I. Survival models and life tables

- Let the random variable $X$ (r.v.) represent the age-at-death of a newborn (mb) $\equiv$ lifetime of mb $\equiv$ time-until-death

- $X$: continuous, positive r.v.

- **Survival function**: $\Lambda(x) = P_x[X > x]$
  
  Probability that mb survives to age $x$

  Properties of $\Lambda(x)$
  i) $\Lambda(0) = 1$
  ii) $\Lambda(w) = 0$ $w$: terminal age (in practice around 110; may be $\infty$ with certain distributions)
  iii) non-increasing function of $x$ - GRAPH.

- **Cumulative distribution function (cdf)**
  
  $F_x(x) = P_x[X \leq x] = 1 - \Lambda(x)$  GRAPH

  Probability that mb dies before age $x$.

- **Probability density function (pdf)**
  
  $f_x(x) = \frac{d}{dx} F_x(x) = -\frac{d}{dx} \Lambda(x).$
• Conditional probability that a male dies between ages \( n + t \), given that he survives to age \( n \):

\[
P_x \left[ \gamma < X < \gamma + t \mid X > \gamma \right] = \frac{P_x \left[ \gamma < X < \gamma + t \right]}{P_x \left[ X > \gamma \right]}
\]

\[
= \frac{F_x (\gamma + t) - F_x (\gamma)}{1 - F_x (\gamma)} = \frac{\rho (\gamma) - \Lambda (\gamma + t)}{\rho (\gamma)} = 1 - \frac{\Lambda (\gamma + t)}{\rho (\gamma)}
\]

\[
\sim \text{Probability of dying between ages } n + t \text{ for a} \text{ male aged } n.
\]

• Force of mortality (hazard rate) at age \( n \)

\[
M_n = \frac{f_x (n)}{1 - F_x (n)} = -\frac{\rho (n)}{\rho (n)} = -\frac{d}{dx} \ln \Lambda (n) \quad (*)
\]

Properties of \( M_n \):

i) \( M_n \geq 0 \quad \forall n \).

ii) From (*) \( \Lambda (n) = \exp \left[ -\int_0^n M_z \, dz \right] \).

iii) As \( \lim_{n \to \infty} \rho (n) = 0 \), \( \lim_{n \to \infty} \int_0^n M_z \, dz = \infty \) (inter).

iv) From (*) \( -\rho' (n) = \Lambda (n) M_n \).

Integrate between \( y \) and \( y + m \)

\[
\Rightarrow \rho (y) - \rho (y + m) = \int_y^{y + m} \Lambda (x) M_x \, dx.
\]

v) Graph

vi) Interpretation.
• **Def of** \( X \)
\[
F_X(x) = 1 - F_X(x) = 1 - e^{-\int_0^x \mu_z \, dz}
\]
\( X \): continuous r.v., so \( f_X(x) = \frac{d}{dx} F_X(x) \)
\[
f_X(x) = -e^{-\int_0^x \mu_z \, dz} \cdot \mu_x
= \mu_x \cdot F_X(x).
\]

**ACTUARIAL NOTATION**

\((n)\): a person aged \( n \).

\( m_g_{n} = P_n [(n) \text{ dies within } m \text{ years}] \)

\( m_j_{n} = P_n [(n) \text{ survives the next } m \text{ years}] \)

N.B. If \( m = 1 \), it is omitted: \( g_{n} \) and \( j_{n} \).

\( t \mu g_{n} = P_n [(n) \text{ dies between ages } n+t \text{ and } n+t+m] \)

If \( m = 1 \), it is omitted.

Ex: \( 10 g_{50} = P_5 [(50) \text{ dies between ages 60 and 61}] \).
Future lifetime of \( \alpha \)

Let \( T(\alpha) \) be the r.v. representing the time-until-death of a person now aged \( \alpha \).

\( T(\alpha) \) : continuous positive r.v.

Then \( \alpha + T(\alpha) = X \rightarrow X - \alpha = T(\alpha) \)

let \( \alpha = 0 \) \( \Rightarrow \) \( T(0) = X \)

\[ \begin{align*}
\cdot \text{cdf of } & T(\alpha) \\
\Pr \left[ \alpha < X \leq \alpha + t \mid X > \alpha \right] = \Pr \left[ 0 < T(\alpha) < t \right] \\
\frac{\alpha(t) - \alpha(\alpha + t)}{\alpha(\alpha)} = F_{T(\alpha)}(t) \\
\Rightarrow \quad t_{B+} &= 1 - \frac{\alpha(\alpha + t)}{\alpha(\alpha)} = 1 - e^{-t} \\
= 1 - 2 \chi \left[ \int_{\alpha}^{\alpha + t} M_3 \, d\gamma \right] &= 1 - 2 \chi \left[ -\int_0^t M_{\alpha + \gamma} \, d\gamma \right] \\
\cdot \text{pdf of } & X \text{ continuous r.v., so} \\
\quad f_{T(\alpha)}(t) = \frac{d}{dt} F_{T(\alpha)}(t) \quad \Rightarrow \\
(\text{N.B. } \alpha = 0 \quad \ldots) 
\end{align*} \]
• Life expectancy at age $x \equiv \text{Mean of } T(x)$
  
  $E[T(x)]$ is denoted $\bar{e}_x$ \hspace{1cm} (\bar{e}_0 : \text{life expectancy at birth})

  Useful to compare 2 populations (ex: Canada vs USA, men vs women, smokers vs non-smokers, white vs black)

  $\bar{e}_0, \bar{e}_{65} : \text{socio-economic index.}$

  By definition, $\bar{e}_x = E[T(x)] = \int_0^\infty t \cdot f(t) \cdot \mu(t) \, dt$

  $= \ldots = \int_0^\infty t \cdot f(t) \, dt.$

• Variance of $T(x)$

  $\text{Var}(T(x)) = E(T(x)^2) - E^2(T(x))$

  $= \ldots = \int_0^\infty 2t \cdot f(t) \, dt - \bar{e}_x^2.$

• Median of $T(x)$

  Find $m(\bar{e}_x)$ such that

  $m(\bar{e}) \int_0^\infty f(\bar{e}) = 0.5$

• Mode of $T(x)$: value of $t$ maximizing

  the pdf $f_T(t) = \frac{1}{M} \cdot M + t.$
Some parametric laws of mortality used over the years - GRAPHS

• **De Moivre (1729)**
  \[ X \sim U[0, W] \quad p(x) = 1 - \frac{x}{W}, \quad 0 \leq x \leq W \]
  Find the distribution of \( T(x) \)
  Find \( M_x \).

• **Gompertz (1825)**
  \[ M_x = Be^{cx} \quad \text{Exponential fit of } x. \]

• **Makeham (1860)**
  \[ M_x = A + Be^{cx} \quad \text{A: force of accidental death,} \]
  \[ \text{Be}^{cx}: \text{force of ageing} \]

• **Weibull (1939) - engineering**
  \[ M_x = k x^n \quad \text{Polynomial fit of } x. \]

• **Perks (1932) - British actuary**
  \[ M_x = \frac{A + Be^{cx}}{1 + Ce^{cx}} \quad \text{logistic fit of } x \]
  \[ \text{as } n \to \infty, M_x \to \]
  \[ \text{- Band (1963): } A = 0 \]

• **Kannisto - demographic (survival to old ages)**
  \[ M_x = \frac{Be^{cx}}{1 + Be^{cx}} \]
Human Mortality: Graph of $M_x$.
No simple function of $x$ (with 3 or 4 parameters) can capture all the characteristics of human mortality over the whole range [0, 110].

Use instead a mortality table which contains, for integer values of $x = 0, 1, 2, \ldots, 110$, some basic functions, e.g. a multiple of $p(x)$.

Let $l_x = l_0 \cdot p(x)$ where $l_0 =$ radix of table (ex: $l_0 = 100,000$, arbitrary number)

Illustrative life table (Browne et al.)

<table>
<thead>
<tr>
<th>$x$</th>
<th>$l_x$</th>
<th>$d_x$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>100,000</td>
<td>42</td>
</tr>
<tr>
<td>1</td>
<td>97,958</td>
<td>132</td>
</tr>
<tr>
<td>2</td>
<td>97,826</td>
<td>70</td>
</tr>
<tr>
<td>3</td>
<td>97,756</td>
<td>160</td>
</tr>
<tr>
<td>4</td>
<td>97,596</td>
<td></td>
</tr>
<tr>
<td>110</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

1. Calculate $P_n[2]$ survives to age 4

$$2 \cdot \frac{d_2}{l_0} = \frac{p(4)}{p(2)} \cdot \frac{l_0}{l_2} = 0.99765$$

2. Calculate $P_n[1]$ dies between ages 2 and 3

$$1 \cdot \frac{d_1}{l_0} = \frac{p(2) - p(3)}{p(1)} \cdot \frac{l_0}{l_0} = \frac{l_2 - l_3}{l_1} = 0.0007145$$
let \( d_x = l_x - l_{x+1} \) be the number of deaths between ages \( x \) and \( x+1 \).

Then \( \mu_1 = \frac{d_x}{l_x} \quad \gamma_x = \frac{l_x - l_{x+1}}{l_x} = \frac{d_x}{l_x} \).

\[
\min \gamma_x = \frac{l_{x+m} - l_{x+m+1}}{l_x}.
\]

**Discrete Model**

Let the r.v. \( K(x) \) be the integral number of future years lived by \( (x) \equiv \text{current}\-\text{future}\-\text{lifetime of}(x) \).

Then \( K(x) = \lfloor T(x) \rfloor \quad K(x) \in \{0, 1, 2, 3, \ldots\} \)

\[P_n[K(x) = 0] = q_x = \frac{d_x}{l_x} \]

\[P_n[K(x) = 1] = P_n[1 < T(x) \leq 2]\]

\[= P_n[X + 1 < X + 2 | X > x] = \frac{\Delta^x(x+1) - \Delta^x(x+2)}{\Delta^x(x)} = \frac{l_{x+1} - l_{x+2}}{l_x} = \frac{d_{x+1}}{l_x} \]

\[= \frac{d_{x+1}}{l_x} \cdot \frac{l_{x+1}}{l_{x+1}} = \frac{d_{x+1}}{l_x} \cdot \frac{\gamma_{x+1}}{l_x} = \gamma_x \cdot \gamma_{x+1} = \gamma_x \cdot \gamma_{x+1} \]

In general, probability function of r.v. \( K(x) \)

\[P_n[K(x) = k] = k \cdot \gamma_x \cdot \gamma_{x+k} = k \cdot \gamma_x, \quad k = 0, 1, 2, \ldots \]

**Curtate life expectancy**: \( E[K(x)] = e_x \)

\[e_x = \sum_{k=0}^{\infty} k \cdot k \cdot \gamma_x = \ldots = \sum_{k=1}^{\infty} k \cdot d_x. \]
Hypothesis for fractionary ages

To calculate probabilities like \( \frac{1}{2} \delta_{x+1} + \frac{1}{2} \delta_{x+2} \) from a life table (or \( 2.7 \delta_{33.5} \)), we need an hypothesis on the behaviour of the curve \( l_x \) between 2 integral ages.

H1: UDD (uniform distribution of death) hypothesis.

Let \( n \) be an integer,

\( l_{x+n} \) is a linear function for \( t \in [0, 1] \) (\( n \) fixed)

\[
l_{x+n} = (1-t) l_x + t l_{x+1}, \quad 0 \leq t \leq 1
\]

(\( n \) fixed, \( t \) varies)

Linear interpolation, for \( l_{x+n} \) between \( l_x \) and \( l_{x+1} \).

From (*) , \( l_x - l_{x+n} = t (l_x - l_{x+1}) = t \cdot d_x \).

Then, \( 1 - \frac{q_x}{1-q_x} = \frac{l_x - l_{x+n}}{l_x} = \frac{t \cdot d_x}{l_x} = t \cdot q_x \)

\[
2 - \quad M_{x+n} = \frac{\frac{d}{dt} P_{x+1}}{p_{x+1}} = -\frac{d}{dt} l_{x+n} = \frac{1}{l_{x+n}} \frac{d}{dt} (l_x - t \cdot d_x)
\]

\[
= \frac{d_x}{l_{x+n}} = \frac{d_x}{l_x - t \cdot d_x} = \frac{q_x}{1-t \cdot q_x}
\]

\[
3 - \quad q_{x+y} = \ldots = \frac{q_x \cdot q_y}{1-t \cdot q_x}, \quad 0 \leq t \leq 1.
\]

\[
4 - \quad e^{(y \cdot d_x) M_{x+n}} = \ldots = \frac{q_x}{1-t \cdot q_x} \quad \text{(constant, independent of \( t \))}
\]

\( \Rightarrow \) UDD

Important result (If const.)

directly from 1.

Under UDD, \( e_{x+n} = e_x + 0.5 \)
H2: CFM (constant force of mortality) hypothesis.

\[ M_{x+t} = M_x \quad \text{for} \quad 0 < t < 1 \]

If \( M_{x+t} = M_x \), then

\[ \delta_x = \exp \left[ -\int_0^t M_{x+s} \, ds \right] = e^{-t \cdot M_x} \]

\[ \delta_x = e^{-M_x} \quad \text{and} \quad M_x = -\ln \delta_x = -\ln \frac{1}{\delta_x} \]

\[ t \delta_x = \exp \left[ -t \cdot M_x \right] = (e^{-M_x})^t = \delta_x^t \]

\[ \ln_{x+t} = \ln_x \cdot e^{-t \cdot M_x} = \ln_x \cdot \delta_x^t \]

\[ = \ln_x \left( \frac{\ln_{x+t}}{\ln_x} \right)^t = \ln_x \cdot \ln_{x+t}^t \quad \text{Geometric interpolation.} \]

\[ t \delta_x \cdot M_{x+t} = M_x \cdot e^{-t \cdot M_x} \Rightarrow \text{Exponential distribution.} \]

**Numerical comparisons**

If \( l_{30} = 96477 \quad \text{and} \quad l_{31} = 96350 \), calculate \( \frac{1}{2} l_{30} \)

Under UDD + CFM

\[ \frac{1}{2} l_{30} = \frac{l_{30/2}}{l_{30}} = \frac{1}{2} (l_{30} + l_{31}) = 0.99934181 \]

arithmetic average

CFM

\[ \frac{1}{2} l_{30} = \frac{l_{30/2}}{l_{30}} = (l_{30})^{1/2} = 0.99934160 \]

gometric average \( l_{30}^{1/2} \)

Little numerical differences in practice!
Select and ultimate mortality tables

Insurance companies select their insured lives after a medical exam: underwriting = selection of risks (there are uninsurable lives).

Mortality of insured lives better than that of general population (produced by Statistics Canada).

Mortality does not depend only on attained age, but also on number of years since policy issue (i.e. since medical exam or tests).
Review - Theory of interest (Ref: Kilgour)

\( i \): interest rate (\( \equiv \) effective rate of interest)

\( v \): discount factor

\( d \): discount rate

\( S \): face of interest

Relations:
\[
\begin{align*}
    v &= \frac{1}{1+i} \\
    d &= 1 - v = \frac{i}{1+i} \\
    S &= \ln(1+i) \\
    \delta &= -\ln v \\
    v &= e^{-\delta}
\end{align*}
\]

Accumulation function (in \( t \) years)
- simple interest: \( a(t) = 1 + it \)
- compound interest: \( a(t) = (1+i)^t \)

Present value of 1 payable in \( t \) years (compound int.)
\[
v^t = (1+i)^{-t} = e^{2\delta t}
\]

Interest compounded \( m \) times a year: \( i^{(m)} \)

Accumulated value of \( \$1 \) at end of year
\[
\left(1+\frac{i^{(m)}}{m}\right)^m = 1+i = e^\delta
\]

Present value of \( \$1 \) at end of year
\[
\left(1-\frac{d^{(m)}}{m}\right)^m = 1-d = e^{-\delta}
\]

If \( m \to \infty \), \( i^{(m)} \) and \( d^{(m)} \to \delta \).

P.V. of \( \$1 \) per year at BOY for \( m \) years
\[
\dd = 1 + v + \ldots + v^{m-1} = \frac{1-v^m}{\delta}
\]

\[
\overline{a}_{\infty} = \int_0^\infty e^{-\delta t} \, dt = \frac{1}{\delta}
\]
II - Life insurance
A. Benefits payable at the moment of death.
   a) Whole-life insurance

   Let us suppose an insurance policy paying $1 at death is issued to $(x)$. If the interest rate is constant in the future, the r.v. of the present value, at policy issue (i.e. age $(x)$) of the death benefit is

   $$Z = 1 \cdot v^{T(x)} = v^T, \quad T > 0.$$  

   The mean of r.v. $Z$ is called the actuarial present value (APV) of the death benefit (or net single premium)

   $$E(Z) = A_x = \int_0^\infty v^t f_T(t) dt$$

   $$= \int_0^\infty v^t t g_x M_{x+t} dt.$$  

   Variance of r.v. $Z$ = measure of risk to insurer.

   $$\text{Var}(Z) = E(Z^2) - E(Z)^2$$

   $$= E(v^{2T}) - A_x^2$$

   $$E(v^{2T}) = \int_0^\infty v^{2t} t g_x M_{x+t} dt$$

   N.B. $v = e^{-s}$, $v^2 = e^{-2s}$

   $$E(v^{2T}) = \int_0^\infty e^{-2st} t g_x M_{x+t} dt$$

   $$(\text{let } s^* = 2s)$$

   $$= \int_0^\infty e^{-s' \cdot t} s^* g_x M_{x+t} dt$$

   $$= A_x^* \quad (i.e. \text{calculated at } s^* = 2s)$$

   $$\text{denoted } 2^\infty A_x.$$  

   $$\text{Var}(Z) = 2^\infty A_x - A_x^2.$$
Ex. If $T(x) \sim$ De Moivre ($w = 100$) and $x = 50$, calculate at $S$, for a whole life insurance of 1000 payable at death:

a) the APV of the death benefit.

b) the variance of $Z$.

c) the pdf of $Z$.

d) the median of $Z$.

b) $m$-year term insurance

The insurance pays 1 at death of $x$ if death occurs within the next $m$ years; 0 if not.

$Z = \begin{cases} 
\sqrt{T}, & 0 < T \leq m, \\
0, & T > m.
\end{cases}$

APV $E(Z) = \overline{A}_{x: m} = \int_0^m \sqrt{t} g_{x+t} \mu_{x+t} dt$.

- Principle of notation
- Interpolation.

$Var(Z) = E(Z^2) - E(Z)^2$

$= \int_0^m \sqrt{2t} g_{x+t} \mu_{x+t} dt - \overline{A}_{x: m}^2$

$= \overline{A}_{x: m}^2$ calculated at $2S$ - $\overline{A}_{x: m}^2$

$= ^2 \overline{A}_{x: m} - \overline{A}_{x: m}^2$
c) m-year endowment insurance

Policy pays $1 at death of $x$, if $x$ dies within $m$ years; if $x$ survives
m years, $1$ is paid at time $m$.

\[ Z = \begin{cases} v^T, & 0 < T \leq m \\ v^m, & T > m \end{cases} \]

\[ A.P.V. \quad E(Z) = \int_0^m v^t d\mu_{x+t} dt + \int_m^\infty v^m d\mu_{x+t} dt. \]

Benefit composed of 2 parts:
m-year term insurance + survival benefit.

\[ E(Z) = \overline{A}_{x:m} = \overline{A}_{x:m} + v^m \int_0^\infty t d\mu_{x+t} dt \]

\[ P[T(x) > m] = 1 - P[T(x) \leq m] \]

\[ m_{\overline{x}} = 1 - m_{\overline{x}} \]

\[ \overline{A}_{x:m} = \overline{A}_{x:m} + v^m d\mu_x. \]

\[ \text{Var}(Z) = v^2 \overline{A}_{x:m} - \overline{A}_{x:m}^2. \]

Consider \( Z = Z_1 + Z_2 \)

\[ Z = \begin{cases} v^T, & 0 < T \leq m \\ 0, & T > m \end{cases} + \begin{cases} 0, & 0 < T \leq m \\ v^m, & T > m \end{cases} \]

\[ \text{Term insurance + Survival benefit.} \]

\[ E(Z) = E(Z_1) + E(Z_2) \]
\( Z_2 \) also called pure endowment. 
\( Z_1 \) and \( Z_2 \) not independent (in fact, negatively correlated).

Calculate correlation coefficient \( \rho \) between \( Z_1 \) and \( Z_2 \).

d) \( m \)-year defined insurance

\[
Z = \begin{cases} 
0, & T \leq m \\
\sqrt{T}, & T > m
\end{cases}
\]

\[
E(Z) = \bar{A}_x = \int_m^\infty \sqrt{t} \mu_{x+t} dt
\]

\[
= \int_0^\infty \frac{1}{\bar{m}} - \int_{-m}^0 = \bar{A}_x - \bar{A}_{x+m}/\bar{m}
\]

\[
\sigma^2 = \int_m^\infty \sqrt{t} \mu_{x+t} dt \\
\quad \text{(let } t = m + u) \\
\quad \text{Change of variable}
\]

\[
\int_0^{m+p} \sqrt{m+u} \mu_{x+m+u} du = \sqrt{m} \gamma_x \int_0^\infty \sqrt{u} \mu_{x+m} \mu_{x+m+u} du
\]

\[
\bar{m} \bar{A}_x = \sqrt{m} \gamma_x \bar{A}_{x+m}; \quad \text{Var}(Z)
\]

e) \( m \)-year defined, \( m \)-year term insurance

\[
Z = \begin{cases} 
0, & T \leq m \\
\sqrt{T}, & m < T < m + m \\
0, & T > m + m
\end{cases}
\]

\[
E(Z) = m/\bar{m} \bar{A}_x = m/\bar{A}_{x+m}/\bar{m}
\]

\[
\Rightarrow \cdots \quad \text{Var}(Z) = \cdots
\]
13. Benefit payable at the end of year of death of (x) → discrete model.

a) Whole life insurance.

If 1 is paid at the end of year of death of (x).

$Z$: r.v. of present value of death benefit

$Z = v^{K+1}, \quad K \in \{0, 1, 2, \ldots \}$

where $P_x^{\{K = k\}} = \frac{q_x^k}{v^k}$.

$P_T$ for whole life ins. $E(Z)$

$E(Z) = A_x = E(v^{K+1}) = \sum_{k=0}^{\infty} v^{k+1} \frac{q_x^k}{v^k}$.

$Var(Z) = E(Z^2) - E^2(Z)$

$= \sum_{k=0}^{\infty} v^{2(k+1)} \frac{q_x^k}{v^k} - A_x^2$

$= A_x^2 - A_x^2$.

b) $n$-year term insurance ($n$: integer)

$Z = \begin{cases} v^{K+1} \quad K = 0, 1, \ldots, n-1 \\ 0 \quad K \geq n. \end{cases}$

$E(Z) = A_x^{k:n} = \sum_{k=0}^{n-1} v^{k+1} \frac{q_x^k}{v^k}$.

$Var(Z) = A_x^{k:n} - A_x^{k:n}$.
c) \( m \)-year endowment ins.

\[
Z = \begin{cases} 
 v^{k+1}, & K = 0, \ldots, m-1 \\
 v^m, & K \geq m.
\end{cases}
\]

\[
E(Z) = A_{x \over m} \cdot \overline{m} = \sum_{k=0}^{m-1} v^{k+1} \cdot \overline{m} + \sum_{k=m}^{\infty} v^m \cdot \overline{m}
\]

\[
= A_{x \over m} + v^m \cdot \overline{m} \cdot \overline{m}
\]

\[
\text{Var}(Z) = \ldots
\]

\[
P(Z_1, Z_2) \quad \text{where} \quad Z_1: m\text{-year ins.}
\]

\[
Z_2: m\text{-year endowment}
\]

d) \( m \cdot A_x \) \quad m\text{-year deferred ins.}

e) \( m \cdot m \cdot A_x \) \quad m\text{-yr deferred, m-yr term ins.}

\( m + m \) are integers.
Ex 1

Consider an endowment insurance policy issued to (20) of $1000, where $1000 is paid at end of yr if death occurs before age 25 or at age 25 if (20) survives 5 years.

a) Calculate the probability that \( Z \) will die before age 25.

b) Calculate the APV of benefits.

c) Calculate the standard error of \( Z \).

Solution: ...

Ex 2

\( b_x : \text{death benefit} \)

\[
\begin{array}{c|cccc|c}
N & l_x & d_x & b_x & k \\
20 & 1000 & 1000 & 0 & 0 \\
21 & 995 & 100 & 1 & 1 \\
22 & 990 & 100 & 2 & 2 \\
23 & 984 & 1500 & 3 & 3 \\
24 & 977 & 2000 & 4 & 4 \\
25 & 969 & 0 & 5 & 5 \\
26 & 959 & 0 & 6 & 6 \\
\end{array}
\]

d) Calculate the probability that \( Z \) will die before age 25.

e) Calculate the APV of benefits.

f) If \( n \) similar policies are issued, which risk loading \( \theta \) will make the portfolio profitable with 95% probability.

Solution: ...
Approximation for $\overline{A}_x$ in terms of $A_x$.

1- Approximation method.

$A_x > A_x$ since benefit paid sooner.

Higher NSP.

If mortality not taken into account, on average benefit paid 6 months

($= \frac{1}{2}$ year sooner)

With simple interest for a half-year

$\overline{A}_x \approx (1 + \frac{1}{2}) A_x$.

2- Under UDD between ages $x + n$ and $x + n + 1$.

We know $\sum_{t=n}^{x} v^{t+x+n}= q_{x+n}$ for $0 \leq t < 1$.

$\overline{A}_x = \int_0^\infty v^t \, dm_{n+t} \, dt$

$= \int_0^{\infty} v^t \, dm_{n+t} \, dt + \int_1^{\infty} v^t \, dm_{n+t} \, dt + \ldots$

$= \int_0^{\infty} v^t \, dm_{n+t} \, dt + \int_1^{\infty} v^t \, dm_{n+t} \, dt + \ldots$

Let $t = 1 + n$,

$= \int_0^{\infty} v^t \, dm_{n+t} \, dt + \int_0^{\infty} v^t \, dm_{n+t} \, dt + \ldots$

Under UDD,

$\overline{A}_x = q_{x+n} \int_0^{\infty} v^t \, dt + \int_0^{\infty} v^t \, dm_{n+t} \, dt + \ldots$

$\int_0^{\infty} v^t \, dm_{n+t} = \ldots = \alpha = \frac{1-v}{2} = \frac{d}{g}$
\[ A_n \xrightarrow{\text{UDD}} \frac{d}{d} \delta_n + \frac{vd}{d} \delta_n \delta_{n+1} + v^2 \frac{d}{d} \delta_n \delta_{n+2} + \ldots \]

As \( d = iv \)

\[ A_n = \frac{i}{d} \left[ \delta \left( \frac{v \delta_n + v^2 \delta_{n+1} + v^3 \delta_{n+2} + \ldots} {1} \right) \right] = \frac{i}{d} \sum_{k=0}^{\infty} \frac{v^k \delta_{n+k}}{k!} = \frac{i}{d} A_n \]

N.B. \( \frac{i}{d} = \frac{i}{\ln(1+i)} = \frac{i}{i\pi} = \frac{i}{\pi} - \frac{i^2}{2} + \frac{i^3}{3} - \ldots \)

\[ = \left[ 1 - \frac{i}{2} + \frac{i^2}{3} - \ldots \right]^{-1} \]

\[ = 1 + \frac{i}{2} + i^2 \left( -\frac{1}{3} + \frac{1}{4} \right) + \ldots = 1 + \frac{i}{2} - \frac{i^2}{12} \]

\[ \sim 1 + \frac{i}{2} \quad \left( \text{pl} \quad i = 0.1, \frac{i^2}{12} < 0.001 \right) \]

Show that under UDD

\[ A_n \xrightarrow{\text{UDD}} \frac{i}{d} A_n \]

\[ A_n \xrightarrow{\text{UDD}} \frac{i}{d} A_n + v^m \delta_n \]

\[ m \mid A_n \]

\[ m \mid m \quad A_n \]
Recursive relationship

1- between \( e_x \) and \( e_{x+1} \)
\[
e_{x} = \sum_{k=1}^{\infty} k d_{x} = d_{x} + 2d_{x} + 3d_{x} + \ldots
\]
\[
= d_{x} + d_{x} \left[ d_{x+1} + 2d_{x+1} + \ldots \right]
\]
\[
e_{x} = d_{x} + d_{x} e_{x+1}
\]

With a mortality table and 1 value \( e_{x} \), all values \( e_{0}, \ldots, e_{n} \) can be calculated recursively.

2- between \( A_x \) and \( A_{x+1} \).
\[
A_{x} = A_{x} + 11 A_{x}
\]
\[
A_{x} = v q_{x} + v d_{x} A_{x+1}
\]
Same remark + interest rate \( i \).

1- and 2- of the same form
\[
\nu_{x} = c_{x} + v d_{x} \nu_{x+1}
\]
\[
(1- \text{ with } i = 0 \text{ or } v = 1).
\]
3- between \( \ddot{a}_{x} \) and \( \ddot{a}_{x+1} \). (of the form above)
III Life annuities

A - Introduction
life insurance: payment made if person dies
life annuity: payment made if person survives.

Life annuity: series of payments made at periodic intervals as long as a person survives.

Types of annuities
- Temporary vs life annuity.
- Payments starting immediately vs deferred (ex. to age 65).
- Payments made at beginning of interval (annuity due) vs payments made at end (annuity immediate).

Same terminology and notation as used for annuities certain but survival is now a condition for payment.

(Reminder: Pure endowment)

\[ A_{x:\overline{n}|} = v^m n g_x \]

(discount for interest and mortality).
B. Continuous annuity

Life annuity

Let us consider a life annuity of $1 for annuity payable continuously, as long as \( \alpha(x) \) lives.

The n.v. of the present value of payments of this life annuity (until the death of \( \alpha(x) \))

\[
Y = \frac{\overline{a}_{\alpha\alpha}}{t_{\alpha\alpha}} = \overline{a}_{\alpha\alpha} = 1 - \frac{v}{s}
\]  

(*)

The APV of a continuous life annuity issued to \( \alpha(x) \):

\[
\overline{\alpha}_{\alpha} = E(Y) = E\left(\overline{a}_{\alpha\alpha}\right)
\]

\[
\text{df of } T : \frac{dx}{M_{\alpha\alpha} t}
\]

\[
= \int_0^\infty \overline{a}_{\alpha\alpha} \cdot \frac{dx}{t_{\alpha\alpha} + \partial_x M_{\alpha\alpha} + dt}
\]

Integrate by parts:

\[
v = -\frac{dx}{t} \quad \text{d}u = v^t
\]

\[
\overline{\alpha}_{\alpha} = \left[-\overline{a}_{\alpha\alpha} \cdot \frac{dx}{t_{\alpha\alpha}}\right]_{t=0}^{t=\infty} + \int_0^\infty v^t \cdot \frac{dx}{t_{\alpha\alpha} + \partial_x M_{\alpha\alpha} + dt}
\]

\[
\overline{\alpha}_{\alpha} = \int_0^\infty v^t \cdot \frac{dx}{t_{\alpha\alpha} + \partial_x M_{\alpha\alpha} + dt}
\]

Interpretation: \( v^t \cdot \frac{dx}{t_{\alpha\alpha} + \partial_x M_{\alpha\alpha} + dt} \) is APV (discounted for interest and mortality) of payment at time \( t \) to sum of these values over all possible values of \( t \).

(analogy to \( \overline{a}_{\alpha\alpha} = \int_0^\infty v^t dt \).
Relation between $\overline{a}_n = \overline{A}_n$

\[
y = \frac{\overline{a}_n}{\overline{A}_n} = \frac{1 - \overline{v}^T}{\delta}
\]

Take expectation on each side:

\[
E(y) = E\left(\frac{1 - \overline{v}^T}{\delta}\right)
\]

\[
\overline{a}_n = \frac{1 - E(\overline{v}^T)}{\delta} = 1 - \frac{\overline{A}_n}{\delta}
\]

\[
\Rightarrow \overline{A}_n = 1 - \delta \overline{a}_n \quad (1 = \delta \overline{a}_n + \overline{A}_n)
\]

Variance of $\overline{a}_n$

\[
\text{Var}(\overline{a}_n) = \text{Var}\left(\frac{1 - \overline{v}^T}{\delta}\right) = \frac{1}{\delta^2} \text{Var}(\overline{v}^T)
\]

\[
= \frac{1}{\delta^2} \left( \overline{A}_n^2 - \overline{A}_n^2 \right)
\]

\[
= \frac{1}{\delta^2} \left[ (1 - 2\delta \cdot \overline{a}_n) - (1 - \delta \overline{a}_n)^2 \right]
\]

\[
= \frac{1}{\delta^2} \left[ 1 - 2\delta \cdot \overline{a}_n - 1 + 2\delta \overline{a}_n - \delta^2 \overline{a}_n^2 \right]
\]

\[
= \frac{2}{\delta} \left[ \overline{a}_n - \overline{a}_n^2 \right] - \overline{a}_n^2
\]
\[ Y = \frac{a}{T} \quad \text{continuous positive r.v.} \]

To find jdf of \( Y \), let us find first its cdf.

\[
F_Y(y) = P_Y(Y \leq y) = P_T(\frac{a}{T} \leq y) \\
= P_T(1 - vT \leq sy) \\
= P_T(vT \geq 1 - sy) \\
= P_T(T \leq \frac{\ln(1 - sy)}{-s})
\]

\[
F_Y(y) = F_T(-\frac{1}{s} \ln(1 - sy))
\]

So \( 0 < t < \infty \quad \Rightarrow \frac{a}{T} < \frac{1}{s} \)

jdf of \( Y \)

\[
f_Y(y) = \frac{d}{dy} F_Y(y) = f_T(-\frac{1}{s} \ln(1 - sy)) \cdot \frac{1}{1 - sy},
\]

for \( 0 < y < \frac{1}{s} \)

If \( s = 0 \)

\[
T = \int_0^\infty \delta_x \ dx = \delta_{\infty}.
\]
Ex. If \( \mu_x = \mu \neq \mu \) (i.e. \( T(\mu) \sim \text{Ex}(\mu) \)), calculate \( S \)

\begin{align*}
\text{a)} & \quad \bar{a}_x \\
\text{b)} & \quad \text{Var}(\bar{a}_T) \\
\text{c)} & \quad \text{the probability that } \bar{a}_T \text{ exceeds } \bar{a}_x, \\
& \quad P_a[\bar{a}_T > \bar{a}_x] \\
\text{d)} & \quad \text{the pdf of } Y = \bar{a}_T \\
\text{e)} & \quad \text{make a graph of } f_Y(y) \text{ if } \mu = 0.04, S = 0.06
\end{align*}

\*  \[ \text{m-year temporary annuity} \]

The APV of an \( n \)-year temporary annuity paying \( \$1 \) per year continuously while \( x \) survives during the next \( m \) years is denoted \( \bar{a}_{x:m} \).

The r.v. of the present value of benefits

\[ Y = \begin{cases} 
\bar{a}_T & \text{if } 0 < T < m \\
\bar{a}_m & \text{if } T \geq m
\end{cases} \]

\[ \bar{a}_{x:m} = E(Y) = \int_{0}^{m} \bar{a}_t \, dt + \int_{m}^{\infty} \bar{a}_m \cdot e^{-y} \, dy \]

integrate by parts

\[ \int_{0}^{m} \bar{a}_t \, dt + \int_{m}^{\infty} \bar{a}_m \cdot e^{-y} \, dy \]

The result is
\[ \bar{a}_x \cdot m \cdot d_x + \int_0^m v^t d_x \, dt + \bar{a}_x \cdot m \cdot d_x \]

**Interpretation:** \( \bar{a}_x \cdot m \cdot d_x = \int_0^m v^t d_x \, dt \).

\[ Y = \frac{1-Z}{\delta} \quad \text{where} \quad Z = \begin{cases} \sqrt{m}, & 0 < T \leq m \\ \sqrt{m}, & T > m \end{cases} \]

Take \( E(\ ) \)

\[ \bar{a}_x \cdot m = E \left( \frac{1-Z}{\delta} \right) = \frac{1-E(Z)}{\delta} = \frac{1-A_{x:m}}{\delta} \]

\[ \Rightarrow A_{x:m} = 1 - \delta \bar{a}_x \cdot m \]

\[ \text{Var}(Y) = \text{Var} \left( \frac{1-Z}{\delta} \right) = \frac{1}{\delta^2} \text{Var}(Z) \]

\[ = \frac{1}{\delta^2} \left[ \bar{a}_x \cdot m - \bar{a}_x \cdot m \right] \]

Express in terms of annuity symbols!

(especially case: \( m \to \infty \)).

**Define annuity**

\[ Y = \begin{cases} 0 & \text{if } T \leq m \\ \bar{a}_T - \bar{a}_m & \text{if } T > m \end{cases} \]

\[ E(Y) = m \bar{a}_x = \bar{a}_x - \bar{a}_{x:m} = \int_0^m v^t d_x \, dt \]

\[ = m \int_0^m v^t d_x \cdot \bar{a}_{x+m} \]

Find an expression for \( \text{Var}(Y) \).
n-year certain and life annuity
\[ Y = \begin{cases} \alpha m & \text{if } T \leq m \\ \frac{\alpha}{1 + \alpha m} & \text{if } T > m \end{cases} \]
\[ \max [T(x), m] \]
\[ \text{Var} \left( \frac{\alpha}{m} \right) = \text{int. by parts} \]
\[ \text{Var}(Y) = \text{Var. for n-yr deferred annuity.} \]
\[ Y = \left( \alpha m \right) + \left( \frac{\alpha}{1 + \alpha m} \right) \quad T \leq m \]
\[ \quad \left( \frac{\alpha}{1 + \alpha m} \right) \quad T > m. \]
\[ Y = \text{cst} + \left( \alpha \text{ v. for deferred annuity} \right) \]

C - Annuity due with annual payments

Life annuity: Payment of $1 at beginning of each year if \((X)\) is then alive.

r.v. of P.V. of payments
\[ Y = \sum_{k=0}^{\infty} \frac{\alpha}{1 + \alpha m} \cdot k^1 \delta_x \quad K = K(x) \]
\[ \text{E}(Y) = \sum_{k=0}^{\infty} \frac{\alpha}{1 + \alpha m} \cdot k^1 \delta_x = \text{E} \left( \frac{\alpha}{1 + \alpha m} \right) \]
\[ \hat{\alpha}_x = \sum_{k=0}^{\infty} \left( \frac{\alpha}{1 + \alpha m} \right) \cdot k^1 \delta_x \quad \text{Develop series} \ldots \]
\[ \hat{\alpha}_x = \sum_{k=0}^{\infty} \frac{\alpha}{k^1 \delta_x} \quad \text{Analogous to } \hat{\alpha}_x = \int_{0}^{\infty} v^x \cdot d\delta_x. \]
relation between $\bar{a}_n$ and $A_n$.

$$\bar{a}_n = E\left(\frac{1 - v^{K+1}}{d}\right) = \frac{1 - A_n}{d}$$

$$A_n = 1 - d \bar{a}_n$$ (□) Interpretation.

$$\text{Var}\left(\frac{\bar{a}_n}{K+1}\right) = \text{Var}\left(\frac{1 - v^{K+1}}{d}\right) = \frac{1}{d^2} \text{Var}(v^{K+1})$$

$$= \frac{2A_n - A_n^2}{d^2}$$ Use (□).

• $n$-year temporary annuity

$$Y = \begin{cases} \frac{\bar{a}_n}{K+1}, & K=0,1,...,n-1 \\ \frac{\bar{a}_n}{m}, & K=n \\ \end{cases} \hookrightarrow \begin{cases} (1 - v^{K+1})/d \\ \frac{1 - v^n}{d} \\ \end{cases}$$

$$\bar{a}_n^{\cdot\cdot\cdot\cdot} = E(Y) = ... = \sum_{k=0}^{n-1} v^k \frac{1}{d}$$

$$Y = \frac{1 - Z}{d} \quad \text{where} \quad Z = \begin{cases} v^{K+1}, & \text{if } K=0,...,n-1 \\ v^m, & \text{if } K=n \\ \end{cases}$$

$$E(Z) = A_{n:\cdot\cdot\cdot\cdot}$$

$$A_{n:\cdot\cdot\cdot\cdot} = 1 - d \bar{a}_n^{\cdot\cdot\cdot\cdot}$$

$$\text{Var}(Y) = \frac{1}{d^2} \left[ \frac{2A_{n:\cdot\cdot\cdot\cdot}}{d^2} - A_{n:\cdot\cdot\cdot\cdot}^2 \right]$$

• $n$-year deferred

$$n! \bar{a}_n = \sum_{k=n}^{\infty} v^k \frac{1}{d} = \bar{a}_n \bar{a}_n^{\cdot\cdot\cdot\cdot} = v^n \frac{1}{d} \cdot \bar{a}_n^{\cdot\cdot\cdot\cdot}$$

• $n$-year certain and life

$$\bar{a}_n^{\cdot\cdot\cdot\cdot} = \bar{a}_n \bar{a}_n^{\cdot\cdot\cdot\cdot} + n! \bar{a}_n$$
D - Annuity-immediate (payment at the end of yr)

\[ E(Y) = a_{x+1} = a_x - 1 \]

\[ Var \left( \frac{a_x}{k+1} \right) = \frac{\ddot{a}_{x+m}}{m} = \frac{\ddot{a}_{x+1}}{k+1} - 1 \]

E - Annuities payable m times a year (payments of \( \frac{1}{m} \))

\[ \ddot{a}_x^{(m)} = \sum_{j=0}^{m-1} \frac{1}{m} \frac{\ddot{a}}{j} \sum_{j=0}^{m-1} \frac{1}{m} \frac{\ddot{a}}{j} \]

\[ \ddot{a}_x^{(m)} \text{ UDD} = a_x^{(m)} \ddot{a}_x - \beta(m) \]

where \( \alpha(m) = \frac{i_d}{i^{(m)}_d} \)

\( \beta(m) = \frac{1-i^{(m)}}{i^{(m)}_d} \)

\( \alpha(m), \beta(m) \) depend only on \( m \) and \( i_d \)

\( \alpha(1) = 1, \beta(1) = 0 \)

\( m = 1 \)

\( \ddot{a}_1 = \ddot{a}_1(1) = 0 \)

\( m \to \infty \)

\[ \lim_{m \to \infty} \ddot{a}_x^{(m)} = \ddot{a}_x = \frac{\ddot{a}}{s^2} \ddot{a}_x = \frac{i_d}{\ddot{a}} - \frac{\ddot{a}}{s^2} \]

End of period \( a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m} \)

Temporary \( \ddot{a}_x^{(m)} = \ddot{a}_x^{(m)} \)

\[ a_x^{(m)} = \ddot{a}_x^{(m)} - \frac{1}{m} + \frac{1}{m} \sqrt{m} \]
Ex. Temporary annuity with varying payments
Benefit $b_{k+1}$ paid at the end of year $k+1$ that [person] survives.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$b_{k+1}$</th>
<th>$b_{k+1} \cdot p_k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1000</td>
<td>0.90</td>
</tr>
<tr>
<td>1</td>
<td>2000</td>
<td>0.75</td>
</tr>
<tr>
<td>2</td>
<td>3000</td>
<td>0.55</td>
</tr>
<tr>
<td>3</td>
<td>3000</td>
<td>0.30</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

If $v = 0.9$

find 
1. the net. fct of $y$
2. the APV of benefits
3. standard error of $Y$.

Solution:

Recursive relationship between $\ddot{a}_x$ and $\ddot{a}_{x+r}$

$$\ddot{a}_x = \sum_{k=0}^{\infty} v^k \cdot k \cdot \ddot{a}_x$$

$$= 1 + \ddot{a}_{x+r} \cdot v^{r} \cdot f_r.$$  

With one value $\ddot{a}_x$ and a mortality table, one can generate whole table of $\ddot{a}_x$ at all other ages.

Add relation to Chapter III, 1.10.
IV Net premiums

Whole life insurance policy usually paid by annual premiums, not a N.S.P.

Let us define, for a contract, the loss r.v. \( L \) equal to r.v. of present value of benefits to be paid by insurer minus r.v. of present value of premiums to be paid by insured.

Equivalence principle to determine premium is such that \( E(L) = 0 \).

Net premiums calculated with equivalence principle:

\[
L = (P.V. \ of \ benefit) - (P.V. \ of \ net \ premiums)
\]

\[
E(L) = 0 \iff E(P.V. \ of \ benefit) = E(P.V. \ of \ premiums)
\]

\[
A.P.V. \ of \ benefit = A.P.V. \ of \ premiums.
\]

A - Continuous premiums

Ex. Whole life insurance, payable at death of \( x \), with continuous premium, payable for life; r.v. of P.V. of loss to insurer

\[
L(T) = \sqrt{T} - \bar{P}aT^{-1}, \quad T > 0.
\]

\( L(T) \): decreasing fct of \( T \)

\[
L(0) = 1 \quad \text{if } T \to \infty, \ L(T) \to 0 - P_8
\]

\[
L(t^*) = 0
\]

negative loss = profit to insurer
According to equivalence principle, net premium is such that

\[ E[L(T)] = 0 \]

Notation for net premium: \( \overline{P}(\overline{A}_x) \)

\[ E[L(T)] = E[vT - \overline{P}(\overline{A}_x)] \]

\[ = \overline{A}_x - \overline{P}(\overline{A}_x) \overline{a}_x = 0 \]

\[ \overline{P}(\overline{A}_x) = \frac{\overline{A}_x}{\overline{a}_x} = \frac{1 - \delta \overline{a}_x}{\overline{a}_x} = \frac{1}{\overline{a}_x} - \delta \]

To measure variability of losses on a whole life insurance policy (because of the r.v. Time-until-death of \( x \)), we calculate the variance of \( L(T) \).

\[ \text{Var} (L(T)) = \text{Var} [vT - \overline{P} \overline{a}_x] \]

\[ = \text{Var} [vT - \overline{P} (\frac{1 - vT}{\delta})] \]

\[ = \text{Var} [vT (1 + \frac{\overline{P}}{\delta}) - \frac{\overline{P}}{\delta}] \]

\[ = \left( 1 + \frac{\overline{P}}{\delta} \right)^2 \text{Var} (vT) \]

\[ = \left( 1 + \frac{\overline{P}}{\delta} \right)^2 (\overline{A}_x - \overline{a}_x^2). \]

If \( \overline{P} \) is net prem (calculated with equivalence principle)

1. \( \text{Var}(L) = E(L^2) \) since \( E(L) = 0 \).

2. \( \overline{P} = \overline{P}(\overline{A}_x) = \frac{1}{\overline{a}_x} - \delta \) so \( \text{Var}[L(T)] = \frac{\overline{A}_x - \overline{a}_x^2}{(\delta \overline{a}_x)^2} \)
Ex. If \( M_x = m \quad \forall x \), find \( \bar{P}(A_x) \) and \( \text{Var}(L(T)) \)

In general, if \( b_t \) is the benefit if death occurs at time \( t \), \( \bar{P} \) is a general symbol for a continuous sum, \( Y \) is the r.v. for a continuous annuity.

\[ L(T) = b_T v^T - \bar{P} Y \]

Equivalence principle: \( E(L(T)) = 0 \)

\[ \Rightarrow E(b_T v^T) - \bar{P} E(Y) = 0 \]

\[ \bar{P} = \frac{E(b_T v^T)}{E(Y)} \]

---

**Table 6.2.1**

<table>
<thead>
<tr>
<th>Fully Continuous Benefit Premiums</th>
<th>Premium Formula ( \bar{P} = \frac{E[b_T v_T]}{E(Y)} )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Plan</strong></td>
<td><strong>Loss Components</strong> ( b_T v_T )</td>
</tr>
<tr>
<td>Whole life insurance</td>
<td>( 1 v^T )</td>
</tr>
<tr>
<td>( n )-Year term insurance</td>
<td>( 1 v^T )</td>
</tr>
<tr>
<td></td>
<td>( 0 )</td>
</tr>
<tr>
<td>( n )-Year endowment insurance</td>
<td>( 1 v^T )</td>
</tr>
<tr>
<td></td>
<td>( 1 v^n )</td>
</tr>
<tr>
<td>( h )-Payment* whole life insurance</td>
<td>( 1 v^T )</td>
</tr>
<tr>
<td></td>
<td>( 1 v^T )</td>
</tr>
<tr>
<td>( h )-Payment, ( n )-year endowment insurance</td>
<td>( 1 v^T )</td>
</tr>
<tr>
<td></td>
<td>( 1 v^T )</td>
</tr>
<tr>
<td></td>
<td>( 1 v^n )</td>
</tr>
<tr>
<td>( n )-Year pure endowment</td>
<td>( 0 )</td>
</tr>
<tr>
<td></td>
<td>( 1 v^n )</td>
</tr>
<tr>
<td>( n )-Year deferred whole life annuity</td>
<td>( 0 )</td>
</tr>
<tr>
<td></td>
<td>( \bar{\Pi}_T, T &gt; n )</td>
</tr>
</tbody>
</table>

*The insurances described in the fourth and fifth rows provide for a premium paying period that is shorter than the period over which death benefits are paid.
†The annuity product described above provides no death benefits and has a level premium with premiums payable for \( n \) years. A different, perhaps more realistic, design for an \( n \)-year level premium-deferred annuity is given in Example 6.6.2.
Ex. For a whole life insurance paying $1 at moment of death, with continuous premium $p$, find:

(a) the cdf of $L$

(b) the pdf of $L$

(c) the probability that a policy is profitable (to insure).

(d) the premium $p^*$ such that there is a 90% probability that a policy is profitable.

For $n$-year endowment insurance

$L(T) = Z - \bar{P}(\bar{A}_{x:\overline{m}}) \cdot Y$, where $Z = \begin{cases} T \leq m & \text{if } T < m \\ \overline{m}, & \text{if } T > m \end{cases}$

$\text{Var}(L(T)) = \text{Var} \left[ Z \left( 1 + \bar{P}(\bar{A}_{x:\overline{m}}) \right) - \bar{P}(\bar{A}_{x:\overline{m}}) \right]$

$= \left( 1 + \bar{P}(\bar{A}_{x:\overline{m}}) \right)^2 \left( 2 \bar{A}_{x:\overline{m}} - \bar{A}_{x:\overline{m}}^2 \right)$

Since $\bar{A}_{x:\overline{m}} = 1 - \delta a_{x:\overline{m}}$, $\bar{P}(\bar{A}_{x:\overline{m}}) = \frac{1}{\overline{a}_{x:\overline{m}}} - \delta$

$\text{Var}(L(T)) = \frac{2 \bar{A}_{x:\overline{m}} - \bar{A}_{x:\overline{m}}^2}{\delta \overline{a}_{x:\overline{m}}}$
B- Fully discrete premiums

- benefit paid at the end of year of death
- premiums paid at the beginning of each year

$P_x$ denotes the net annual prem for a whole life insurance policy of 1 issued at age $x$ calculated under the equivalence principle.

Loss r.v.

$L(K) = v^{K+1} - P_x \bar{a}_{K+1}$, $K=0, 1, 2, ...$

Equivalence principle: $E(L(K)) = 0$

$E\left[v^{K+1} - P_x \bar{a}_{K+1}\right]$

$= A_x - P_x \ddot{a}_x = 0 \quad \Rightarrow \quad P_x = \frac{A_x}{\ddot{a}_x}$

$P_x = \frac{1 - \ddot{a}_x}{\ddot{a}_x} = \frac{1}{\ddot{a}_x} - 1$

$\text{Var}(L(K)) = \text{Var}\left[v^{K+1}(1 + P_x) - P_x \frac{d}{d}\right]$

$= (1 + P_x)^2 \text{Var}(v^{K+1}) = \frac{2A_x - A_x^2}{(d\ddot{a}_x)^2}$

General formula for the loss

$L(K) = b_{K+1}v^{K+1} - P \cdot Y$, where

$b_{K+1}$: death benefit in year $K+1$ (end of year)
$P$: general symbol for NAP paid at beginning of yr.
$Y$: r.v. for discrete annuity

Equivalence principle: $E(L(K)) = 0$

$\Rightarrow \quad P = \frac{E(b_{K+1}v^{K+1})}{E(Y)}$
### Fully Discrete Annual Benefit Premiums

<table>
<thead>
<tr>
<th>Plan</th>
<th>Loss Components</th>
<th>Premium Formula</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Whole life insurance</strong></td>
<td>$b_{x+k+1}v_{k+1}$</td>
<td>$P = \frac{E[b_{x+k+1}v_{k+1}]}{E[Y]}$</td>
</tr>
<tr>
<td>$n$-Year term insurance</td>
<td>$1\ u^{K+1}$, $\ddot{a}_{x+k+1}$, $K = 0, 1, 2, \ldots$</td>
<td>$P_x = \frac{A_x}{\ddot{a}_x}$</td>
</tr>
<tr>
<td></td>
<td>$0$</td>
<td>$P_{x+k+1} = \frac{A_{x+k+1}}{\ddot{a}_{x+k+1}}$</td>
</tr>
<tr>
<td><strong>n-Year endowment insurance</strong></td>
<td>$1\ u^{K+1}$, $\ddot{a}_{x+k+1}$, $K = 0, 1, \ldots, n$</td>
<td>$P_{x+k} = \frac{A_{x+k}}{\ddot{a}_{x+k}}$</td>
</tr>
<tr>
<td></td>
<td>$1\ u^n$</td>
<td>$P_{x+k+1} = \frac{A_{x+k+1}}{\ddot{a}_{x+k+1}}$</td>
</tr>
<tr>
<td><strong>h-Payment whole life insurance</strong></td>
<td>$1\ u^{K+1}$, $\ddot{a}_{x+h+1}$, $K = 0, 1, \ldots, h$</td>
<td>$P_x = \frac{A_x}{\ddot{a}_x}$</td>
</tr>
<tr>
<td></td>
<td>$1\ u^{K+1}$</td>
<td>$P_{x+k} = \frac{A_{x+k}}{\ddot{a}_{x+k}}$</td>
</tr>
<tr>
<td></td>
<td>$1\ u^n$</td>
<td>$P_{x+k+1} = \frac{A_{x+k+1}}{\ddot{a}_{x+k+1}}$</td>
</tr>
<tr>
<td><strong>n-Year endowment insurance</strong></td>
<td>$1\ u^{K+1}$, $\ddot{a}_{x+n+1}$, $K = 0, 1, \ldots, n$</td>
<td>$P_x = \frac{A_x}{\ddot{a}_x}$</td>
</tr>
<tr>
<td></td>
<td>$1\ u^n$</td>
<td>$P_{x+k} = \frac{A_{x+k}}{\ddot{a}_{x+k}}$</td>
</tr>
<tr>
<td><strong>n-Year pure endowment</strong></td>
<td>$0$</td>
<td>$P_{x+k+1} = \frac{A_{x+k+1}}{\ddot{a}_{x+k+1}}$</td>
</tr>
<tr>
<td></td>
<td>$1\ u^n$</td>
<td>$P_{x+k+1} = \frac{A_{x+k+1}}{\ddot{a}_{x+k+1}}$</td>
</tr>
<tr>
<td><strong>n-Year deferred whole life annuity</strong></td>
<td>$0\ \ddot{a}_{x+k+1-n}u^n$</td>
<td>$P_{x+k+1} = \frac{A_{x+k+1}}{\ddot{a}_{x+k+1}}$</td>
</tr>
</tbody>
</table>

*** Example ***

Consider a 3-year endowment insurance policy of $1000 issued to (30); death benefit is payable at the end of year of death. If $v = 0.9$

$\ddot{a}_{30} = 0.010$, $\ddot{a}_{30} = 0.011$, $\ddot{a}_{30} = 0.012$

1. Determine the premium under the equivalence principle
2. Determine the net single premium $\ddot{L}(K)$ with $\ddot{a}_{30} = \frac{1}{2}$
3. Determine $\Pi$, the sum such that the probability of a financial loss is positive for a portfolio of 100 independent policies is 0.05
4. Determine $\Pi^*$, the sum such that $Pr[\text{financial loss on 1 policy}]$ is 0.021.
C- Semi-continuous premiums
- premiums paid annually (BOY)
- death benefit paid at moment of death.

\[
P(A_{x}) = \frac{A_{x}}{\dd{A}_{x}} \overset{UDD}{=} \frac{1}{\delta} \dd{A_{x}} = \frac{1}{\delta} P_{x}
\]

\[
P(A_{x:1}) = \frac{A_{x:1}}{\dd{A}_{x:1}} \overset{UDD}{=} \frac{1}{\delta} P_{x:1}
\]

\[
P(A_{x:1}) = \frac{1}{\delta} P_{x:1} + P_{x:1}
\]

D- True m-th payment premium

\[P_{x}^{(m)}: NAP, payable in m instalments for a whole life ins. of 1, payable at end of year of death.\]

---

<table>
<thead>
<tr>
<th>Plan</th>
<th>At End of Policy Year</th>
<th>At Moment of Death</th>
</tr>
</thead>
<tbody>
<tr>
<td>Whole life insurance</td>
<td>( p_{x}^{(m)} = \frac{A_{x}}{\dd{A}_{x}^{(m)}} )</td>
<td>( p_{x}^{(m)}(A_{x}) = \frac{A_{x}}{\dd{A}_{x}^{(m)}} )</td>
</tr>
<tr>
<td>n-Year term insurance</td>
<td>( p_{x}^{(1)<em>{x:1}} = \frac{A</em>{x:1}}{\dd{A}_{x:1}} )</td>
<td>( p_{x}^{(1)<em>{x:1}}(A</em>{x}) = \frac{A_{x:1}}{\dd{A}_{x:1}} )</td>
</tr>
<tr>
<td>n-Year endowment insurance</td>
<td>( p_{x}^{(m)} = \frac{A_{x:1}}{\dd{A}_{x:1}} )</td>
<td>( p_{x}^{(m)}(A_{x:1}) = \frac{A_{x:1}}{\dd{A}_{x:1}} )</td>
</tr>
<tr>
<td>h-Payment years, whole life insurance</td>
<td>( h_{x}^{(m)} = \frac{A_{x}}{\dd{A}_{x}^{(m)}} )</td>
<td>( h_{x}^{(m)}(A_{x}) = \frac{A_{x}}{\dd{A}_{x}^{(m)}} )</td>
</tr>
<tr>
<td>h-Payment years, n-year endowment insurance</td>
<td>( h_{x}^{(m)} = \frac{A_{x:1}}{\dd{A}_{x:1}} )</td>
<td>( h_{x}^{(m)}(A_{x:1}) = \frac{A_{x:1}}{\dd{A}_{x:1}} )</td>
</tr>
</tbody>
</table>

*The actual amount of each fractional premium, payable \( m \) times each policy year, during the premium paying period and the survival of \( x \), is \( P_{x}^{(m)}/m \). Note that here \( h \) refers to the number of payment years, not to the number of payments.*
Under UDD,
\[ P_x^{(m)} = \frac{A_x}{\hat{a}_x^{(m)}} = \frac{A_x}{\alpha(m)\hat{a}_x - \beta(m)} \]
\[ P^{(m)}(\overline{A}_x) = \frac{i^x A_x}{\alpha(m)\hat{a}_x - \beta(m)} = \frac{i^x P^{(m)}}{\alpha(m)\hat{a}_x - \beta(m)} \]

**Exercise**
1. Express \( P_x^{(m)} \) in terms of \( P_x \) (under UDD).
2. Find a recursive relation between \( P_x \) and \( P_{x+1} \).
3. Order (in increasing order) the premiums
   \[ P_{x:m}, \overline{P}(\overline{A}_{x:m}), P^{(m)}(\overline{A}_{x:m}), P(\overline{A}_{x:m}) \].