\partial}{\partial y_j}.$$

This maps a basis element of  $T_p(M)$  to a basis element of  $T_{f(p)}(N)$ . Hence, in local coordinates, the linear transformation

$$df_p : T_p(M) \rightarrow T_{f(p)}(N)$$

is represented by the matrix:

$$df_p = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \cdot & \cdots & \cdot \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}.$$

The coefficients  $\frac{\partial f_j}{\partial x_i}$  are smooth functions of the local coordinate  $x$ . The mapping  $df_p$  has the following names: the derivative mapping, the differential, the tangent mapping, the Jacobian of  $f$  at  $a$ . The tangent mapping at  $p$  is the linear approximation of the smooth mapping  $f$ .

I would like our definition of the tangent space  $T_p(M)$  at a point  $p$  of a smooth manifold  $M$  to correspond to my intuitive notion of what it should be. The only situation in which I do have an intuitive notion is when I have an intuitive notion of  $M$  itself, that is, when  $M$  is a smooth submanifold of some Euclidean space. In this case, I think of  $T_p(M)$  as the space of all vectors with base point  $p$  which are tangent to  $M$  at  $p$ . For our definition, it is preferable to think of these vectors as having the origin as base point, so that  $T_p(M)$  is a vector subspace of the ambient Euclidean space. Tangent vectors generally are not contained in  $M$ , even if the base point  $p$  is. To obtain an intrinsic definition of  $T_p(M)$ , we note that there is a bijection between vectors  $a$  at the origin and derivatives  $\sum a_j \frac{\partial}{\partial x_j}$  with respect to  $a$ . Moreover, this correspondence is preserved by smooth mappings, and in particular, by smooth change of charts. Namely, if  $f$  is a smooth mapping,  $\sigma$  is a smooth curve passing through  $p$ , and the vector  $a$  is tangent to  $\sigma$  at  $p$ , then since  $df_p$  is the linear approximation of  $f$  at  $p$ , the vector  $df_p(a)$  is tangent to the curve  $f \circ \sigma$  at  $f(p)$ .

Having discussed the tangent space to a smooth (real) manifold, we now introduce the (complex) tangent space to a complex manifold. Let  $p$  be a point of a complex manifold  $M$  and let  $\mathcal{O}_p$  be the  $\mathbb{C}$ -algebra of germs of holomorphic functions at  $p$ . The complex (or holomorphic) tangent space  $T_p(M)$  to  $M$  at  $p$  is the complex vector space of all derivations of the  $\mathbb{C}$ -algebra  $\mathcal{O}_p$ , hence the complex vector space homomorphisms  $D : \mathcal{O}_p \rightarrow \mathbb{C}$  such that

$$D(fg) = f(p) \cdot D(g) + D(f) \cdot g(p).$$

In local coordinates, we note that  $T_p(M) = T_z(\mathbb{C}^n)$  and that the partial derivatives  $\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\}$  form a basis of  $T_z(\mathbb{C}^n)$ . If  $M$  is a submanifold of

some complex Euclidean space, then the complex tangent space to  $M$  at  $p$  is the largest complex vector subspace contained in the real tangent space to  $M$  at  $p$ . Having defined the (complex) tangent space  $T_p(M)$  to a complex manifold at a point  $p \in M$ , we define the (complex) tangent space  $T(M)$  of  $M$ :

$$T(M) = \bigcup_{p \in M} T_p(M).$$

## 14 Runge domains

A domain  $G$  in a complex manifold  $M$  is called a Runge domain in  $M$  if every function holomorphic in  $G$  can be approximated by functions holomorphic on all of  $M$ . For example, a domain  $G$  in  $\mathbb{C}$  is a Runge domain in  $\mathbb{C}$  if and only if  $G$  is simply connected.

**Theorem 39.** *If  $G_1, \dots, G_n$  are Runge domains in  $\mathbb{C}$ , then  $G = G_1 \times \dots \times G_n$  is a Runge domain in  $\mathbb{C}^n$ .*

It is easy to see that the property of being a Runge domain in  $M$  is invariant under biholomorphic mappings of  $M$ . That is, if  $\varphi$  is an automorphism of  $M$ , then it maps Runge domains in  $M$  to Runge domains in  $M$ . However, it is not invariant under biholomorphic mappings of the domains themselves. That is, if  $G$  is a Runge domain in  $M$  and  $\varphi$  maps  $G$  biholomorphically onto a domain  $W$  in  $M$ , then  $W$  need not be a Runge domain in  $M$ . Wermer gives an example of polynomial mapping  $\varphi$  of  $\mathbb{C}^3$  into  $\mathbb{C}^3$  which maps a Runge domain of  $\mathbb{C}^3$  biholomorphically onto a non Runge domain in  $\mathbb{C}^3$ .

## 15 Meromorphic functions

It is not so simple to define a meromorphic function on a complex manifold  $M$ . For one thing, it is not quite a function on  $M$ . But at least it turns out to be a function on 'most of'  $M$ . In one variable, meromorphic functions are quotients of holomorphic functions. In several variables, this is too restrictive, so we shall define them to be locally quotients of holomorphic functions.

If  $p$  is a point of a complex manifold  $M$ , the ring  $\mathcal{O}_p$  of germs of holomorphic functions at  $p$ , is an integral domain and so we may form the quotient field, which we denote by  $\mathcal{M}_p$ . This field is called the field of germs of meromorphic functions at  $p$ . Thus, a meromorphic germ  $f_p$  at  $p$  can be represented

as a quotient  $f_p = g_p/h_p$ , where  $g_p, h_p \in \mathcal{O}_p$ , and  $h_p \neq 0$ . We now define a meromorphic function  $f$  on  $M$  as a mapping  $f$  which assigns to each  $p \in M$  a meromorphic germ  $f_p$  at  $p$ . We impose the following compatibility between these germs. For every  $p \in M$ , there is a connected neighborhood  $U$  of  $p$  and holomorphic functions  $g, h \in \mathcal{O}(U)$  with  $h \neq 0$ , such that  $f_q = g_q/h_q$  for all  $q \in U$ . It turns out that we can (and shall) assume the following coherence property: for each  $q \in U$ ,  $g_q$  and  $h_q$  are relatively prime.

Now we would like a meromorphic function to indeed be a function, that is, taking complex values. Let  $p \in M$  and  $U, g$  and  $h$  be as in the above definition. We set

$$f(p) = \begin{cases} g(p)/h(p) & \text{if } h(p) \neq 0, \\ \infty & \text{if } h(p) = 0, g(p) \neq 0, \\ 0/0 & \text{if } h(p) = g(p) = 0. \end{cases}$$

The point  $p$  is called a point of holomorphy in the first case, a pole in the second case and a point of indeterminacy in the third case. This trichotomy is well defined, that is, does not depend on the choice of  $U, g$  and  $h$ . Moreover, the value  $f(p)$  is also well defined in the first two cases. The first two cases form an open dense set  $G$  of  $M$ . In this sense, a meromorphic function is a well defined function on most of  $M$ . In a neighborhood of each point of  $G$ , either  $f$  or  $1/f$  is holomorphic. Note that  $f$  is being considered as a mapping in two ways. First of all,  $f$  was originally defined as a mapping which assigns to each  $p \in M$ , a germ  $f_p$ , with a compatibility condition between germs. Now we are also considering  $f$  as a mapping which assigns to each  $p \in G$  a finite or infinite value  $f(p)$ , which may be considered as the value of the germ  $f_p$  at  $p$ . If  $p$  is a point of indeterminacy, then, for each complex value  $\zeta$ , there are points of holomorphy  $q$  arbitrarily close to  $p$  such that  $f(q) = \zeta$ . For proofs of these claims, see [2].

## 16 Compact Riemann surfaces and algebraic curves

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