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Spectral sequences in Conley's theory

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Abstract. In this paper, we analyse the dynamics encoded in the spectral sequence (E^r, d^r) associated with certain Conley theory connection maps in the presence of an 'action' type filtration. More specifically, we present an algorithm for finding a chain complex C and its differential; the method uses a connection matrix Δ to provide a system that spans E^r in terms of the original basis of C and to identify all of the differentials $d^r_p : E^r_p \to E^r_{p-r}$. In exploring the dynamical implications of a non-zero differential, we prove the existence of a path that joins the singularities generating E^0_p and E^0_{p-r} in the case where a direct connection by a flow line does not exist. This path is made up of juxtaposed orbits of the flow and of the reverse flow, and proves to be important in some applications.

1. *Introduction*

The role played by algebraic-topological tools in the study of dynamical systems has always been quite significant. This is exemplified by classical topics such as Lusternik–Schnirelmann theory and Morse theory, as well as by more recent advances such as the theory developed by Conley [**Co**].

A key concept in Conley's theory is the notion of Morse decomposition; this provides, by means of appropriate attractor–repeller pairs, a decomposition of an invariant set inside a flow into smaller and smaller components. The basic idea is that if one can understand the smallest invariant sets in the flow, then one can proceed to investigate slightly more complex ones consisting of attractor–repeller pairs that are given by a pair of invariant sets of the first type together with all the flow lines joining them. The process can then be continued to deal with invariant sets of the next level of complexity, and so on, by taking into account 'longer' and 'longer' flow lines.

From an algebraic-topological point of view, this process bears a strong resemblance to that which is encoded algebraically by the concept of *spectral sequence*. After

Leray introduced them in the 1950s, spectral sequences have been used extensively in homological algebra, algebraic topology and geometry as an efficient tool for computation. One version of this concept is defined when one has a chain complex (C, ∂) endowed with an increasing filtration F^pC so that $\partial(F^pC) \subset F^pC$ (here we assume that $F^{-1}C = 0$). The associated spectral sequence is a (generally infinite) sequence of chain complexes (E^r, d^r) so that, roughly, each successive stage contains information about longer and longer parts of the differential: the differential d^0 at the first stage in the complex is the part of ∂ which does not decrease filtration, while d^1 concerns the part of ∂ which reduces filtration by no more than 1, and so on. Moreover, $H(E^r, d^r) = E^{r+1}$.

The two points of view come together in the presence of a flow with an associated Lyapunov function or action functional which provides an appropriate filtration. The simplest such case is that of negative gradient flow associated to a Morse function on a finite-dimensional manifold, where the level sets of the function provide a filtration of the associated Morse complex. More refined spectral sequences appear in Morse theory; see [**C3**]. See also [**BaC**] for spectral sequences in the context of Floer theory. The key point here is that these spectral sequences are not merely computational tools, but are also interesting objects in themselves: their higher differentials encode algebraically significant information on 'long' trajectories of the system. Therefore, it is important to gain as deep an understanding as possible of the algebra–geometry dictionary in this setting. The purpose of this paper is precisely to start exploring this issue systematically.

We address two main issues. The first concerns the detection of cycles. More precisely, in practice, the generators of the complex C mentioned above are very specific: they are singularities in the Morse case (or periodic orbits in the Floer case). The domain E^r of d^r is a certain quotient of a subgroup of C. Elements in this domain are represented by elements of C, called $(r-1)$-cycles, whose appropriate classes are in the kernels of all previous differentials d^s with $s < r$. Finding a system of $(r-1)$-cycles that span E^r in terms of the original basis of C is a non-trivial matter; however, it is a necessity in applications, such as in investigations relating to spectral numbers in symplectic topology (see [**L**]). We shall provide an algorithm, which we refer to as the sweeping method, that produces such a system. Theorem 1.1 makes the construction explicit, that is, the E^r are determined and the long differentials identified.

An application of this algorithm brings up a second, very natural, problem: assuming that a long differential can be identified in such a 'dynamical spectral sequence', what geometric consequences can one infer from it? Is it true that there are 'long orbits' that relate some invariant set to another, distant one? This is an important question because, in applications, long orbits have high energy, in the sense that the variation of the action functional along such an orbit is large, and detecting high-energy orbits is geometrically significant; see [**BaC**]. It is not hard to see that the existence of long orbits relating distant invariant sets does not hold in general. Nevertheless, we show here that there exists a path joining two invariant sets which is made up of curves that geometrically coincide with flow lines, where some of these arcs in the path are flow-reversing; this is Theorem 1.2, which we call the 'zig-zag theorem'.

1.1. *The spectral sequence for a Morse complex.* Let M be an n-dimensional compact Riemannian manifold and let $\mathcal{D}(M) = \{M_p\}_{p=1}^m$ be a *Morse decomposition*† of M. In this article, we focus on the case in which a *filtered* Conley chain complex‡ with finest filtration is, in fact, a Morse complex, i.e. each Morse set M_p is a non-degenerate singularity of the gradient flow φ of a Morse function $f : M \to \mathbb{R}$.

Given non-degenerate singularities x and y of indices k and $k-1$, respectively, the set of connecting orbits is finite. By orienting the unstable and stable manifolds, we define the *intersection number* $n(x, y)$ to be the number of connecting orbits counted with orientation. To count orbits with orientation, choose a regular value c of f so that $f(y) < c < f(x)$; then $n(x, y)$ is the number of intersections between the spheres $S^{k-1} = W^u(x) \cap f^{-1}(c)$ and $S^{n-k} = W^s(y) \cap f^{-1}(c)$.

Let $C = \{C_k\}$ be the $\mathbb{Z}$-module generated by the singularities and graded by their indices, that is,

$$C_k = \bigoplus_{x \in \text{crit}_k f} \mathbb{Z}\langle x \rangle,$$

where $\text{crit}_k(f)$ is the set of index-k critical points of f.

The *connection matrix* $\Delta : C \to C$ associated to $\mathcal{D}(M)$ is defined to be the differential of the graded Morse chain complex $C = \mathbb{Z}\langle \text{crit} f \rangle$, i.e. it is determined by the maps $\Delta_k : C_k \to C_{k-1}$ via

$$\Delta_k(x) = \sum_{y \in \text{crit}_{k-1} f} n(x, y)\langle y \rangle,$$

where $n(x, y)$ is the intersection number. Moreover, Δ is an upper triangular matrix with $\Delta \circ \Delta = 0$.

We use the same notation for the map Δ_k as for the associated submatrices of Δ; see Figure 1.

The columns of the matrix Δ need not be ordered with respect to k. We only require that the map Δ_k be filtration preserving.

We denote this filtered graded Morse chain complex by

$$(C, \Delta) = (\mathbb{Z}\langle \text{crit} f \rangle, \Delta).$$

We will write the boundary operator ∂ and its matrix Δ interchangeably.

Note that the rth auxiliary diagonal of Δ that intersects Δ_k has entries $\Delta_{p+1-r,p+1}$, which represent the intersection numbers of the unstable and stable spheres determined by connections between the unstable and stable manifolds of M_{p+1} and M_{p+1-r} for $p \in \{r, \dots, m-1\}$. Clearly, if the $(p+1)$st column intersects the submatrix Δ_k, then M_{p+1} and M_{p+1-r} are, respectively, singularities of Morse indices k and $k-1$, which we shall denote by h_k and h_{k-1}. These singularities are in the filtrations $F_p \setminus F_{p-1}$ and $F_{p-r} \setminus F_{p-r-1}$, respectively. Hence we say that the pair (h_k, h_{k-1}) has *gap* r. In summary,

† A Morse decomposition of M is a collection $\mathcal{D}(M) = \{M_p\}_{p=1}^m$ of mutually disjoint compact invariant subsets of M such that if $\gamma \in M \setminus \bigcup_{p=1}^m M_p$, then there exists $p < p'$ with $\gamma \in C(M_p, M_{p'})$. In other words, $\mathcal{D}(M)$ contains the recurrent behavior of the flow. A subset of M which belongs to some Morse decomposition is called a *Morse set*.

‡ A filtration $F = \{F_p\}$ on a chain complex C is a sequence of subcomplexes F_pC, $p \in \mathbb{Z}$, such that $F_pC \subset F_{p+1}C$ for each p.

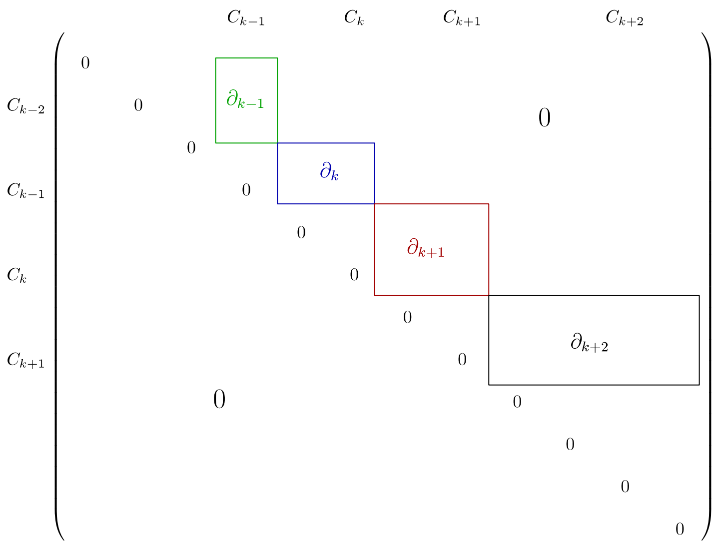

FIGURE 1. The connection matrix.

the rth auxiliary diagonal, when intersected with Δ_k, registers information on numerically consecutive singularities of Morse indices k and $k-1$ with gap r. We will use the same notation to indicate an elementary chain of C.

It will be helpful to associate with the $(p+1)$st column of Δ the elementary chain h_k such that† $h_k \in F_pC \setminus F_{p-1}C$.

In this paper we will explain how the connection matrix Δ determines the spectral sequence, i.e. how it determines the spaces E^r and induces the differentials d^r.

A *bigraded module* E^r over a principal ideal domain‡ R is an indexed collection of R-modules $E^r_{p,q}$, for $p, q \in \mathbb{Z}$. A *differential* d^r *of bidegree* $(-r, r-1)$ is a collection of homomorphisms $d^r : E_{p,q} \to E_{p-r,q+r-1}$, for $p, q \in \mathbb{Z}$, such that $d^r \circ d^r = 0$. The *homology module* $H(E^r)$ is the bigraded module

$$H_{p,q}(E^r) = \frac{\operatorname{Ker} d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}}{\operatorname{Im} d^r : E^r_{p+r,q-r+1} \to E^r_{p,q}}.$$

A *spectral sequence* $\{E^r, d^r\}$, $r \geq 0$, is a sequence of chain complexes where each chain complex E^r is the homology module of the previous one, that is:

- E^r is bigraded module, and d^r is a differential with bidegree $(-r, r-1)$ in E^r;
- for each $r \geq 0$ there exists an isomorphism $H(E^r) \approx E^{r+1}$.

In general, we will omit reference to q throughout this section. Its role will be important when considering more general Morse sets of a Morse decomposition; but in our case, when the Morse set is a singularity of index k, the only q for which $E^r_{p,q}$ is non-zero is $q = k - p$. Hence, it will be understood that E^r_p is, in fact, $E^r_{p,k-p}$.

† Note that the numbering on the columns is shifted by one with respect to the subindex p of the filtration F_p.
‡ Throughout this article we work with $R = \mathbb{Z}$.

For a filtered graded chain complex (C, ∂), we can define a spectral sequence

$$E_p^r = Z_p^r/(Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1})$$

where

$$Z_p^r = \{c \in F_pC \mid \partial c \in F_{p-r}C\}.$$

Hence the module Z_p^r consists of chains in F_pC with boundary in $F_{p-r}C$. Thus it is natural to look at chains associated to the columns of the connection matrix to the left of and including the $(p+1)$st column. This guarantees that any linear combination of chains will respect the filtration. Furthermore, since the boundary of the chains must be in F_{p-r}, we must consider columns or linear combinations which respect the filtration and have the property that the entries in rows $i > (p - r + 1)$ are all zeros. Therefore, the significant entry in the connection matrix is determined by the element on the rth auxiliary diagonal in the $(p - r + 1)$st row and $(p + 1)$st column. This will be made precise later.

However, as r increases, the $\mathbb{Z}$-modules E_p^r change generators. Our main result will connect this algebraic change of generators of the $\mathbb{Z}$-modules of the spectral sequence to a particular family of changes of basis over $\mathbb{Q}$ of the connection matrix Δ. We will make use of a recursive sweeping method in §2 that singles out important non-zero entries, which we will refer to as primary pivots and change-of-basis pivots, of the rth auxiliary diagonal of Δ^r, in order to define a matrix Δ^{r+1}. At each step, Δ^{r+1} is a change of basis of Δ^r. Hence, all of the Δ^r 'represent', in some sense, the initial connection matrix (that is, they all represent the same linear transformation). We will also show how the rth auxiliary diagonal of Δ^r induces d^r.

THEOREM 1.1. *The matrices Δ^r obtained by applying the sweeping method to Δ determine the spectral sequence (E_p^r, d^r). Moreover, if E_p^r and E_{p-r}^r are both non-zero, then the map $d_p^r : E_p^r \to E_{p-r}^r$ is induced by Δ^r; specifically, it is multiplication by the entry $\Delta^r_{p-r+1,p+1}$ whenever this entry is a primary pivot, a change-of-basis pivot or a zero with a column of zero entries below it.*

For clarity, we subdivide Theorem 1.1 into Theorems 4.4 and 5.7.

In §6 we prove a theorem on the existence of a path of flow lines in φ connecting consecutive singularities. Given a non-zero entry $\Delta_{p-r+1,p+1}$ in Δ, there exists a connecting orbit that joins two singularities. On the other hand, if $\Delta_{p-r+1,p+1}$ is zero, we will prove in the zig-zag theorem that there exists a path† joining the singularities $h_k \in F_p$ and $h_{k-1} \in F_{p-r}$ whenever $\Delta_{p-r+1,p+1}$ corresponds to a non-zero d_p^r.

THEOREM 1.2. (Zig-zag theorem) *Let (E^r, d^r) be a spectral sequence induced by a Morse Conley chain complex $(C\Delta, \Delta)$ of a flow φ, where Δ is a connection matrix over $\mathbb{Z}$. Given a non-zero $d^r : E_{p,q}^r \to E_{p-r,q+r-1}^r$, there exists a path of connecting orbits of φ joining $h_k \in F_p \setminus F_{p-1}$ to $h_{k-1} \in F_{p-r} \setminus F_{p-r-1}$.*

Take $\overline{h}_k \in F_s$ and $\overline{h}_{k-1} \in F_{s-\ell}$, with $p > s$ and $r > \ell$, such that there exist connecting orbits between h_k and $\overline{h}_{k-1}$, $\overline{h}_k$ and h_{k-1}, and $\overline{h}_k$ and $\overline{h}_{k-1}$. Furthermore, suppose that there are no singularities between $\overline{h}_k$ and $\overline{h}_{k-1}$. See Figure 2.

† See Definition 6.2.

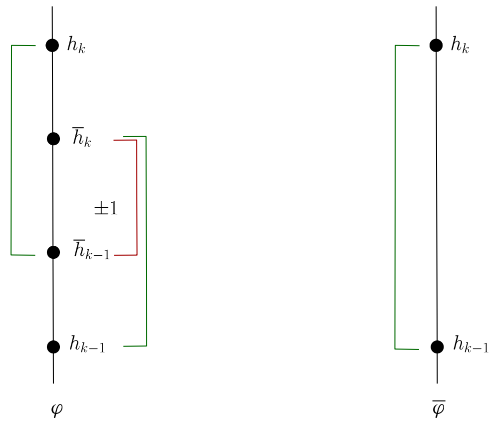

FIGURE 2. The perturbed flow $\overline{\varphi}$ after cancellation.

A particular case of interest occurs when the map d_s^ℓ is an isomorphism and corresponds to an entry ± 1 which is a primary pivot (or change-of-basis pivot) in the connection matrix. Since these maps are isomorphisms, they imply algebraic cancellations in the spectral sequence. On the other hand, they also correspond to dynamical cancellation of consecutive index singularities $\overline{h}_k$ and $\overline{h}_{k-1}$ in φ. By Reineck's theorem [**R3**], there is a continuation of the flow φ to $\overline{\varphi}$ which corresponds to the dynamical cancellation associated to the primary pivot $\Delta^\ell_{s-\ell+1,s+1} = \Delta_{s-\ell+1,s+1}$ on the ℓth auxiliary diagonal of Δ^ℓ.

A certain choice of path in φ will admit a reversal of the flow along the orbit which will cancel $\overline{h}_k$ and $\overline{h}_{k-1}$, creating a new orbit that connects h_k and h_{k-1} in the perturbed flow $\overline{\varphi}$. Hence, the orbit connecting $\overline{h}_k$ and $\overline{h}_{k-1}$ can be viewed as a bridge responsible for the creation of the orbit connecting h_k and h_{k-1} in $\overline{\varphi}$. Since the bridge, i.e. orbit connecting $\overline{h}_k$ and $\overline{h}_{k-1}$, ceases to exist in $\overline{\varphi}$, we are justified in allowing this orbit to be traversed in the reverse direction when we construct the path connecting h_k and h_{k-1} in the flow φ. In this case, the path in φ indicates the birth of an orbit in $\overline{\varphi}$.

On the other hand, connecting orbits of a flow φ^ℓ that correspond to non-zero d^ℓ are associated with a path of connecting orbits in φ by the zig-zag theorem. By the same arguments as above, the connecting orbits in φ^ℓ associated with isomorphisms d_s^ℓ which correspond to primary pivots ± 1 in the connection matrix are algebraic cancellations in the spectral sequence; hence they also correspond to dynamical cancellation of consecutive index singularities in φ^ℓ. By Reineck's theorem, there is a continuation of the flow φ^ℓ to $\overline{\varphi}^\ell$ which corresponds to the dynamical cancellation associated with the primary pivot $\Delta^\ell_{s-\ell+1,s+1}$ on the ℓth auxiliary diagonal of Δ^ℓ. Once again, this justifies why we allow this orbit in φ^ℓ that corresponds to a path in φ to be traversed in the reverse direction.

Inspired by this particular case of algebraic–dynamical correspondence, we consider more general paths in Δ^r where traversal in the reverse direction will be allowed along orbits that correspond to primary as well as change-of-basis pivots which are not necessarily equal to ± 1. Our idea is motivated by the fact that, owing to the zig-zag theorem, certain change-of-basis pivots correspond to non-zero differentials in the spectral sequence.

Although, in this case, it is not clear what a dynamical counterpart to the algebraic behavior is, the zig-zag theorem suggests a correspondence between orbits in the flow φ^ℓ associated to a non-zero differential d^ℓ of the spectral sequence and paths in the flow φ.

2. *Sweeping method*

In this section we present the sweeping method, which constructs recursively a family of matrices $\{\Delta^r\}_{r\geq 0}$, with $\Delta^0 = \Delta$, by considering at each stage the rth auxiliary diagonal. This family of matrices will be used to determine the spectral sequence (E^r, d^r).

We remark that the sweeping method and all the other theorems in this article do not require that the columns of the matrix Δ be ordered with respect to k or, equivalently, that the singularities h_k be ordered with respect to the filtration. Without loss of generality, we will assume the singularities to be ordered with respect to the filtration so as to simplify notation and permit the indices that refer to the columns to increase by increments of one. Otherwise, in a more general setting, we would have to introduce subsequence notation for the columns in order to consider the intersection of the auxiliary diagonals with the index-k columns. For clarity, in our examples we will also keep the singularities ordered with respect to the filtration.

For a fixed auxiliary diagonal r, the method described below must be applied for all k simultaneously.

A: Initial step.

(1) Consider all columns h_k together with all rows h_{k-1} in Δ. Let $\Delta_{k_{i,j}}$ be the entries in Δ for which the ith row is h_{k-1} and the jth column is h_k.
Let ξ_1 be the first auxiliary diagonal of Δ that contains non-zero entries $\Delta_{k_{i,j}}$, which will be called *index-k primary pivots*. It follows that for each non-zero $\Delta_{k_{i,j}}$ on ξ_1, the entries $\Delta_{k_{s,j}}$ for $s > i$ will all be zero. These entries must be zero, or else they would have been detected as primary pivots on a ξ auxiliary diagonal for $\xi < \xi_1$.
We end this first step by defining Δ^{ξ_1} to be Δ with the index-k primary pivots on the ξ_1th auxiliary diagonal marked.

(2) Consider the matrix Δ^{ξ_1}, and let $\Delta^{\xi_1}_{k_{i,j}}$ be the entries in Δ^{ξ_1} for which the ith row is h_{k-1} and the jth column is h_k. Let ξ_2 be the first auxiliary diagonal greater than ξ_1 which contains non-zero entries $\Delta^{\xi_1}_{k_{i,j}}$. We now construct a matrix Δ^{ξ_2} following the procedure below.
Given a non-zero entry $\Delta^{\xi_1}_{k_{i,j}}$ on the ξ_2th auxiliary diagonal of Δ^{ξ_1}:
   (a) if there are no primary pivots in the ith row and the jth column, mark the given entry as an index-k primary pivot and keep the same numerical value, i.e. let $\Delta^{\xi_2}_{k_{i,j}} = \Delta^{\xi_1}_{k_{i,j}}$;
   (b) if case (a) does not hold, then consider the entries in the jth column and in the sth row, with $s > i$, of Δ^{ξ_1}.
      (b1) If there is an index-k primary pivot in an entry in the jth column and in a row s with $s > i$, then the numerical value of the entry remains the same, i.e. $\Delta^{\xi_2}_{k_{i,j}} = \Delta^{\xi_1}_{k_{i,j}}$, and the entry is left unmarked.

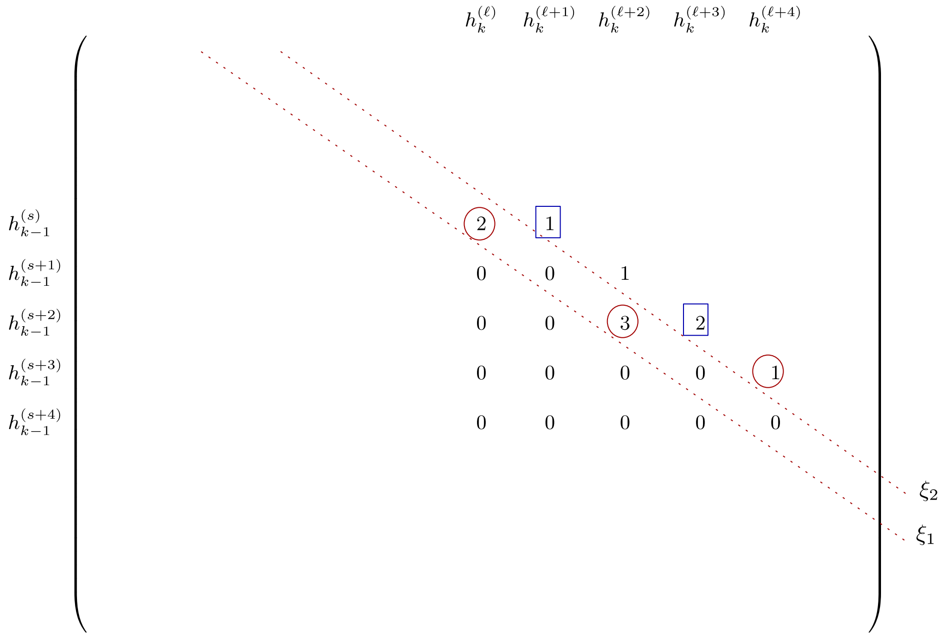

FIGURE 3. Auxiliary diagonals ξ_1 and ξ_2.

(b2) If there are no primary pivots in the jth column below $\Delta^{\xi_1}_{k_{i,j}}$, then there must be an index-k primary pivot in the ith row, say in the tth column of Δ^{ξ_1}, where $t < j$. In this case, the numerical value of the entry remains the same, i.e. $\Delta^{\xi_2}_{k_{i,j}} = \Delta^{\xi_1}_{k_{i,j}}$, but the entry $\Delta^{\xi_1}_{k_{i,j}}$ is marked as a *change-of-basis pivot*.

Note that we have defined a matrix Δ^{ξ_2} which is actually equal to Δ^{ξ_1} except that the ξ_2th diagonal is marked with primary and change-of-basis pivots. See Figure 3.

B: Intermediate step. Consider a matrix Δ^r with the primary and change-of-basis pivots marked on the ξth auxiliary diagonal, for all $\xi \leq r$. We now describe how Δ^{r+1} is defined. Without loss of generality, we can suppose that there is at least one change-of-basis pivot on the rth auxiliary diagonal. If this is not the case, then let $\Delta^{r+1} = \Delta^r$ with the $(r+1)$st auxiliary diagonal marked with primary and change-of-basis pivots as in B.2.

B.1: Change of basis. Suppose that $\Delta^r_{k_{i,j}}$ is a change-of-basis pivot. Then perform a change of basis on Δ^r by adding a linear combination over $\mathbb{Q}$ of all the h_k columns ℓ of Δ^r with $\kappa \leq \ell < j$ where κ is the first column of Δ^r associated with a k-chain, to a positive integer multiple $\mathfrak{u} \neq 0$ of the jth column of Δ^r, in order to zero out the entry $\Delta^r_{k_{i,j}}$ without introducing non-zero entries in $\Delta^r_{k_{s,j}}$ for $s > i$. Moreover, the resulting linear combination should be of the form $\beta^\kappa h_k^{(\kappa)} + \cdots + \beta^{j-1} h_k^{(j-1)} + \beta^j h_k^{(j)}$ where β^ℓ are integers for $\ell = \kappa, \ldots, j$. We use $h_k^{(\ell)}$ to denote the elementary k-chain associated to the ℓth column of Δ.

The integer $\mathfrak{u}$ is called *leading coefficient* of the change of basis. If more than one linear combination is possible, we will choose the one which minimizes $\mathfrak{u}$. Let u be the minimal leading coefficient of a change of basis. Once the change of basis has been performed, we obtain a k-chain associated to the jth column of Δ^{r+1}. This is a linear combination over $\mathbb{Q}$ of the ℓth h_k columns, $\kappa \leq \ell < j$, of Δ^r plus an integer multiple u of the jth column of Δ^r such that $\Delta^{r+1}_{k_{i,j}} = 0$. It is also an integer linear combination of h_k columns of Δ on and to the left of the jth column.

Observe that if the $\overline{\ell}$th column of Δ^r is an h_k column, it corresponds to an integer linear combination $\sigma_k^{(\overline{\ell}),r} = \sum_{\ell=\kappa}^{\overline{\ell}} c_\ell^{\overline{\ell},r} h_k^{(\ell)}$ of h_k columns of Δ, where the κth h_k column is the first column in Δ associated to a k-chain. The expression $\sigma_k^{(\overline{\ell}),r}$ stands for the $\overline{\ell}$th column of Δ^r with Morse index k. Hence, if the jth column of Δ^{r+1} is an h_k column, it will be

$$\begin{aligned}\sigma_k^{(j),r+1} = u \underbrace{\sum_{\ell=\kappa}^{j} c_\ell^{j,r} h_k^{(\ell)}}_{\sigma_k^{(j),r}} + q_{j-1} \underbrace{\sum_{\ell=\kappa}^{j-1} c_\ell^{j-1,r} h_k^{(\ell)}}_{\sigma_k^{(j-1),r}} + \cdots \\ + q_{\kappa+1} \underbrace{(c_\kappa^{\kappa+1,r} h_k^{(\kappa)} + c_{\kappa+1}^{\kappa+1,r} h_k^{(\kappa+1)})}_{\sigma_k^{(\kappa+1),r}} + q_\kappa \underbrace{c_\kappa^{\kappa,r} h_k^{(\kappa)}}_{\sigma_k^{(\kappa),r}} \qquad (1)\end{aligned}$$

or, equivalently,

$$\begin{aligned}&(u c_\kappa^{j,r} + q_{j-1} c_\kappa^{j-1,r} + \cdots + q_\kappa c_\kappa^{\kappa,r}) h_k^{(\kappa)} \\ &\quad + (u c_{\kappa+1}^{j,r} + q_{j-1} c_{\kappa+1}^{j-1,r} + \cdots + q_{\kappa+1} c_{\kappa+1}^{\kappa+1,r}) h_k^{(\kappa+1)} + \cdots \\ &\quad + (u c_{j-1}^{j,r} + q_{j-1} c_{j-1}^{j-1,r}) h_k^{(j-1)} + u c_j^{j,r} h_k^{(j)} \qquad (2)\end{aligned}$$

with $c_\kappa^{\kappa,r} = 1$ and

$$c_\kappa^{j,r+1} = u c_\kappa^{j,r} + q_{j-1} c_\kappa^{j-1,r} + \cdots + q_\kappa c_\kappa^{\kappa,r} \in \mathbb{Z}, \qquad (3)$$

$$c_{\kappa+1}^{j,r+1} = u c_{\kappa+1}^{j,r} + q_{j-1} c_{\kappa+1}^{j-1,r} + \cdots + q_{\kappa+1} c_{\kappa+1}^{\kappa+1,r} \in \mathbb{Z}, \qquad (4)$$

$$\vdots$$

$$c_{j-1}^{j,r+1} = u c_{j-1}^{j,r} + q_{j-1} c_{j-1}^{j-1,r} \in \mathbb{Z}, \qquad (5)$$

$$c_j^{j,r+1} = u c_j^{j,r} \in \mathbb{Z}. \qquad (6)$$

It is clear that the first column of any Δ_k cannot undergo any change of basis since there is no column to its left, and this explains why $c_\kappa^{\kappa,r} = 1$.

Note that $q_{\overline{\ell}} = 0$ in $q_{\overline{\ell}} \sum_{\ell=1}^{\overline{\ell}} c_\ell^{\overline{\ell},r} h_k^{(\ell)}$ whenever the $\overline{\ell}$th column has a primary pivot in a row s with $s > i$.

If the primary pivot of the ith row is in the tth column, then the rational number q_t is non-zero in $q_t \sum_{\ell=1}^{t} c_\ell^{t,r} h_k^{(\ell)}$ and is such that

$$\Delta^{r+1}_{k_{i,j}} = u \Delta^r_{k_{i,j}} + q_t \Delta^r_{k_{i,t}} = 0.$$

Since $u \geq 1$ is unique, q_t is uniquely defined.

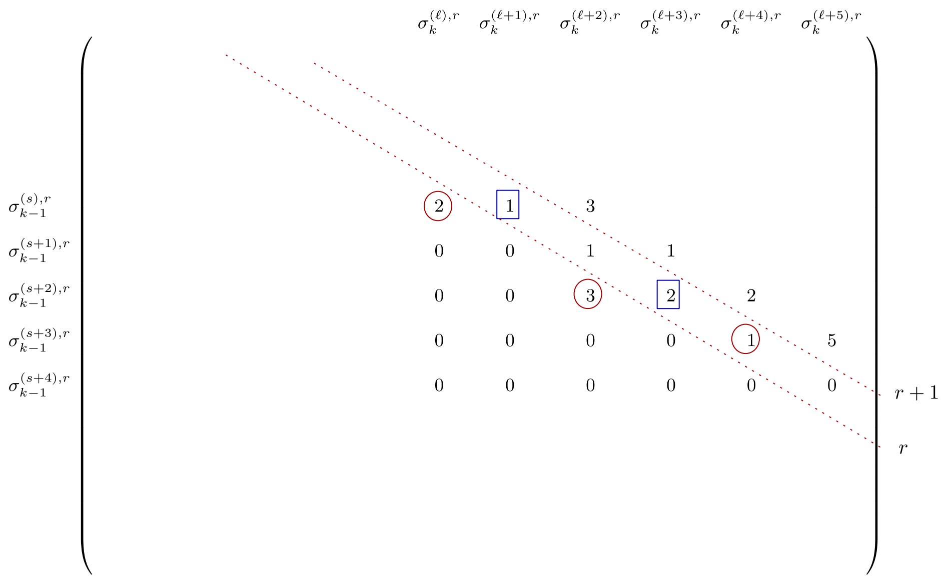

FIGURE 4. Sweeping method: Δ^r.

After the above procedure has been performed for all change-of-basis pivots of the rth diagonal of Δ^r, we can define a change-of-basis matrix.

Therefore, the matrix Δ^{r+1} has numerical values determined by the change of basis over $\mathbb{Q}$ of Δ^r. In particular, all the change-of-basis pivots on the rth auxiliary diagonal Δ^r are zero in Δ^{r+1}. See Figures 4 and 5.

B.2: Marking the $(r+1)$th auxiliary diagonal of Δ^{r+1}. Consider the matrix Δ^{r+1} defined in the previous step. We will now mark the $(r+1)$st auxiliary diagonal with primary and change-of-basis pivots as follows.

Given a non-zero entry $\Delta^{r+1}_{k_{i,j}}$:

(1) if there are no primary pivots in the ith row and the jth column, mark this entry as an index-k primary pivot;

(2) if case (1) does not hold, consider the entries in the jth column and in the sth row, with $s > i$, of Δ^{r+1}.

  (b1) If there is an index-k primary pivot in the entries in the jth column below $\Delta^{r+1}_{k_{i,j}}$, then leave the entry unmarked.

  (b2) If there are no primary pivots in the jth column below $\Delta^{r+1}_{k_{i,j}}$, then there must be an index-k primary pivot in the ith row, say in the tth column of Δ^{r+1}, with $t < j$. Mark it as a change-of-basis pivot; see Figure 5.

C: Final step. We repeat the above procedure until all auxiliary diagonals have been considered.

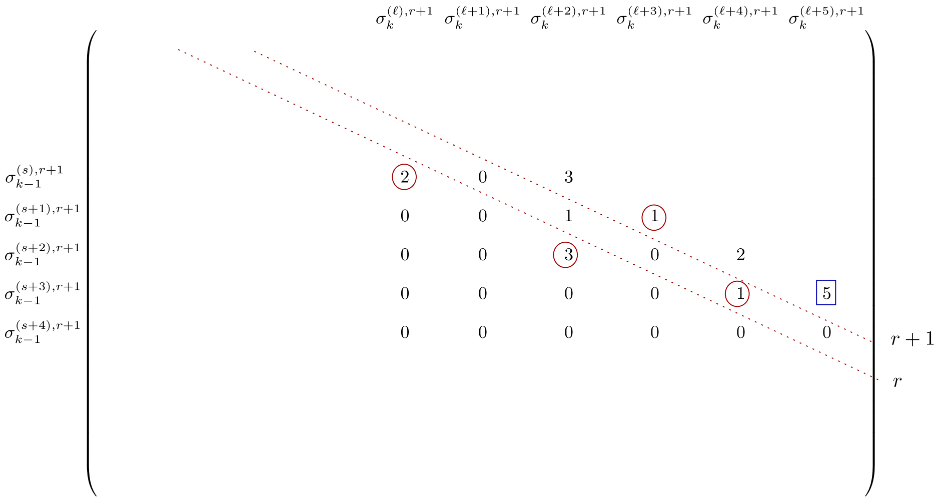

FIGURE 5. Sweeping method: Δ^{r+1}.

	F_0 h_0	F_1 $h_{k-1}^{(2)}$	F_2 $h_{k-1}^{(3)}$	F_3 $h_k^{(4)}$	F_4 $h_k^{(5)}$	F_5 $h_k^{(6)}$	F_6 $h_k^{(7)}$	F_7 $h_k^{(8)}$	F_8 $h_k^{(9)}$	F_9 $h_{k+1}^{(10)}$	F_{10} $h_{k+1}^{(11)}$	F_{11} $h_{k+1}^{(12)}$	F_{12} $h_{k+1}^{(13)}$	F_{13} h_n
F_0 h_0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
F_1 $h_{k-1}^{(2)}$	0	0	0	2	3	2	1	0	0	0	0	0	0	0
F_2 $h_{k-1}^{(3)}$	0	0	0	2	3	1	0	2	1	0	0	0	0	0
F_3 $h_k^{(4)}$	0	0	0	0	0	0	0	0	0	0	1	-3	1	0
F_4 $h_k^{(5)}$	0	0	0	0	0	0	0	0	0	1	0	2	0	0
F_5 $h_k^{(6)}$	0	0	0	0	0	0	0	0	0	-3	-2	1	-3	0
F_6 $h_k^{(7)}$	0	0	0	0	0	0	0	0	0	3	2	-2	4	0
F_7 $h_k^{(8)}$	0	0	0	0	0	0	0	0	0	-1	1	-2	1	0
F_8 $h_k^{(9)}$	0	0	0	0	0	0	0	0	0	2	-2	3	-1	0
F_9 $h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
F_{10} $h_{k+1}^{(11)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
F_{11} $h_{k+1}^{(12)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
F_{12} $h_{k+1}^{(13)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
F_{13} h_n	0	0	0	0	0	0	0	0	0	0	0	0	0	0

FIGURE 6. Δ: the matrix for Example 2.1.

Example 2.1. Let Δ be as in Figure 6. Applying the sweeping method to Δ, we obtain the matrices Δ^1, Δ^2, Δ^3, Δ^4, Δ^5, Δ^6, Δ^7 and Δ^8 given by Figures 7–14.

	h_0	$h^{(2)}_{k-1}$	$h^{(3)}_{k-1}$	$h^{(4)}_k$	$h^{(5)}_k$	$h^{(6)}_k$	$h^{(7)}_k$	$h^{(8)}_k$	$h^{(9)}_k$	$h^{(10)}_{k+1}$	$h^{(11)}_{k+1}$	$h^{(12)}_{k+1}$	$h^{(13)}_{k+1}$	h_n
h_0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$h^{(2)}_{k-1}$	0	0	0	2	3	2	1	0	0	0	0	0	0	0
$h^{(3)}_{k-1}$	0	0	0	2	3	1	0	2	1	0	0	0	0	0
$h^{(4)}_k$	0	0	0	0	0	0	0	0	0	0	1	-3	1	0
$h^{(5)}_k$	0	0	0	0	0	0	0	0	0	1	0	2	0	0
$h^{(6)}_k$	0	0	0	0	0	0	0	0	0	-3	-2	1	-3	0
$h^{(7)}_k$	0	0	0	0	0	0	0	0	0	3	2	-2	4	0
$h^{(8)}_k$	0	0	0	0	0	0	0	0	0	-1	1	-2	1	0
$h^{(9)}_k$	0	0	0	0	0	0	0	0	0	2	-2	3	-1	0
$h^{(10)}_{k+1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$h^{(11)}_{k+1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$h^{(12)}_{k+1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$h^{(13)}_{k+1}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
h_n	0	0	0	0	0	0	0	0	0	0	0	0	0	0

FIGURE 7. Δ^1: marking primary pivots.

As is easily perceived, the computation of the family of matrices produced by the sweeping method is laborious. Hence, we will illustrate several of our results in this paper using this one example.

3. *Properties of* Δ^r

The propositions in this section describe basic properties of the Δ^r's produced by the sweeping method and will be used in the proof of the main theorems. More specifically, our attention will be directed towards characterizing properties associated with the primary and change-of-basis pivots which are essential in determining the spectral sequence.

It is easy to see that all the Δ^r's are upper triangular and that $\Delta^r \circ \Delta^r = 0$, since they are obtained recursively from the initial connection matrix Δ via changes of basis over $\mathbb{Q}$.

It is also straightforward to see that if $\Delta^r_{k_{i,j}}$ is a primary pivot, then there can be no linear combination of columns to the left of the jth column which, added to the jth column, would zero that entry while maintaining all entries $\Delta^r_{k_{s,j}}$ equal to zero for $s > i$. This is because there are three kinds of columns to the left of the jth column. The primary pivot is either above the ith row or below it, or the column does have not a primary pivot in Δ^r. In the latter case, the column has all entries below the rth diagonal equal to zero. This is also true when the primary pivot is above the ith row, since all entries below it are zero. Hence, these three types of columns cannot contribute to a linear combination that aims to zero the entry $\Delta^r_{k_{i,j}}$.

	h_0	$h_{k-1}^{(2)}$	$h_{k-1}^{(3)}$	$h_k^{(4)}$	$h_k^{(5)}$	$h_k^{(6)}$	$h_k^{(7)}$	$h_k^{(8)}$	$h_k^{(9)}$	$h_{k+1}^{(10)}$	$h_{k+1}^{(11)}$	$h_{k+1}^{(12)}$	$h_{k+1}^{(13)}$	h_n
h_0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$h_{k-1}^{(2)}$	0	0	0	2	3	2	1	0	0	0	0	0	0	0
$h_{k-1}^{(3)}$	0	0	0	2	3	1	0	2	1	0	0	0	0	0
$h_k^{(4)}$	0	0	0	0	0	0	0	0	0	0	1	-3	1	0
$h_k^{(5)}$	0	0	0	0	0	0	0	0	0	1	0	2	0	0
$h_k^{(6)}$	0	0	0	0	0	0	0	0	0	-3	-2	1	-3	0
$h_k^{(7)}$	0	0	0	0	0	0	0	0	0	3	2	-2	4	0
$h_k^{(8)}$	0	0	0	0	0	0	0	0	0	-1	1	-2	1	0
$h_k^{(9)}$	0	0	0	0	0	0	0	0	0	2	-2	3	-1	0
$h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$h_{k+1}^{(11)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$h_{k+1}^{(12)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$h_{k+1}^{(13)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
h_n	0	0	0	0	0	0	0	0	0	0	0	0	0	0

FIGURE 8. Δ^2: marking primary and change-of-basis pivots.

	$\sigma_0^{(1),3}$	$\sigma_{k-1}^{(2),3}$	$\sigma_{k-1}^{(3),3}$	$\sigma_k^{(4),3}$	$\sigma_k^{(5),3}$	$\sigma_k^{(6),3}$	$\sigma_k^{(7),3}$	$\sigma_k^{(8),3}$	$\sigma_k^{(9),3}$	$\sigma_{k+1}^{(10),3}$	$\sigma_{k+1}^{(11),3}$	$\sigma_{k+1}^{(12),3}$	$\sigma_{k+1}^{(13),3}$	$\sigma_n^{(14),3}$
$\sigma_0^{(1),3} = h_0$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k-1}^{(2),3} = h_{k-1}^{(2)}$	0	0	0	2	0	2	1	0	0	0	0	0	0	0
$\sigma_{k-1}^{(3),3} = h_{k-1}^{(3)}$	0	0	0	2	0	1	0	2	1	0	0	0	0	0
$\sigma_k^{(4),3} = h_k^{(4)}$	0	0	0	0	0	0	0	0	0	$3/2$	$5/2$	0	1	0
$\sigma_k^{(5),3} = 2h_k^{(5)} - 3h_k^{(4)}$	0	0	0	0	0	0	0	0	0	$1/2$	$1/2$	1	0	0
$\sigma_k^{(6),3} = h_k^{(6)}$	0	0	0	0	0	0	0	0	0	-3	-5	1	-3	0
$\sigma_k^{(7),3} = h_k^{(7)}$	0	0	0	0	0	0	0	0	0	3	5	-2	4	0
$\sigma_k^{(8),3} = h_k^{(8)}$	0	0	0	0	0	0	0	0	0	-1	0	-2	1	0
$\sigma_k^{(9),3} = h_k^{(9)}$	0	0	0	0	0	0	0	0	0	2	0	3	-1	0
$\sigma_{k+1}^{(10),3} = h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k+1}^{(11),3} = h_{k+1}^{(11)} + h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k+1}^{(12),3} = h_{k+1}^{(12)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k+1}^{(13),3} = h_{k+1}^{(13)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_n^{(14),3} = h_n$	0	0	0	0	0	0	0	0	0	0	0	0	0	0

FIGURE 9. Δ^3: change of basis and marking pivots.

	$\sigma_0^{(1),4}$	$\sigma_{k-1}^{(2),4}$	$\sigma_{k-1}^{(3),4}$	$\sigma_k^{(4),4}$	$\sigma_k^{(5),4}$	$\sigma_k^{(6),4}$	$\sigma_k^{(7),4}$	$\sigma_k^{(8),4}$	$\sigma_k^{(9),4}$	$\sigma_{k+1}^{(10),4}$	$\sigma_{k+1}^{(11),4}$	$\sigma_{k+1}^{(12),4}$	$\sigma_{k+1}^{(13),4}$	$\sigma_n^{(14),4}$
$\sigma_0^{(1),4} = h_0$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k-1}^{(2),4} = h_{k-1}^{(2)}$	0	0	0	2	0	1	1	0	0	0	0	0	0	0
$\sigma_{k-1}^{(3),4} = h_{k-1}^{(3)}$	0	0	0	2	0	0	0	2	1	0	0	0	0	0
$\sigma_k^{(4),4} = h_k^{(4)}$	0	0	0	0	0	0	0	0	0	0	0	1	$-1/2$	0
$\sigma_k^{(5),4} = 2h_k^{(5)} - 3h_k^{(4)}$	0	0	0	0	0	0	0	0	0	-1	-2	6	$-3/2$	0
$\sigma_k^{(6),4} = h_k^{(6)} - h_k^{(5)} + h_k^{(4)}$	0	0	0	0	0	0	0	0	0	-3	-5	11	-3	0
$\sigma_k^{(7),4} = h_k^{(7)}$	0	0	0	0	0	0	0	0	0	3	5	-13	4	0
$\sigma_k^{(8),4} = h_k^{(8)}$	0	0	0	0	0	0	0	0	0	-1	0	-1	1	0
$\sigma_k^{(9),4} = h_k^{(9)}$	0	0	0	0	0	0	0	0	0	2	0	0	-1	0
$\sigma_{k+1}^{(10),4} = h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k+1}^{(11),4} = h_{k+1}^{(11)} + h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k+1}^{(12),4} = 2h_{k+1}^{(12)} - 3h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k+1}^{(13),4} = h_{k+1}^{(13)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_n^{(14),4} = h_n$	0	0	0	0	0	0	0	0	0	0	0	0	0	0

FIGURE 10. Δ^4 for Example 2.1.

	$\sigma_0^{(1),5}$	$\sigma_{k-1}^{(2),5}$	$\sigma_{k-1}^{(3),5}$	$\sigma_k^{(4),5}$	$\sigma_k^{(5),5}$	$\sigma_k^{(6),5}$	$\sigma_k^{(7),5}$	$\sigma_k^{(8),5}$	$\sigma_k^{(9),5}$	$\sigma_{k+1}^{(10),5}$	$\sigma_{k+1}^{(11),5}$	$\sigma_{k+1}^{(12),5}$	$\sigma_{k+1}^{(13),5}$	$\sigma_n^{(14),5}$
$\sigma_0^{(1),5} = h_0$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k-1}^{(2),5} = h_{k-1}^{(2)}$	0	0	0	2	0	1	1	0	0	0	0	0	0	0
$\sigma_{k-1}^{(3),5} = h_{k-1}^{(3)}$	0	0	0	2	0	0	0	2	1	0	0	0	0	0
$\sigma_k^{(4),5} = h_k^{(4)}$	0	0	0	0	0	0	0	0	0	0	0	1	0	0
$\sigma_k^{(5),5} = 2h_k^{(5)} - 3h_k^{(4)}$	0	0	0	0	0	0	0	0	0	-1	-2	6	1	0
$\sigma_k^{(6),5} = h_k^{(6)} - h_k^{(5)} + h_k^{(4)}$	0	0	0	0	0	0	0	0	0	-3	-5	11	1	0
$\sigma_k^{(7),5} = h_k^{(7)}$	0	0	0	0	0	0	0	0	0	3	5	-13	-1	0
$\sigma_k^{(8),5} = h_k^{(8)}$	0	0	0	0	0	0	0	0	0	-1	0	-1	0	0
$\sigma_k^{(9),5} = h_k^{(9)}$	0	0	0	0	0	0	0	0	0	2	0	0	0	0
$\sigma_{k+1}^{(10),5} = h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k+1}^{(11),5} = h_{k+1}^{(11)} + h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k+1}^{(12),5} = 2h_{k+1}^{(12)} - 3h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k+1}^{(13),5} = h_{k+1}^{(13)} + h_{k+1}^{(12)} - h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_n^{(14),5} = h_n$	0	0	0	0	0	0	0	0	0	0	0	0	0	0

FIGURE 11. Δ^5 for Example 2.1.

In order to simplify notation, reference to the index k of the matrix Δ_k^r will be omitted whenever this is unlikely to cause confusion.

PROPOSITION 3.1. *If the entry $\Delta^r_{p-r+1,p+1}$ has been identified by the sweeping method as a primary pivot or a change-of-basis pivot, then $\Delta^r_{s,p+1} = 0$ for all $s > p-r+1$.*

Proof. By the sweeping method, $\Delta^r_{s,p+1}$ cannot be a primary pivot for all $s > p-r+1$. Since non-zero entries below the rth diagonal of Δ^r which are not primary pivots occur only in columns above a primary pivot, we have that $\Delta^r_{s,p+1} = 0$ for all $s > p-r+1$. □

	$\sigma_0^{(1),6}$	$\sigma_{k-1}^{(2),6}$	$\sigma_{k-1}^{(3),6}$	$\sigma_k^{(4),6}$	$\sigma_k^{(5),6}$	$\sigma_k^{(6),6}$	$\sigma_k^{(7),6}$	$\sigma_k^{(8),6}$	$\sigma_k^{(9),6}$	$\sigma_{k+1}^{(10),6}$	$\sigma_{k+1}^{(11),6}$	$\sigma_{k+1}^{(12),6}$	$\sigma_{k+1}^{(13),6}$	$\sigma_n^{(14),6}$
$\sigma_0^{(1),6} = h_0$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k-1}^{(2),6} = h_{k-1}^{(2)}$	0	0	0	2	0	1	0	−2	0	0	0	0	0	0
$\sigma_{k-1}^{(3),6} = h_{k-1}^{(3)}$	0	0	0	2	0	0	0	0	1	0	0	0	0	0
$\sigma_k^{(4),6} = h_k^{(4)}$	0	0	0	0	0	0	0	0	0	−1	0	−3	0	0
$\sigma_k^{(5),6} = 2h_k^{(5)} - 3h_k^{(4)}$	0	0	0	0	0	0	0	0	0	−1	−2	5	1	0
$\sigma_k^{(6),6} = h_k^{(6)} - h_k^{(5)} + h_k^{(4)}$	0	0	0	0	0	0	0	0	0	0	0	−2	0	0
$\sigma_k^{(7),6} = h_k^{(7)} - h_k^{(6)} + h_k^{(5)} - h_k^{(4)}$	0	0	0	0	0	0	0	0	0	3	5	−13	−1	0
$\sigma_k^{(8),6} = h_k^{(8)} - h_k^{(4)}$	0	0	0	0	0	0	0	0	0	−1	0	−1	0	0
$\sigma_k^{(9),6} = h_k^{(9)}$	0	0	0	0	0	0	0	0	0	2	0	0	0	0
$\sigma_{k+1}^{(10),6} = h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k+1}^{(11),6} = h_{k+1}^{(11)} + h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k+1}^{(12),6} = 2h_{k+1}^{(12)} - 3h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k+1}^{(13),6} = h_{k+1}^{(13)} + h_{k+1}^{(12)} - h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_n^{(14),6} = h_n$	0	0	0	0	0	0	0	0	0	0	0	0	0	0

FIGURE 12. Δ^6 for Example 2.1.

	$\sigma_0^{(1),7}$	$\sigma_{k-1}^{(2),7}$	$\sigma_{k-1}^{(3),7}$	$\sigma_k^{(4),7}$	$\sigma_k^{(5),7}$	$\sigma_k^{(6),7}$	$\sigma_k^{(7),7}$	$\sigma_k^{(8),7}$	$\sigma_k^{(9),7}$	$\sigma_{k+1}^{(10),7}$	$\sigma_{k+1}^{(11),7}$	$\sigma_{k+1}^{(12),7}$	$\sigma_{k+1}^{(13),7}$	$\sigma_n^{(14),7}$
$\sigma_0^{(1),7} = h_0$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k-1}^{(2),7} = h_{k-1}^{(2)}$	0	0	0	2	0	1	0	0	−1	0	0	0	0	0
$\sigma_{k-1}^{(3),7} = h_{k-1}^{(3)}$	0	0	0	2	0	0	0	0	0	0	0	0	0	0
$\sigma_k^{(4),7} = h_k^{(4)}$	0	0	0	0	0	0	0	0	0	0	0	−5	0	0
$\sigma_k^{(5),7} = 2h_k^{(5)} - 3h_k^{(4)}$	0	0	0	0	0	0	0	0	0	0	−2	5	3	0
$\sigma_k^{(6),7} = h_k^{(6)} - h_k^{(5)} + h_k^{(4)}$	0	0	0	0	0	0	0	0	0	2	0	0	0	0
$\sigma_k^{(7),7} = h_k^{(7)} - h_k^{(6)} + h_k^{(5)} - h_k^{(4)}$	0	0	0	0	0	0	0	0	0	3	5	−13	0	0
$\sigma_k^{(8),7} = h_k^{(8)} + 2h_k^{(6)} - 2h_k^{(5)} + h_k^{(4)}$	0	0	0	0	0	0	0	0	0	−1	0	−1	0	0
$\sigma_k^{(9),7} = h_k^{(9)} - h_k^{(5)} + h_k^{(4)}$	0	0	0	0	0	0	0	0	0	2	0	0	0	0
$\sigma_{k+1}^{(10),7} = h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k+1}^{(11),7} = h_{k+1}^{(11)} + h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k+1}^{(12),7} = 2h_{k+1}^{(12)} - 3h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k+1}^{(13),7} = 5h_{k+1}^{(13)} + 5h_{k+1}^{(12)} + h_{k+1}^{(11)} - 4h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_n^{(14),7} = h_n$	0	0	0	0	0	0	0	0	0	0	0	0	0	0

FIGURE 13. Δ^7 for Example 2.1.

Proposition 3.2 asserts that we cannot have more than one primary pivot in a fixed row or column. Moreover, if there is a primary pivot in row i, then there is no primary pivot in column i.

PROPOSITION 3.2. *Let $\{\Delta^r\}$ be the family of matrices that results from applying the sweeping method to a connection matrix Δ. Given any two primary pivots $\Delta^r_{k_{i,j}}$ and $\Delta^r_{\bar{k}_{m,\ell}}$, we have that $\{i, j\} \cap \{m, \ell\} = \emptyset$.*

Proof. The only non-trivial case that needs to be considered is where $\bar{k} = k + 1$, and we have to prove that $j \neq m$ in this case. Suppose that there exists a primary pivot in the jth

	$\sigma_0^{(1),8}$	$\sigma_{k-1}^{(2),8}$	$\sigma_{k-1}^{(3),8}$	$\sigma_k^{(4),8}$	$\sigma_k^{(5),8}$	$\sigma_k^{(6),8}$	$\sigma_k^{(7),8}$	$\sigma_k^{(8),8}$	$\sigma_k^{(9),8}$	$\sigma_{k+1}^{(10),8}$	$\sigma_{k+1}^{(11),8}$	$\sigma_{k+1}^{(12),8}$	$\sigma_{k+1}^{(13),8}$	$\sigma_n^{(14),8}$
$\sigma_0^{(1),8} = h_0$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k-1}^{(2),8} = h_{k-1}^{(2)}$	0	0	0	2	0	1	0	0	0	0	0	0	0	0
$\sigma_{k-1}^{(3),8} = h_{k-1}^{(3)}$	0	0	0	2	0	0	0	0	0	0	0	0	0	0
$\sigma_k^{(4),8} = h_k^{(4)}$	0	0	0	0	0	0	0	0	0	0	0	-5	0	0
$\sigma_k^{(5),8} = 2h_k^{(5)} - 3h_k^{(4)}$	0	0	0	0	0	0	0	0	0	0	-2	5	3	0
$\sigma_k^{(6),8} = h_k^{(6)} - h_k^{(5)} + h_k^{(4)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_k^{(7),8} = h_k^{(7)} - h_k^{(6)} + h_k^{(5)} - h_k^{(4)}$	0	0	0	0	0	0	0	0	0	3	5	-13	0	0
$\sigma_k^{(8),8} = h_k^{(8)} + 2h_k^{(6)} - 2h_k^{(5)} + h_k^{(4)}$	0	0	0	0	0	0	0	0	0	-1	0	-1	0	0
$\sigma_k^{(9),8} = h_k^{(9)} + h_k^{(6)} - 2h_k^{(5)} + 2h_k^{(4)}$	0	0	0	0	0	0	0	0	0	2	0	0	0	0
$\sigma_{k+1}^{(10),8} = h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k+1}^{(11),8} = h_{k+1}^{(11)} + h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k+1}^{(12),8} = 2h_{k+1}^{(12)} - 3h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_{k+1}^{(13),8} = 5h_{k+1}^{(13)} + 5h_{k+1}^{(12)} + h_{k+1}^{(11)} - 4h_{k+1}^{(10)}$	0	0	0	0	0	0	0	0	0	0	0	0	0	0
$\sigma_n^{(14),8} = h_n$	0	0	0	0	0	0	0	0	0	0	0	0	0	0

FIGURE 14. Δ^8 for Example 2.1.

column and another primary pivot in the jth row of Δ^r, i.e. $\Delta^r_{k_{i,j}}$ and $\Delta^r_{k+1_{j,\ell}}$ are primary pivots. Therefore, $\Delta^r_{k_{s,j}} = 0$ for all $s > i$ and $\Delta^r_{k+1_{s,\ell}} = 0$ for all $s > j$.

Let $\sigma_k^{(j),r}$, $\sigma_{k-1}^{(i),r}$ and $\sigma_{k+1}^{(\ell),r}$ be chains associated to the jth, ith and ℓth columns of Δ^r, respectively.

Since $\Delta^r \circ \Delta^r = 0$, $V_1 = \{\sigma_{k-1}^{(i),r}, \sigma_k^{(j),r}, \sigma_{k+1}^{(\ell),r}\}$ cannot be an interval because $\Delta^r(V_1)^2 \neq 0$. Hence there must exist $\sigma_k^{(j_2),r}$, associated to the j_2th column of Δ^r, such that $\sigma_k^{(j_2),r} \neq \sigma_k^{(j),r}$, $\Delta^r_{k_{i,j_2}} \neq 0$ and $\Delta^r_{k+1_{j_2,\ell}} \neq 0$. Note that $j_2 < j$, since $\sigma_k^{(j_2),r} \neq \sigma_k^{(j),r}$ and all entries below a primary pivot are zero.

The entry $\Delta^r_{k_{i,j_2}}$ cannot be a primary pivot, since the ith row already has a primary pivot. Thus, the primary pivot of the j_2th column must be below the entry $\Delta^r_{k_{i,j_2}}$, i.e. there exists $\sigma_{k-1}^{(i_2),r}$ associated to the i_2th row of Δ^r, with $i_2 > i$, such that $\Delta^r_{k_{i_2,j_2}}$ is a primary pivot. Therefore, $\Delta^r_{k_{s,j_2}} = 0$ for all $s > i_2$. See Figure 15.

Once again, since $\Delta^r \circ \Delta^r = 0$ and $\Delta^r(V_2)^2 \neq 0$ for $V_2 = \{\sigma_{k-1}^{(i_2),r}, \sigma_k^{(j_2),r}, \sigma_{k+1}^{(\ell),r}\}$, it follows that V_2 cannot be an interval, i.e. there exists $\sigma_k^{(j_3),r}$ in the j_3th column of Δ^r, with $j_3 \leq j$, such that $\sigma_k^{(j_3),r} \neq \sigma_k^{(j_2),r}$, $\Delta^r_{k_{i_2,j_3}} \neq 0$ and $\Delta^r_{k+1_{j_3,\ell}} \neq 0$.

We must show that $\sigma_k^{(j_3),r} \neq \sigma_k^{(j),r}$. By the construction of $\sigma_k^{(j_3),r}$ we have that $\Delta^r_{k_{i_2,j_3}} \neq 0$ where $i_2 > i$. Thus, if j_3 were equal to j, then we would have the entry $\Delta^r_{k_{i_2,j}} \neq 0$ lying below the primary pivot $\Delta^r_{k_{i,j}}$; this contradicts the fact that $\Delta^r_{k_{s,j}} = 0$ for all $s > i$.

Upon repeating the above steps and always using the fact that $\Delta^r \circ \Delta^r = 0$, we eventually run out of rows or columns to continue the above arguments; see Figure 16. If there are no more h_k columns, we will have an interval V with $\Delta(V)^2 \neq 0$, which contradicts the fact that $\Delta^r \circ \Delta^r = 0$. On the other hand, if there are no more h_{k-1} columns, we will have a non-zero entry in Δ^r below the rth auxiliary diagonal which is neither a primary pivot nor an entry above a primary pivot; this contradicts the fact that

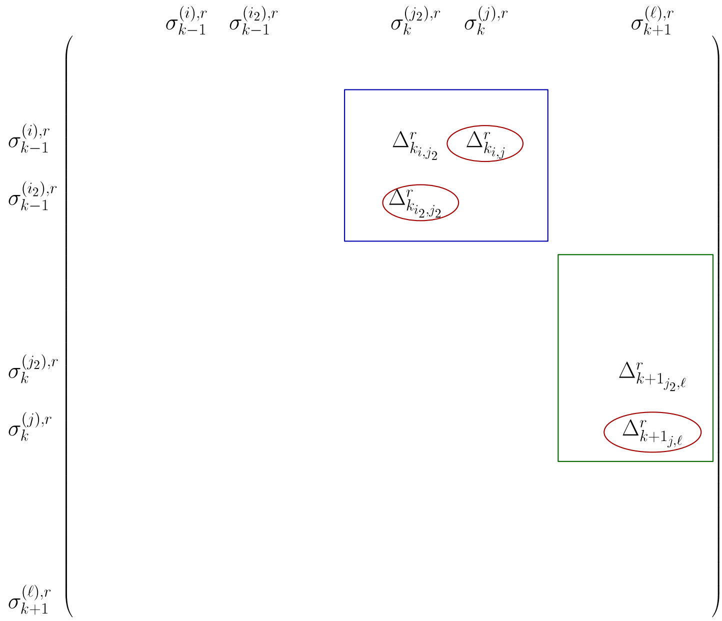

FIGURE 15. Impossibility of primary pivots occurring simultaneously in the jth row and the jth column.

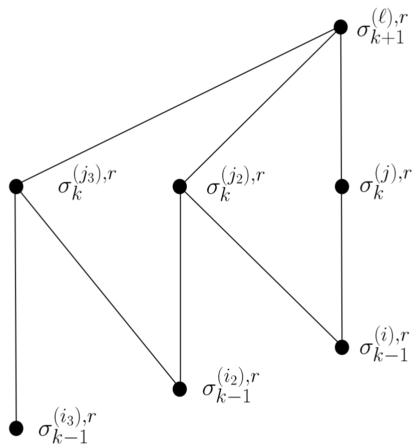

FIGURE 16. Construction of a finite sequence of singularities to ensure that there are no intervals $\Delta^r(V)$ in Δ^r with $\Delta^r(V)^2 = 0$.

the only non-zero entries in Δ^r below the rth auxiliary diagonal are primary pivots and entries above primary pivots. □

4. *The modules E^r_p of the spectral sequence*

In this section, we show how the $\mathbb{Z}$-modules E^r_p are determined when we apply the sweeping method to a matrix Δ. The primary and change-of-basis pivots of Δ^r produced by the sweeping method play an important role in determining the generators of Z_p^r, hence the necessity of proving that the pivots are always integers.

Recall that

$$E_p^r = Z_p^r/(Z_{p-1}^{r-1} + \partial Z_{p+r-1}^{r-1}),$$

where

$$Z_p^r = \{c \in F_p C \mid \partial c \in F_{p-r} C\}.$$

Each h_k column of the connection matrix Δ represents the connections of an elementary chain h_k of C_k to an elementary chain h_{k-1} of C_{k-1}.

The $\mathbb{Z}$-module $Z_{p,k-p}^r = \{c \in F_p C_k \mid \partial c \in F_{p-r} C_{k-1}\}$ is generated by k-chains contained in F_p with boundaries in F_{p-r}. In the matrix Δ, this corresponds to all h_k columns to the left of the $(p+1)$st column, or linear combinations thereof, such that their boundaries (non-zero entries) are above the $(p-r+1)$st row†.

Similarly, in the matrix Δ,

$$Z_{p-1,k-(p-1)}^{r-1} = \{c \in F_{p-1} C_k \mid \partial c \in F_{p-r} C_{k-1}\}$$

corresponds to all h_k columns to the left of the pth column, or linear combinations thereof, such that their boundaries are above the $(p-r+1)$st row.

Finally,

$$\partial Z_{p+r-1,(k+1)-(p+r-1)}^{r-1} = \partial\{c \in F_{p+r-1} C_{k+1} \mid \partial c \in F_p C_k\}$$

is the set of all the boundaries of elements in $Z_{p+r-1,(k+1)-(p+r-1)}^{r-1}$, which, in the matrix Δ, corresponds to all h_k columns to the left of the $(p+1)$st column (or, equivalently, all h_k rows above the $(p+1)$st row) that are boundaries of h_{k+1} columns to the left of the $(p+r)$th column.

The index-k singularity in $F_p \setminus F_{p-1}$ corresponds to the k-chain associated to the $(p+1)$st column of Δ. Hence we denote this singularity by $h_k^{(p+1)}$.

The next proposition establishes a formula for $Z_{p,k-p}^r$.

PROPOSITION 4.1.

$$Z_{p,k-p}^r = \mathbb{Z}[\mu^{(p+1),r}\sigma_k^{(p+1),r}, \mu^{(p),r-1}\sigma_k^{(p),r-1}, \ldots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}],$$

where κ is the first column in Δ associated to a k-chain, and $\mu^{(j),\zeta} = 0$ whenever the primary pivot of the jth column is below the $(p-r+1)$st row and $\mu^{(j),\zeta} = 1$ otherwise.

Proof. Note that $\sigma_k^{(p+1-\xi),r-\xi}$ is associated to the $(p+1-\xi)$th column of the matrix Δ^ξ. By definition, $\mu^{(p+1-\xi),r-\xi} = 1$ if and only if the primary pivot on the $(p+1-\xi)$th column is above the row $(p+1-\xi)-(r-\xi) = p-r+1$. It is easy to verify that chains associated to columns with primary pivots below the $(p-r+1)$st row do not correspond to generators of $Z_{p,k-p}^r$. Consider a k-chain $\sigma_k^{(p+1-\xi),r-\xi}$, with $\xi \in \{0, \ldots, p+1-\kappa\}$, associated to the $(p+1-\xi)$th column of $\Delta^{r-\xi}$ such that the primary pivot of the

† The expressions 'above the row' and 'to the left of the column' shall include the row or column in question, whereas the expressions 'below the row' and 'to the right of the column' do not include the row or column in question.

$(p+1-\xi)$th column of $\Delta^{r-\xi}$ is above the $(p-r+1)$st row. For the latter primary pivots, we show that $\sigma_k^{(p+1-\xi),r-\xi}$ is a k-chain which corresponds to a generator of Z_p^r. It is easy to see that $\sigma_k^{(p+1-\xi),r-\xi}$ is in F_pC_k for $\xi \geq 0$. Furthermore, the $(r-\xi)$th step in the sweeping method has zeroed out all change-of-basis pivots below the $(r-\xi)$th auxiliary diagonal. In other words, all non-zero entries of the $(p+1-\xi)$th column of $\Delta^{r-\xi}$ are above the $(p+1-\xi)-(r-\xi)=(p-r+1)$st row. Hence the boundary of $\sigma_k^{(p+1-\xi),r-\xi}$ is in $F_{p-r}C_{k-1}$.

We now show that any element in Z_p^r is a linear integer combination of $\mu^{(p+1-\xi),r-\xi}\sigma_k^{(p+1-\xi),r-\xi}$ for $\xi = 0,\ldots,p+1-\kappa$. This is achieved by multiple induction on p and r.

- Consider $F_{\kappa-1}$, where κ is the first column of Δ associated to a k-chain. Let ξ be such that the boundary of $h_k^{(\kappa)}$ is in $F_{\kappa-1-\xi}C_k \setminus F_{\kappa-1-\xi-1}C_k$.
  (1) $Z_{\kappa-1}^r$ is generated by a k-chain in $F_{\kappa-1}C_k$ with boundaries in $F_{\kappa-1-r}C_{k-1}$. Note that there exists only one chain $h_k^{(\kappa)}$ in $F_{\kappa-1}C_k$. Hence we have the following possibilities.
    (a) If $\xi < r$, then $\partial h_k^{(\kappa)} \notin F_{\kappa-1-r}C_{k-1}$; thus $Z_{\kappa-1}^r = 0$.
    (b) If $\xi > r$, then $\partial h_k^{(\kappa)} \in F_{\kappa-1-r}C_{k-1}$; thus $Z_{\kappa-1}^r = [h_k^{(\kappa)}]$.
  (2) On the other hand, $\sigma_k^{(\kappa),r}$ is a k-chain associated to the κth column of Δ^r. Since there is no change of basis caused by the sweeping method that affects the first column of Δ_k, we have that $\sigma_k^{(\kappa),r} = h_k^{(\kappa)}$. Furthermore, $\mu^{(\kappa),r} = 1$ if and only if the boundary of $h_k^{(\kappa)} = \sigma_k^{(\kappa),r}$ is above the rth auxiliary diagonal. Hence the following hold.
    (a) If $\xi < r$, then $\mu^{(\kappa),r} = 0$; thus $[\mu^{(\kappa),r}\sigma_k^{(\kappa),r}] = 0$.
    (b) If $\xi > r$, then $\mu^{(\kappa),r} = 1$; thus $[\mu^{(\kappa),r}\sigma_k^{(\kappa),r}] = [\sigma_k^{(\kappa),r}] = [h_k^{(\kappa)}]$.

  Hence $Z_{\kappa-1}^r = [\mu^{(\kappa),r}\sigma_k^{(\kappa),r}]$.
- Let the ξ_1th auxiliary diagonal be the first in Δ that intersects Δ_k. All the columns of Δ corresponding to the chains $h_k^{(p+1)},\ldots,h_k^{(\kappa)}$ have non-zero entries above the ξ_1th auxiliary diagonal and, thus, above the $(p-\xi_1+1)$st row of Δ.
  (1) By definition, $Z_p^{\xi_1}$ is generated by k-chains contained in F_pC_k with boundary in $F_{p-\xi_1}C_{k-1}$. Since the columns of Δ associated to the chains $h_k^{(p+1)},\ldots,h_k^{(\kappa)}$ have non-zero entries above the $(p-\xi_1+1)$st row, this implies that the boundaries are in $F_{p-\xi_1}C_{k-1}$, that is,
  $$Z_p^{\xi_1} = [h_k^{(p+1)},\ldots,h_k^{(\kappa)}].$$
  (2) Since non-zero entries in the columns of Δ associated to the chains $h_k^{(p+1)},\ldots,h_k^{(\kappa)}$ are all above the ξ_1th auxiliary diagonal, it follows that $\sigma_k^{(j),\xi_1} = h_k^{(j)}$ for $j=\kappa,\ldots p+1$ and $\mu^{(j),\xi_1} = 1$ for $j=\kappa,\ldots p+1$. Hence,
  $$[\mu^{(p+1),\xi_1}\sigma_k^{(p+1),\xi_1},\ldots,\mu^{(\kappa),\kappa-p+1+\xi_1}\sigma_k^{(\kappa),\kappa-p+1+\xi_1}] = [h_k^{(p+1)},\ldots,h_k^{(\kappa)}].$$

  Therefore,
  $$Z_p^{\xi_1} = [\mu^{(p+1),\xi_1}\sigma_k^{(p+1),\xi_1},\ldots,\mu^{(\kappa),\kappa-p+1+\xi_1}\sigma_k^{(\kappa),\kappa-p+1+\xi_1}].$$

- We assume that the generators of Z_{p-1}^{r-1} correspond to k-chains associated to $\sigma_k^{(p+1-\xi),r-\xi}$, $\xi = 1, \dots, p+1-\kappa$, whenever the primary pivot of the $(p+1-\xi)$th column is above the $(p-r+1)$st row. If the primary pivot of the $(p+1)$st column is below the $(p-r+1)$st row, then $Z_p^r = Z_{p-1}^{r-1}$, and this is the case when $\mu^{(p+1),r} = 0$. Suppose now that the primary pivot of the $(p+1)$st column is above the $(p-r+1)$st row. Let

$$\mathfrak{h}_k = b^{p+1} h_k^{(p+1)} + \cdots + b^{\kappa} h_k^{(\kappa)}$$

be a k-chain corresponding to an element of $Z_{p,k-p}^r$. We know that $\mathfrak{h}_k$ is in F_p and that its boundary is above the $(p-r+1)$st row. If $b^{p+1} = 0$, then $\mathfrak{h}_k \in Z_{p-1}^{r-1}$ and the result follows from the induction hypothesis. So, from now on, suppose that $b^{p+1} \neq 0$.
By the sweeping method, $\sigma_k^{(p+1),r}$ has $c_{p+1}^{p+1,r}$ as the minimal leading coefficient. We will show that since $c_{p+1}^{p+1,r}$ is the minimal leading coefficient, we have

$$b^{p+1} = \alpha_1 c_{p+1}^{p+1,r}, \quad \alpha_1 \in \mathbb{Z}.$$

Suppose that b^{p+1} is not an integer multiple of $c_{p+1}^{p+1,r}$. Let $\upsilon > 0$ be an integer such that $\upsilon c_{p+1}^{p+1,r}$ is the largest multiple of $c_{p+1}^{p+1,r}$ with $\upsilon c_{p+1}^{p+1,r} < b^{p+1}$. Hence

$$\upsilon c_{p+1}^{p+1,r} < b^{p+1} < (\upsilon + 1) c_{p+1}^{p+1,r},$$

i.e. $0 < b^{p+1} - \upsilon c_{p+1}^{p+1,r} < c_{p+1}^{p+1,r}$. It follows that the k-chain $\mathfrak{h}_k - \upsilon \sigma_k^{(p+1),r}$ has leading coefficient $b^{p+1} - \upsilon c_{p+1}^{p+1,r} < c_{p+1}^{p+1,r}$, which contradicts the fact that $c_{p+1}^{p+1,r}$ is the minimal leading coefficient. Therefore $b^{p+1} = \alpha_1 c_{p+1}^{p+1,r}$ for $\alpha_1 \in \mathbb{Z}$.
Thus we can rewrite $\mathfrak{h}_k$ as

$$\mathfrak{h}_k = \alpha_1 \sigma_k^{(p+1),r} + (b^p - \alpha_1 c_p^{p+1,r}) h_k^{(p)} + \cdots + (b^{\kappa} - \alpha_1 c_{\kappa}^{p+1,r}) h_k^{(\kappa)}.$$

Note that

$$\mathfrak{h}_k - \alpha_1 \sigma_k^{(p+1),r} = (b^p - \alpha_1 c_p^{p+1,r}) h_k^{(p)} + \cdots + (b^{\kappa} - \alpha_1 c_{\kappa}^{p+1,r}) h_k^{(\kappa)} \in F_{p-1}.$$

Moreover, since $\mathfrak{h}_k$ and $\sigma_k^{(p+1),r}$ have their boundaries above the $(p-r+1)$st row, the boundary of $\mathfrak{h}_k - \alpha_1 \sigma_k^{(p+1),r}$ is above the $(p-r+1)$st row. It follows that $\mathfrak{h}_k - \alpha_1 \sigma_k^{(p+1),r} \in Z_{p-1}^{r-1}$. By the induction hypothesis,

$$\mathfrak{h}_k - \alpha_1 \sigma_k^{(p+1),r} = \alpha_2 \mu^{(p),r-1} \sigma_k^{(p),r-1} + \cdots + \alpha_{\kappa} \mu^{(\kappa),r-p-1+\kappa} \sigma_k^{(\kappa),r-p-1+\kappa},$$

that is,

$$\mathfrak{h}_k = \alpha_1 \sigma_k^{(p+1),r} + \alpha_2 \mu^{(p),r-1} \sigma_k^{(p),r-1} + \cdots + \alpha_{\kappa} \mu^{(\kappa),r-p-1+\kappa} \sigma_k^{(\kappa),r-p-1+\kappa}. \quad \square$$

Note that the matrices Δ^r can have some entries which are not integers. However, the following proposition shows that pivots in Δ^r are always integers.

PROPOSITION 4.2. *Suppose that $\Delta^r_{p-r+1,p+1}$ is either a primary pivot or a change-of-basis pivot. Then $\Delta^r_{p-r+1,p+1}$ is an integer.*

Proof. Since $\Delta^r_{p-r+1,p+1}$ is either a primary pivot or a change-of-basis pivot, we have $\Delta^r_{s,p+1} = 0$ for all $s > p - r + 1$. Hence $\sigma_k^{(p+1),r} \in Z^r_p$ and

$$\partial\sigma_k^{(p+1),r} = \Delta^r_{p-r+1,p+1}\sigma_{k-1}^{(p-r+1),r} + \cdots + \Delta^r_{\kappa^*,p+1}\sigma_{k-1}^{(\kappa^*),r},$$

where κ^* is the first column associated to a $(k-1)$-chain. It follows that

$$\begin{aligned}\partial\sigma_k^{(p+1),r} &\in \partial Z^r_p \subset Z^{r+1}_{p-r}\\ &= \mathbb{Z}[\mu^{(p-r+1),r+1}\sigma_k^{(p-r+1),r+1}, \mu^{(p-r),r}\sigma_k^{(p-r),r}, \ldots, \mu^{(\kappa),2r-p+\kappa}\sigma_k^{(\kappa),2r-p+\kappa}].\end{aligned}$$

Thus the coefficient $\Delta^r_{p-r+1,p+1}c^{p-r+1,r}_{p-r+1}$ of $h^{(p-r+1)}_{k-1}$ in $\partial\sigma^{(p+1),r}$ has to be a multiple of the coefficient $c^{p-r+1,r+1}_{p-r+1}$ of $h^{(p-r+1)}_{k-1} \in Z^{r+1}_{p-r}$, that is,

$$\Delta^r_{p-r+1,p+1}c^{p-r+1,r}_{p-r+1} = \alpha c^{p-r+1,r+1}_{p-r+1},$$

where $\alpha \in \mathbb{Z} \setminus \{0\}$. Hence we have

$$\Delta^r_{p-r+1,p+1} = \frac{\alpha c^{p-r+1,r+1}_{p-r+1}}{c^{p-r+1,r}_{p-r+1}}.$$

It follows from (6) that $\Delta^r_{p-r+1,p+1}$ is an integer. □

The next lemma will be used in Theorem 4.4; it detects torsion in the spectral sequence.

LEMMA 4.3. *Suppose that $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \not\subseteq Z^{r-1}_{p-1,k-(p-1)}$. Then*

$$\begin{aligned}Z^{r-1}_{p-1,k-(p-1)} &+ \partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}\\ &= \mathbb{Z}[\ell\sigma_k^{(p+1),r}, \mu^{(p),r-1}\sigma_k^{(p),r-1}, \ldots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}],\end{aligned}$$

where

$$\ell = \gcd\{\mu^{(r+p),r-1}c^{p+1,r-1}_{p+1}\Delta^{r-1}_{p+1,r+p}, \ldots, \mu^{(\overline{\kappa}),\overline{\kappa}-p-1}c^{p+1,\overline{\kappa}-p-1}_{p+1}\Delta^{\overline{\kappa}-p-1}_{p+1,\overline{\kappa}}\}/c^{p+1,r}_{p+1},$$

κ is the first column associated to a k-chain and $\overline{\kappa}$ is the first column associated to a $(k+1)$-chain.

Proof. Since $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \not\subseteq Z^{r-1}_{p-1,k-(p-1)}$, we have that

$$Z^{r-1}_{p-1,k-(p-1)} + \partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$$

is a submodule of

$$Z^r_{p,k-p} = \mathbb{Z}[\mu^{(p+1),r}\sigma_k^{(p+1),r}, \mu^{(p),r-1}\sigma_k^{(p),r-1}, \ldots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}]$$

but is not a submodule of

$$\begin{aligned}Z^{r-1}_{p-1,k-(p-1)}\\ &= \mathbb{Z}[\mu^{(p),r-1}\sigma_k^{(p),r-1}, \mu^{(p-1),r-2}\sigma_k^{(p-1),r-2}, \ldots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}].\end{aligned}$$

Then $\mu^{(p+1),r} = 1$ and $Z^{r-1}_{p-1} + \partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ contains an integer multiple of $\sigma_k^{(p+1),r}$, i.e. $Z^{r-1}_{p-1} + \partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ is equal to

$$\mathbb{Z}[\ell\sigma_k^{(p+1),r}, \mu^{(p),r-1}\sigma_k^{(p),r-1}, \mu^{(p-1),r-2}\sigma_k^{(p-1),r-2}, \dots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}]$$

for some integer ℓ. We will now find the integer ℓ. We have

$$Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} = \mathbb{Z}[\mu^{(p+r),r-1}\sigma_{k+1}^{(p+r),r-1}, \dots, \mu^{(\overline{\kappa}),\overline{\kappa}-p-1}\sigma_{k+1}^{(\overline{\kappa}),\overline{\kappa}-p-1}]$$

where $\mu^{(p+r-\xi),r-1-\xi} = 0$ whenever the primary pivot of the $(p+r-\xi)$th column is below the $(p+1)$st row. Hence

$$\begin{aligned}\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \\ = \mathbb{Z}[\mu^{(p+r),r-1}\partial\sigma_{k+1}^{(p+r),r-1}, \mu^{(p+r-1),r-2}\partial\sigma_{k+1}^{(p+r-1),r-2}, \dots, \mu^{(\overline{\kappa}),\overline{\kappa}-p-1}\partial\sigma_{k+1}^{(\overline{\kappa}),\overline{\kappa}-p-1}].\end{aligned} \tag{7}$$

For $\xi = 0, \dots, p+r-\overline{\kappa}$ with $\mu^{(p+r-\xi),r-1-\xi} = 1$, we have $\Delta^{r-1-\xi}_{i,p+r-\xi} = 0$ for all $i > p+1$ and hence

$$\partial\sigma_{k+1}^{(p+r-\xi),r-1-\xi} = \Delta^{r-1-\xi}_{p+1,p+r-\xi}\sigma_k^{(p+1),r-1-\xi} + \dots + \Delta^{r-1-\xi}_{\kappa,p+r-\xi}\sigma_k^{(\kappa),r-1-\xi}.$$

In fact, the boundaries $\partial\sigma_{k+1}^{(p+r-\xi),r-1-\xi}$ with $\Delta^{r-1-\xi}_{i,p+r-\xi} \neq 0$ for some $i > p+1$ correspond exactly to those columns which have the primary pivots below the $(p+1)$st row, and therefore $\mu^{(p+r-\xi),r-1-\xi} = 0$.

Hence, for $\xi = 0, \dots, p+r-\overline{\kappa}$, when $\mu^{(p+r-\xi),r-1-\xi} = 1$ we have

$$\begin{aligned}Z^{r-1}_{p-1} + [\partial\sigma_{k+1}^{(p+r-\xi),r-1-\xi}] \\ = Z^{r-1}_{p-1} + [\Delta^{r-1-\xi}_{p+1,p+r-\xi}\sigma_k^{(p+1),r-1-\xi} + \dots + \Delta^{r-1-\xi}_{\kappa,p+r-\xi}\sigma_k^{(\kappa),r-1-\xi}].\end{aligned} \tag{8}$$

On the other hand, $Z^{r-1}_{p-1} + [\partial\sigma_{k+1}^{(p+r-\xi),r-1-\xi}] \subset Z^{r-1}_{p-1} + \partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ implies that

$$\begin{aligned}Z^{r-1}_{p-1} + [\partial\sigma_{k+1}^{(p+r-\xi),r-1-\xi}] \\ = [\ell_\xi\sigma_k^{(p+1),r}, \mu^{(p),r-1}\sigma_k^{(p),r-1}, \mu^{(p-1),r-2}\sigma_k^{(p-1),r-2}, \dots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}].\end{aligned} \tag{9}$$

The coefficient of $h_k^{(p+1)}$ on the set of generators of the $\mathbb{Z}$-module in (8) is $\Delta^{r-1-\xi}_{p+1,p+r-\xi}c^{p+1,r-1-\xi}_{p+1}$. On the other hand, the coefficient of $h_k^{(p+1)}$ on the set of the generators of the $\mathbb{Z}$-module in (9) is $\ell_\xi c^{p+1,r}_{p+1}$. Hence

$$\ell_\xi = \Delta^{r-1-\xi}_{p+1,p+r-\xi}c^{p+1,r-1-\xi}_{p+1}/c^{p+1,r}_{p+1}.$$

Thus we have that

$$\ell = \gcd\{\mu^{(p+r-\xi),r-1-\xi}\ell_\xi\}$$

where $\xi = 0, \dots, p+r-\overline{\kappa}$, that is,

$$\ell = \gcd\{\mu^{(r+p),r-1}c^{p+1,r-1}_{p+1}\Delta^{r-1}_{p+1,r+p}, \dots, \mu^{(\overline{\kappa}),\overline{\kappa}-p-1}c^{p+1,\overline{\kappa}-p-1}_{p+1}\Delta^{\overline{\kappa}-p-1}_{p+1,\overline{\kappa}}\}/c^{p+1,r}_{p+1}. \quad \square$$

THEOREM 4.4. *The matrix Δ^r obtained by applying the sweeping method to Δ determines E^r_p.*

Proof. We will prove that

$$E^r_{p,k-p} = \frac{Z^r_{p,k-p}}{Z^{r-1}_{p-1,k-(p-1)} + \partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}}$$

is either zero or a finitely generated module whose generator corresponds to a k-chain associated to the $(p+1)$st column of Δ^r.

Note that $\Delta^r_{p-r+1,p+1}$ is on the rth diagonal and plays a crucial role in determining $E^r_{p,k-p}$.

We now proceed to identify the effect that entries on the rth auxiliary diagonal of Δ^r have on determining the generators of the $\mathbb{Z}$-modules E^r_p.

A non-zero entry on the rth auxiliary diagonal can be either a primary pivot, a change-of-basis pivot or in a column above a primary pivot. A zero entry can be in a column above a primary pivot, or all entries below it will also be zero.

(1) Suppose the entry $\Delta^r_{p-r+1,p+1}$ has been identified by the sweeping method as a primary pivot. It follows from Proposition 3.1 that $\Delta^r_{s,p+1} = 0$ for all $s > p-r+1$. Therefore, the chain associated to the $(p+1)$st column in Δ^r corresponds to a generator of $Z^r_{p,k-p}$. This chain is a linear combination over $\mathbb{Q}$ of the chains associated to the h_k columns of Δ^{r-1} on and to the left of the $(p+1)$st column such that the coefficient of the $(p+1)$st h_k column is a non-zero integer. By the sweeping method, this chain is also a linear combination over $\mathbb{Z}$ of the h_k columns of Δ to the left of the $(p+1)$st column. This chain is $\sigma_k^{(p+1),r}$ and, since the coefficient of the $(p+1)$st h_k column is a non-zero integer, $\sigma_k^{(p+1),r}$ is not contained in the generators of $Z^{r-1}_{p-1,k-(p-1)}$.
Claim. If $\Delta^r_{p-r+1,p+1}$ has been identified by the sweeping method as a primary pivot, then $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$.
The generators of $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ must correspond to $(k+1)$-chains associated to h_{k+1} columns with the property that their boundaries are above the $(p+1)$st row; consequently, all entries below the $(p+1)$st row are zero. Hence the entries of these h_{k+1} columns in the $(p+1)$st row must, by the sweeping method, be either a primary pivot or a zero entry. See Figure 17.
By Proposition 3.2, the $(p+1)$st row cannot contain a primary pivot since we have assumed that the $(p+1)$st column has a primary pivot. Therefore, the entries of these h_{k+1} columns in the $(p+1)$st row must be zeros. It follows that $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ does not contain in its set of generators a multiple of the generator $\sigma_k^{(p+1),r}$. The claim is thus verified.
By Proposition 4.1 we have that $E^r_{p,k-p} = \mathbb{Z}[\sigma_k^{(p+1),r}]$.

(2) If the entry $\Delta^r_{p-r+1,p+1}$ is identified by the sweeping method as a change-of-basis pivot, then the sweeping method guarantees that $\Delta^{r+1}_{p-r+1,p+1} = 0$. Furthermore, $\Delta^r_{s,p+1} = 0$ for all $s > p-r+1$ by Proposition 3.1.

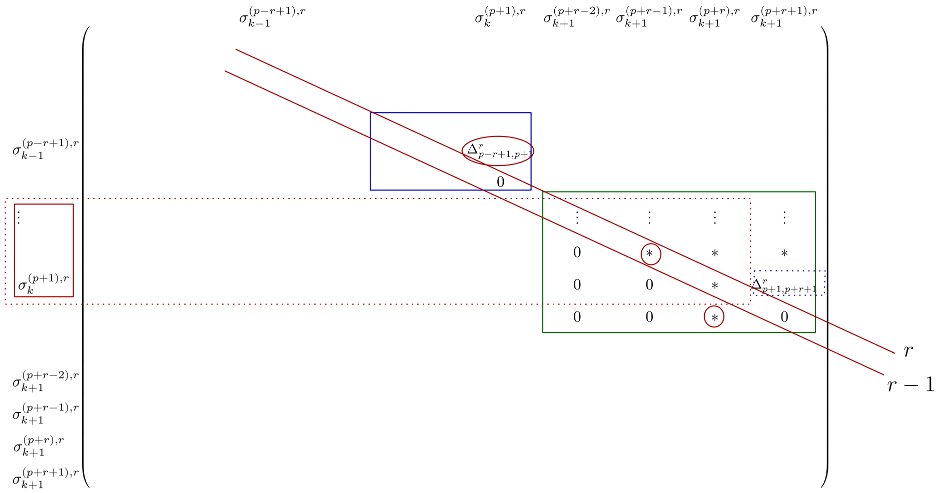

FIGURE 17. $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$.

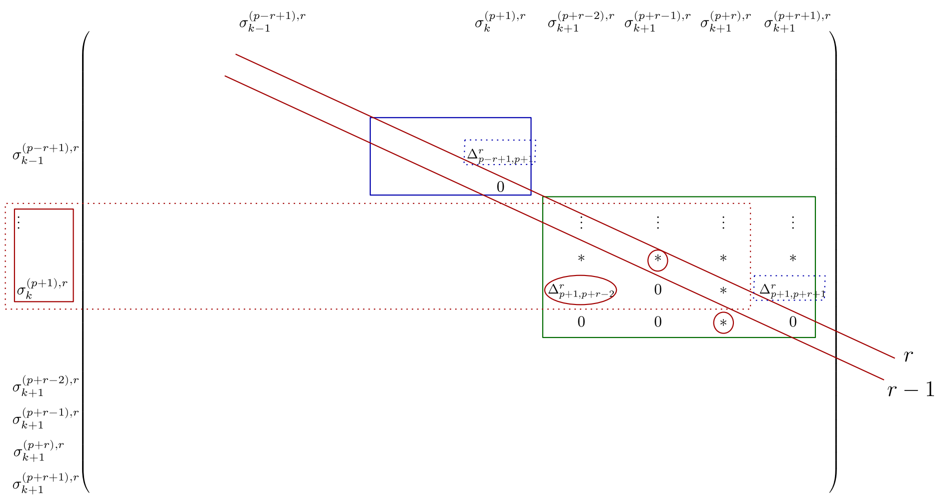

FIGURE 18. $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \not\subseteq Z^{r-1}_{p-1,k-(p-1)}$.

Therefore, as in the previous case, the generator corresponding to the k-chain associated to the $(p+1)$st column $\sigma_k^{(p+1),r}$ in Δ^r is a generator of $Z^r_{p,k-p}$. Thus we have to analyze the $(p+1)$st row. There are two possibilities.

(a) $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$, i.e. all the boundaries of the elements in $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ are above the pth row. In this case, as before, $E^r_{p,k-p} = \mathbb{Z}[\sigma_k^{(p+1),r}]$ by Proposition 4.1.

(b) $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \not\subseteq Z^{r-1}_{p-1,k-(p-1)}$, i.e. there exists an element in $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ whose boundary has a non-zero entry in the $(p+1)$st row, which is then necessarily a primary pivot.

By Proposition 4.1 and Lemma 4.3,

$$E^r_{p,k-p} = \frac{\mathbb{Z}}{\ell\mathbb{Z}}[\sigma_k^{(p+1),r}].$$

(3) If the entry $\Delta^r_{p-r+1,p+1}$ is non-zero but is not a primary pivot or a change-of-basis pivot, then it must be an entry above a primary pivot. In other words, there exists $s > p - r + 1$ such that $\Delta^r_{s,p+1}$ is a primary pivot. It follows that $\sigma_k^{(p+1),r}$ is not in $Z^r_{p,k-p}$. Thus, $Z^{r-1}_{p-1,k-(p-1)} = Z^r_{p,k-p}$ and hence $E^r_{p,k-p} = 0$.

(4) If the entry $\Delta^r_{p-r+1,p+1}$ is a zero entry, we have the following possibilities.

(a) There is a primary pivot below $\Delta^r_{p-r+1,p+1}$, i.e. there exists $s > p - r + 1$ such that $\Delta^r_{s,p+1}$ is a primary pivot. In this case, the generator corresponding to the k-chain associated to the $(p+1)$st column $\sigma_k^{(p+1),r}$ is not a generator of Z^r_p, and hence $Z^{r-1}_{p-1,k-(p-1)} = Z^r_{p,k-p}$. It follows that $E^r_{p,k-p} = 0$.

(b) $\Delta^r_{s,p+1} = 0$ for all $s > p - r + 1$. In this case, the generator corresponding to the k-chain associated to the $(p+1)$st column $\sigma_k^{(p+1),r}$ in Δ^r is a generator of $Z^r_{p,k-p}$. Thus we must analyze the $(p+1)$st row. We have the following possibilities.

(i) $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$, i.e. all the boundaries of the elements in $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ are above the pth row. In this case, as before, $E^r_{p,k-p} = \mathbb{Z}[\sigma_k^{(p+1),r}]$ by Proposition 4.1.

(ii) $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \not\subseteq Z^{r-1}_{p-1,k-(p-1)}$, i.e. there exists an element in $Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ whose boundary has a non-zero entry in the $(p+1)$st row. By Proposition 4.1 and Lemma 4.3,

$$E^r_{p,k-p} = \frac{\mathbb{Z}}{\ell\mathbb{Z}}[\sigma_k^{(p+1),r}], \quad \ell \in \mathbb{Z}.$$

(5) The entry $\Delta^r_{p-r+1,p+1}$ is not in Δ^r_k. This includes the case where $p - r + 1 < 0$, i.e. where $\Delta^r_{p-r+1,p+1}$ is not in the matrix Δ^r.
The analysis for E^r_p is very similar to that in the previous case, i.e. we have two possibilities to consider.

(a) There is a primary pivot in the $(p+1)$st column on an auxiliary diagonal $\overline{r} < r$. In this case, the generator corresponding to the k-chain associated to the $(p+1)$st column $\sigma_k^{(p+1),r}$ is not a generator of $Z^r_{p,k-p}$. Hence

$$Z^{r-1}_{p-1,k-(p-1)} = Z^r_{p,k-p} \quad \text{and} \quad E^r_{p,k-p} = 0.$$

(b) All entries of Δ^r in the $(p+1)$st column on auxiliary diagonals lower than r are zero, i.e. the generator corresponding to the k-chain associated to the $(p+1)$st column $\sigma_k^{(p+1),r}$ in Δ^r is a generator of $Z^r_{p,k-p}$. We then have to analyze the $(p+1)$st row.

(i) If $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)}$, then, by Proposition 4.1, $E^r_{p,k-p} = \mathbb{Z}[\sigma_k^{(p+1),r}]$.

(ii) If $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \not\subseteq Z^{r-1}_{p-1,k-(p-1)}$, then, by Proposition 4.1 and Lemma 4.3,

$$E^r_{p,k-p} = \frac{\mathbb{Z}}{\ell\mathbb{Z}}[\sigma_k^{(p+1),r}].$$

□

5. *The differentials of the spectral sequence*

In this section we will show how the sweeping method applied to Δ induces the differentials $d_p^r : E_p^r \to E_{p-r}^r$ in the spectral sequence. Whenever E_p^r and E_{p-r}^r are both non-zero, the entry $\Delta^r_{p-r+1,p+1}$ in Δ^r will be a primary pivot, a change-of-basis pivot or a zero with a column of zero entries below it, and it induces d_p^r. We will denote by κ the first column of a connection matrix associated to a k-chain and by $\overline{\kappa}$ the first column associated to a $(k+1)$-chain.

In §2 we defined $\sigma_k^{(p+1),r+1}$ as a linear integer combination of h_k's, with $c_{p+1}^{p+1,r}$ being the smallest leading coefficient. The next proposition shows that $\sigma_k^{(p+1),r+1}$ is also a linear combination of $\sigma_k^{(j),\xi} \in \Delta^\xi$, with $j = \kappa, \ldots, p+1$ and $\xi = r-p-1+\kappa, \ldots, r$ such that $j - \xi = p - r + 1$. In each case, the linear combination minimizes u.

PROPOSITION 5.1. *Given a change-of-basis pivot* $\Delta^r_{p-r+1,p+1}$, *there exist integers* $b_{p+1}, b_p, \ldots, b_\kappa$ *such that the boundary of*

$$b_{p+1}\sigma_k^{(p+1),r} + b_p\mu^{(p),r-1}\sigma_k^{(p),r-1} + \cdots + b_\kappa\mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}$$

is above the $(p-r)$*th row. Moreover, the smallest* b_{p+1} *which satisfies this is* u.

Proof. Since $\Delta^r_{p-r+1,p+1}$ is a change-of-basis pivot, $\Delta^r_{s,p+1} = 0$ for all $s > p-r+1$ and $\Delta^{r+1}_{p-r+1,p+1} = 0$. Hence $\sigma_k^{(p+1),r+1} \in Z_p^{r+1} \subset Z_p^r$. By Proposition 4.1,

$$Z_p^r = \mathbb{Z}[\mu^{(p+1),r}\sigma_k^{(p+1),r}, \mu^{(p),r-1}\sigma_k^{(p),r-1}, \ldots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}].$$

In other words,

$$\begin{aligned}\sigma_k^{(p+1),r+1} &= b_{p+1}\mu^{(p+1),r}\sigma_k^{(p+1),r} + b_p\mu^{(p),r-1}\sigma_k^{(p),r-1}\\ &\quad + \cdots + b_\kappa\mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}\end{aligned}$$

where $b_{p+1}, \ldots, b_\kappa$ are integers. Since $c_{p+1}^{p+1,r+1} = uc_{p+1}^{p+1,r}$, we deduce that in this case $b_{p+1} = u$. It follows that the integers $b_{p+1}, b_p, \ldots, b_\kappa$ exist and that u is a possible value for b_{p+1}.

Finally, we will show that u is the smallest positive integer for which $b_p, \ldots, b_\kappa$ exist, i.e. that the smallest b_{p+1} is u. Suppose that $\overline{u} < u$ is a positive integer such that there exist $\overline{b}_p, \ldots, \overline{b}_\kappa$ with

$$\begin{aligned}\sigma_k^{(p+1),r+1} &= \overline{u}\mu^{(p+1),r}\sigma_k^{(p+1),r} + \overline{b}_p\mu^{(p),r-1}\sigma_k^{(p),r-1}\\ &\quad + \cdots + \overline{b}_\kappa\mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}.\end{aligned}$$

Then

$$\begin{aligned}\sigma_k^{(p+1),r+1} &= \overline{u}\mu^{(p+1),r}c_{p+1}^{p+1,r}h_k^{(p+1)} + (\overline{u}\mu^{(p+1),r}c_p^{p+1,r} + \overline{b}_p\mu^{(p),r-1}c_p^{p,r-1})h_k^{(p)}\\ &\quad + \cdots + (\overline{u}\mu^{(p+1),r}c_\kappa^{p+1,r} + \overline{b}_p\mu^{(p),r-1}c_\kappa^{p,r-1}\\ &\quad + \cdots + \overline{b}_\kappa\mu^{(\kappa),r-p-1+\kappa}c_\kappa^{\kappa,r-p-1+\kappa})h_k^{(\kappa)},\end{aligned}$$

which contradicts the minimality property of u as defined in (2). Therefore u is the smallest positive integer such that $b_p, \ldots b_\kappa$ exist. □

The next proposition establishes a formula for the u in Proposition 5.1 in the case where the entry $\Delta^r_{p-r+1,p+1}$ is a change-of-basis pivot. In all other cases, $u = 1$.

PROPOSITION 5.2. *Suppose that $\Delta^r_{p-r+1,p+1}$ is a change-of-basis pivot and let $u = c^{p+1,r+1}_{p+1}/c^{p+1,r}_{p+1}$ be the integer defined in (1). If*

$$v = \gcd\{\mu^{(p),r-1}c^{p-r+1,r-1}_{p-r+1}\Delta^{r-1}_{p-r+1,p}, \dots, \\ \mu^{(\kappa),\kappa-p+r-1}c^{p-r+1,\kappa-p+r-1}_{p-r+1}\Delta^{\kappa-p+r-1}_{p-r+1,\kappa}\}/c^{p-r+1,r}_{p-r+1}$$

and $\lambda = v/\gcd\{\Delta^r_{p-r+1,p+1}, v\}$, then $u = \lambda$.

Proof. We know by Proposition 5.1 that u in (1) is the smallest positive integer such that there exist integers $b_p, \dots, b_\kappa$ with

$$\sigma^{(p+1),r+1}_k = u\mu^{(p+1),r}\sigma^{(p+1),r}_k + b_p\mu^{(p),r-1}\sigma^{(p),r-1}_k \\ + \dots + b_\kappa\mu^{(\kappa),r-p-1+\kappa}\sigma^{(\kappa),r-p-1+\kappa}_k.$$

Since $\Delta^r_{p-r+1,p+1}$ is a change-of-basis pivot, $\Delta^r_{s,p+1} = 0$ for all $s > p - r + 1$ and hence $\mu^{(p+1),r} = 1$. Calculating the boundary of both sides of the equation gives

$$\partial\sigma^{(p+1),r+1}_k = u\partial\sigma^{(p+1),r}_k + b_p\mu^{(p),r-1}\partial\sigma^{(p),r-1}_k \\ + \dots + b_\kappa\mu^{(\kappa),r-p-1+\kappa}\partial\sigma^{(\kappa),r-p-1+\kappa}_k. \tag{10}$$

Since $\Delta^r_{p-r+1,p+1}$ is a change-of-basis pivot, $\Delta^{r+1}_{p-r+1,p+1} = 0$. Hence the coefficient of $h^{(p-r+1)}_{k-1}$ in $\partial\sigma^{(p+1),r+1}_k$ is zero. Moreover,

$$\partial\sigma^{(p+1),r}_k = \Delta^r_{p-r+1,p+1}c^{p-r+1,r}_{p-r+1}h^{(p-r+1)}_{k-1} + \cdots,$$
$$\partial\sigma^{(p),r-1}_k = \Delta^{r-1}_{p-r+1,p}c^{p-r+1,r-1}_{p-r+1}h^{(p-r+1)}_{k-1} + \cdots,$$
$$\vdots$$
$$\partial\sigma^{(\kappa),r-p-1+\kappa}_k = \Delta^{r-p-1+\kappa}_{p-r+1,\kappa}c^{p-r+1,r-p-1+\kappa}_{p-r+1}h^{(p-r+1)}_{k-1} + \cdots.$$

Equating the coefficients of $h^{(p-r+1)}_{k-1}$ on both sides of equation (10) yields

$$0 = u\Delta^r_{p-r+1,p+1}c^{p-r+1,r}_{p-r+1} + b_p\mu^{(p),r-1}\Delta^{r-1}_{p-r+1,p}c^{p-r+1,r-1}_{p-r+1} \\ + \dots + b_\kappa\mu^{(\kappa),r-p-1+\kappa}\Delta^{r-p-1+\kappa}_{p-r+1,\kappa}c^{p-r+1,r-p-1+\kappa}_{p-r+1}.$$

Thus,

$$u\Delta^r_{p-r+1,p+1}c^{p-r+1,r}_{p-r+1} = -\big[b_p\mu^{(p),r-1}\Delta^{r-1}_{p-r+1,p}c^{p-r+1,r-1}_{p-r+1} + \cdots \\ + b_\kappa\mu^{(\kappa),r-p-1+\kappa}\Delta^{r-p-1+\kappa}_{p-r+1,\kappa}c^{p-r+1,r-p-1+\kappa}_{p-r+1}\big]$$

$$u\Delta^r_{p-r+1,p+1} = -\big[b_p\mu^{(p),r-1}\Delta^{r-1}_{p-r+1,p}c^{p-r+1,r-1}_{p-r+1} + \cdots \\ + b_\kappa\mu^{(\kappa),r-p-1+\kappa}\Delta^{r-p-1+\kappa}_{p-r+1,\kappa}c^{p-r+1,r-p-1+\kappa}_{p-r+1}\big]/c^{p-r+1,r}_{p-r+1}.$$

It follows from Proposition 5.1, which asserts the minimality property of u, that

$$u\Delta^{r}_{p-r+1,p+1}c^{p-r+1,r}_{p-r+1} = \gcd\{\mu^{(p),r-1}\Delta^{r-1}_{p-r+1,p}c^{p-r+1,r-1}_{p-r+1},\ldots,\mu^{(\kappa),r-p-1+\kappa}\Delta^{r-p-1+\kappa}_{p-r+1,\kappa}c^{p-r+1,r-p-1+\kappa}_{p-r+1}\},$$

that is,

$$u\Delta^{r}_{p-r+1,p+1} = v.$$

Hence

$$\mathrm{lcm}\{u\Delta^{r}_{p-r+1,p+1},\Delta^{r}_{p-r+1,p+1}\} = \mathrm{lcm}\{\Delta^{r}_{p-r+1,p+1},v\}.$$

Equivalently,

$$u\Delta^{r}_{p-r+1,p+1} = \mathrm{lcm}\{\Delta^{r}_{p-r+1,p+1},v\}.$$

Upon dividing both sides of the equality by the product $\Delta^{r}_{p-r+1,p+1}\cdot v$, the equation becomes

$$\frac{u}{v} = \frac{\mathrm{lcm}\{\Delta^{r}_{p-r+1,p+1},v\}}{\Delta^{r}_{p-r+1,p+1}\cdot v},$$

which is equivalent to

$$\frac{u}{v} = \frac{1}{\gcd\{\Delta^{r}_{p-r+1,p+1},v\}},$$

that is,

$$u = \frac{v}{\gcd\{\Delta^{r}_{p-r+1,p+1},v\}} = \lambda. \qquad \square$$

LEMMA 5.3. *Let $E^r_p = \mathbb{Z}_t[\sigma_k^{(p+1),r}]$ where*

$$t = \frac{\gcd\{\mu^{(r+p),r-1}c^{p+1,r-1}_{p+1}\Delta^{r-1}_{p+1,r+p},\ldots,\mu^{(\overline{\kappa}),\overline{\kappa}-p-1}c^{p+1,\overline{\kappa}-p-1}_{p+1}\Delta^{\overline{\kappa}-p-1}_{p+1,\overline{\kappa}}\}}{c^{p+1,r}_{p+1}},$$

and suppose that $\Delta^r_{p-r+1,p+1}$ is a change-of-basis pivot.

(1) *If $\Delta^r_{p+1,p+r+1}$ is a change-of-basis pivot, then*

$$E^{r+1}_{p,k-p} = \frac{u\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{\gcd\{\Delta^r_{p+1,p+r+1},t\}\mathbb{Z}[\sigma_k^{(p+1),r+1}]}.$$

(2) *If $\Delta^r_{p+1,p+r+1}$ is a zero entry with a column of zeros below it, i.e. $\Delta^r_{s,p+r+1}=0$ for $s>p+1$, then*

$$E^{r+1}_{p,k-p} = \frac{u\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{t\mathbb{Z}[\sigma_k^{(p+1),r+1}]}.$$

Similarly, if $\Delta^r_{p-r+1,p+1}$ is a zero entry with a column of zeros below it, then the formulas above hold for $u=1$.

Proof. Since $\Delta^r_{p-r+1,p+1}$ is a change-of-basis pivot or a zero entry with a column of zeros below it, we have that $\Delta^{r+1}_{p-r+1,p+1}=0$ and hence $\sigma_k^{(p+1),r+1}\in Z^{r+1}_p$. Therefore, by Lemma 4.3,

$$E^{r+1}_{p,k-p} = \frac{\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{s\mathbb{Z}[\sigma_k^{(p+1),r+1}]},$$

where

$$s = \gcd\{\mu^{(p+r+1),r} c_{p+1}^{p+1,r} \Delta^r_{p+1,p+r+1}, \mu^{(r+p),r-1} c_{p+1}^{p+1,r-1} \Delta^{r-1}_{p+1,r+p}, \dots, \mu^{(\overline{\kappa}),\overline{\kappa}-p-1} c_{p+1}^{p+1,\overline{\kappa}-p-1} \Delta^{\overline{\kappa}-p-1}_{p+1,\overline{\kappa}}\} / c_{p+1}^{p+1,r+1}$$

$$= \gcd\left\{ \frac{\mu^{(p+r+1),r} c_{p+1}^{p+1,r} \Delta^r_{p+1,p+r+1}}{c_{p+1}^{p+1,r}}, \frac{\gcd\{\mu^{(r+p),r-1} c_{p+1}^{p+1,r-1} \Delta^{r-1}_{p+1,r+p}, \dots, \mu^{(\overline{\kappa}),\overline{\kappa}-p-1} c_{p+1}^{p+1,\overline{k}-p-1} \Delta^{\overline{\kappa}-p-1}_{p+1,\overline{\kappa}}\}}{c_{p+1}^{p+1,r}} \right\} \times c_{p+1}^{p+1,r} / c_{p+1}^{p+1,r+1}.$$

Since $\Delta^r_{p+1,p+r+1}$ is a change-of-basis pivot or a zero entry with a column of zeros below it, we have that $\mu^{(p+r+1),r} = 1$. Hence

$$s = c_{p+1}^{p+1,r} \frac{\gcd\{\Delta^r_{p+1,p+r+1}, t\}}{c_{p+1}^{p+1,r+1}}.$$

If $\Delta^r_{p-r+1,p+1}$ is a change-of-basis pivot, then

$$\frac{c_{p+1}^{p+1,r}}{c_{p+1}^{p+1,r+1}} = \frac{1}{u}.$$

On the other hand, it is trivial to see that if $\Delta^r_{p-r+1,p+1}$ is a zero entry with a column of zeros below it, then there is no change of basis and hence $c_{p+1}^{p+1,r} = c_{p+1}^{p+1,r+1}$, i.e. $u = 1$. □

Remark 5.4. As a direct consequence of the proof of Lemma 5.3, we have that whenever $\Delta^r_{p-r+1,p+1}$ is a change-of-basis pivot, $u \le \gcd\{\Delta^r_{p+1,p+r+1}, t\} \le t$.

LEMMA 5.5. *Let $E^r_p = \mathbb{Z}[\sigma_k^{(p+1),r}]$ and suppose that $\Delta^r_{p-r+1,p+1}$ is a change-of-basis pivot. Then the following hold.*

(1) *If $\Delta^r_{p+1,p+r+1}$ is a primary pivot, then*

$$E^{r+1}_{p,k-p} = \frac{u\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{\Delta^r_{p+1,p+r+1}\mathbb{Z}[\sigma_k^{(p+1),r+1}]}.$$

(2) *If $\Delta^r_{p+1,p+r+1}$ is a zero entry with a column of zeros below it, then*

$$E^{r+1}_{p,k-p} = u\mathbb{Z}[\sigma_k^{(p+1),r+1}].$$

Similarly, if $\Delta^r_{p-r+1,p+1}$ is a zero entry with a column of zeros below it, then the formulas above hold for $u = 1$.

Proof. Since $\Delta^r_{p-r+1,p+1}$ is a change-of-basis pivot or a zero with a column of zero entries below it, $\Delta^{r+1}_{p-r+1,p+1} = 0$ and thus $\sigma_k^{(p+1),r+1} \in Z^{r+1}_{p,k-p}$. It follows that $Z^r_{p-1,k-(p-1)} \subsetneq Z^{r+1}_{p,k-p}$. Moreover, $E^r_p = \mathbb{Z}[\sigma_k^{(p+1),r}]$ implies that

$$\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)} \subseteq Z^{r-1}_{p-1,k-(p-1)},$$

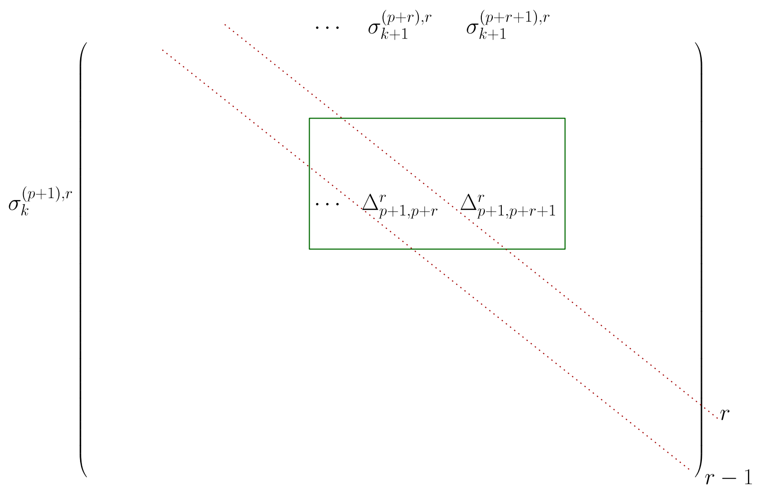

FIGURE 19. The difference between $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ and $\partial Z^r_{p+r,(k+1)-(p+r)}$.

i.e. for all $\sigma_{k+1}^{(p+r-\xi),r-1-\xi}$ with $\xi = 0,\ldots,p+r-\overline{\kappa}$, either $\partial\sigma_{k+1}^{(p+r-\xi),r-1-\xi} \in Z^{r-1}_{p-1,k-(p-1)}$ and hence $\Delta^{r-1-\xi}_{p+1,p+r-\xi} = 0$ or $\sigma_{k+1}^{(p+r-\xi),r-1-\xi}$ has a primary pivot below the $(p+1)$st row and hence $\mu^{(p+r-\xi),r-1-\xi} = 0$. The difference between $\partial Z^{r-1}_{p+r-1,(k+1)-(p+r-1)}$ and $\partial Z^r_{p+r,(k+1)-(p+r)}$ is that the latter includes the boundary of the $(p+r+1)$st column; see Figure 19. But the hypothesis is that the element in the $(p+r+1)$st column and $(p+1)$st row is $\Delta^r_{p+1,p+r+1}$. If $\Delta^r_{p+1,p+r+1}$ is a primary pivot, then

$$E^{r+1}_{p,k-p} = \frac{\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{s\mathbb{Z}[\sigma_k^{(p+1),r+1}]},$$

where

$$\begin{aligned} s &= \gcd\big\{\mu^{(p+r+1),r} c_{p+1}^{p+1,r}\Delta^r_{p+1,p+r+1},\ldots,\\ &\qquad \mu^{(p+r-\xi),r-1-\xi} c_{p+1}^{p+1,r-1-\xi}\Delta^{r-1-\xi}_{p+1,p+r-\xi},\ldots,\\ &\qquad \mu^{(\overline{\kappa}),\overline{\kappa}-p-1} c_{p+1}^{p+1,\overline{\kappa}-p-1}\Delta^{\overline{\kappa}-p-1}_{p+1,\overline{\kappa}}\big\}\big/c_{p+1}^{p+1,r+1}\\ &= \mu^{(p+r+1),r} c_{p+1}^{p+1,r}\Delta^r_{p+1,p+r+1}\big/c_{p+1}^{p+1,r+1}\\ &= \frac{\Delta^r_{p+1,p+r+1}}{u}. \end{aligned}$$

If $\Delta^r_{p+1,p+r+1} = 0$, then $\partial Z^r_{p+r,(k+1)-(p+r)} \subseteq Z^r_{p-1,k-(p-1)}$ and, therefore, $E^{r+1}_p = u\mathbb{Z}[\sigma_k^{(p+1),r}]$. □

We will use the following result, which follows from elementary algebra.

LEMMA 5.6. *Suppose that* $\mathfrak{m}$ *represents multiplication by a non-zero integer* m*, and let* $\lambda = v/\gcd\{m, v\}$.

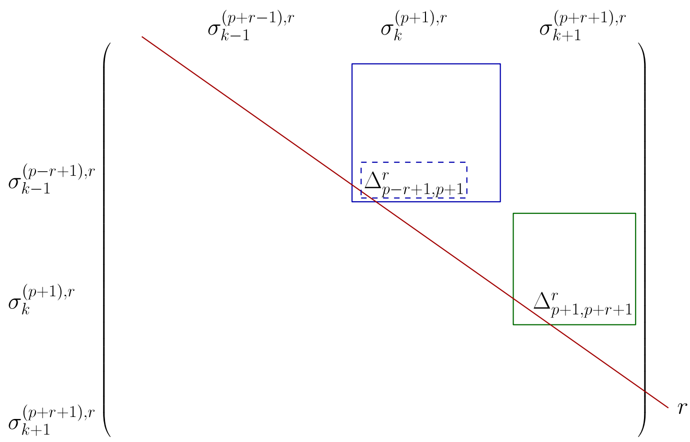

FIGURE 20. Change-of-basis pivot $\Delta^r_{p-r+1,p+1} \neq 0$.

(1) *If* $\mathbb{Z} \xrightarrow{\mathfrak{m}} \mathbb{Z}_v$ *, then*

$$\operatorname{Ker} \mathfrak{m} = \lambda\mathbb{Z} \quad \textit{and} \quad \operatorname{Im} \mathfrak{m} = \frac{\mathbb{Z}}{\lambda\mathbb{Z}} = \frac{\gcd\{m, v\}\mathbb{Z}}{v\mathbb{Z}}.$$

(2) *If* $\mathbb{Z}_t \xrightarrow{\mathfrak{m}} \mathbb{Z}_v$ *and* $t \geq \lambda$*, then*

$$\operatorname{Ker} \mathfrak{m} = \frac{\lambda\mathbb{Z}}{t\mathbb{Z}} \quad \textit{and} \quad \operatorname{Im} \mathfrak{m} = \frac{\mathbb{Z}}{\lambda\mathbb{Z}} = \frac{\gcd\{m, v\}\mathbb{Z}}{v\mathbb{Z}}.$$

THEOREM 5.7. *If E^r_p and E^r_{p-r} are both non-zero, then the map $d^r_p : E^r_p \to E^r_{p-r}$ is induced by δ^r_p, i.e. multiplication by the entry $\Delta^r_{p-r+1,p+1}$, whenever this entry is either a primary pivot, a change-of-basis pivot or a zero with a column of zero entries below it.*

Proof. Suppose that E^r_p and E^r_{p-r} are both non-zero. By definition,

$$E^{r+1}_p = \frac{\operatorname{Ker} d^r_p}{\operatorname{Im} d^r_{p+r}}.$$

We must show in each of the following cases that

$$\frac{\operatorname{Ker} \delta^r_p}{\operatorname{Im} \delta^r_{p+r}} = E^{r+1}_p.$$

We need to analyze the cases where both E^r_p and E^r_{p-r} are non-zero, because otherwise d^r_p would be zero. By Theorem 4.4 this we will lead us to consider three main cases for the entry $\Delta^r_{p-r+1,p+1}$, namely when it is a primary pivot, a change-of-basis pivot or a zero with a column of zeros below it.

(1) $\Delta^r_{p-r+1,p+1}$ is a primary pivot. In this case we know from Theorem 4.4 that $E^r_p = \mathbb{Z}[\sigma_k^{(p+1),r}]$. Moreover, $E^r_{p-r} = \mathbb{Z}[\sigma_{k-1}^{(p-r+1),r}]$. In fact, E^r_{p-r} cannot be $\mathbb{Z}_t[\sigma_{k-1}^{(p-r+1),r}]$, because this would imply the existence of a primary pivot in the $(p-r+1)$st row on a diagonal below the rth auxiliary diagonal.

We have the sequence

$$\cdots \longleftarrow \mathbb{Z}[\sigma_{k-1}^{(p-r+1),r}] \xleftarrow{\delta_p^r} \mathbb{Z}[\sigma_k^{(p+1),r}] \xleftarrow{\delta_{p+r}^r} E_{p+r}^r \longleftarrow \cdots . \tag{11}$$

(a) Suppose $E_{p+r}^r = 0$.
Since $\delta_p^r : \mathbb{Z}[\sigma_k^{(p+1),r}] \to \mathbb{Z}[\sigma_{k-1}^{(p-r+1),r}]$ is multiplication by $\Delta_{p-r+1,p+1}^r \neq 0$, we have Ker $\delta_p^r = 0$. Hence Ker δ_p^r/Im $\delta_{p+r}^r = 0$.

(b) Suppose $E_{p+r}^r \neq 0$. As in the previous case, $\delta_p^r : \mathbb{Z}[\sigma_k^{(p+1),r}] \to \mathbb{Z}[\sigma_{k-1}^{(p-r+1),r}]$ is multiplication by $\Delta_{p-r+1,p+1}^r \neq 0$, hence Ker $\delta_p^r = 0$.
Since $E_{p+r}^r \neq 0$, let us consider the three possibilities for $\Delta_{p+1,p+r+1}^r$: it is either a primary pivot, a change-of-basis pivot or a zero entry with a column of zero entries below it. However, since $\Delta_{p-r+1,p+1}^r$ is a primary pivot, by Proposition 3.2 there can be no primary pivot in the $(p+1)$st row. Hence $\Delta_{p+1,p+r+1}^r$ cannot be a primary pivot or a change-of-basis pivot. Thus, $\Delta_{p+1,p+r+1}^r$ is a zero. It follows that Ker δ_p^r/Im $\delta_{p+r}^r = 0$.

On the other hand, for both of the above cases, because $\Delta_{p-r+1,p+1}^r$ is a primary pivot we have $\sigma_k^{(p+1),r+1} = \sigma_k^{(p+1),r}$. Note that its boundary in the $(p-r+1)$st row is $\Delta_{p-r+1,p+1}^r \neq 0$; hence it does not lie above the $(p-r)$th row. It follows that $\sigma_k^{(p+1),r+1} \notin Z_p^{r+1}$ and thus $Z_p^{r+1} = Z_{p-1}^r$ and $E_p^{r+1} = 0$.

(2) $\Delta_{p-r+1,p+1}^r$ is a change-of-basis pivot. See Figure 20. Then there exists a primary pivot in the $(p-r+1)$st row on a diagonal below the rth auxiliary diagonal. It follows from Theorem 4.4 case (2)(b) that $E_{p-r}^r = \mathbb{Z}_v[\sigma_{k-1}^{(p-r+1),r}]$, where

$$v = \gcd\{\mu^{(p),r-1} c_{p-r+1}^{p-r+1,r-1} \Delta_{p-r+1,p}^{r-1}, \ldots, \mu^{(\kappa),\kappa-p+r-1} \times c_{p-r+1}^{p-r+1,\kappa-p+r-1} \Delta_{p-r+1,\kappa}^{\kappa-p+r-1}\}/c_{p-r+1}^{p-r+1,r}.$$

Let

$$\lambda = \frac{v}{\gcd\{\Delta_{p-r+1,p+1}^r, v\}}.$$

By Proposition 5.2, we have $\lambda = u$.

(a) If $\Delta_{p+1,p+r+1}^r \neq 0$ is a primary pivot, it follows from Proposition 3.2 that there is no primary pivot either in the $(p+1)$st row and column or in the $(p+r+1)$st row and column on a diagonal below the rth auxiliary diagonal. Hence, by Theorem 4.4 cases (2)(a) and (1), $E_p^r = \mathbb{Z}[\sigma_k^{(p+1),r}]$ and $E_{p+r}^r = \mathbb{Z}[\sigma_{k+1}^{(p+r),r}]$. In this case, we have the sequence

$$\cdots \longleftarrow \mathbb{Z}_v[\sigma_{k-1}^{(p-r+1),r}] \xleftarrow{\delta_p^r} \mathbb{Z}[\sigma_k^{(p+1),r}] \xleftarrow{\delta_{p+r}^r} \mathbb{Z}[\sigma_{k+1}^{(p+r),r}] \longleftarrow \cdots . \tag{12}$$

Then

$$\text{Im } \delta_{p+r}^r = \Delta_{p+1,p+r+1}^r \mathbb{Z}[\sigma_k^{(p+1),r}]$$

and, by Lemma 5.6,

$$\text{Ker } \delta_p^r = \lambda \mathbb{Z}[\sigma_k^{(p+1),r}].$$

Hence

$$\frac{\text{Ker } \delta^r_p}{\text{Im } \delta^r_{p+r}} = \frac{\lambda\mathbb{Z}[\sigma_k^{(p+1),r}]}{\Delta^r_{p+1,\,p+r+1}\mathbb{Z}[\sigma_k^{(p+1),r}]} = \frac{u\mathbb{Z}[\sigma_k^{(p+1),r}]}{\Delta^r_{p+1,\,p+r+1}\mathbb{Z}[\sigma_k^{(p+1),r}]}.$$

On the other hand, since $\Delta^r_{p+1,\,p+r+1}$ is a primary pivot, it follows from Lemma 5.5 that

$$E_p^{r+1} = \frac{u\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{\Delta^r_{p+1,\,p+r+1}\mathbb{Z}[\sigma_k^{(p+1),r+1}]}.$$

(b) If $\Delta^r_{p+1,\,p+r+1} = 0$ with a column of zero entries below it, then $\text{Im } \delta^r_{p+r} = 0$. Hence

$$\frac{\text{Ker } \delta^r_p}{\text{Im } \delta^r_{p+r}} = \text{Ker } \delta^r_p.$$

(i) $E^r_p = \mathbb{Z}[\sigma_k^{(p+1),r}]$. In this case Lemma 5.6 gives

$$\text{Ker } \delta^r_p = \lambda\mathbb{Z}[\sigma_k^{(p+1),r}] = u\mathbb{Z}[\sigma_k^{(p+1),r}].$$

On the other hand, it follows from Lemma 5.5 that

$$E_p^{r+1} = u\mathbb{Z}[\sigma_k^{(p+1),r+1}].$$

(ii) $E^r_p = \mathbb{Z}_t[\sigma_k^{(p+1),r}]$. We have from Lemma 5.6 that

$$\text{Ker } \delta^r_p = \frac{\lambda\mathbb{Z}[\sigma_k^{(p+1),r}]}{t\mathbb{Z}[\sigma_k^{(p+1),r}]} = \frac{u\mathbb{Z}[\sigma_k^{(p+1),r}]}{t\mathbb{Z}[\sigma_k^{(p+1),r}]}.$$

On the other hand, it follows from Lemma 5.3 that

$$E_p^{r+1} = u\mathbb{Z}_t[\sigma_k^{(p+1),r+1}].$$

(c) If $\Delta^r_{p+1,\,p+r+1} \neq 0$ is a change-of-basis pivot, then there exists a primary pivot in the $(p+1)$st row on a diagonal below the rth auxiliary diagonal. It follows from Theorem 4.4 case (2)(b) that $E^r_{p,k-p} = \mathbb{Z}_t[\sigma_k^{(p+1),r}]$ where

$$t = \gcd\big\{\mu^{(r+p),r-1} c^{p+1,r-1}_{p+1} \Delta^{r-1}_{p+1,r+p}, \ldots, \\ \mu^{(\bar\kappa),\bar\kappa-p-1} c^{p+1,\bar\kappa-p-1}_{p+1} \Delta^{\bar\kappa-p-1}_{p+1,\bar\kappa}\big\}/c^{p+1,r}_{p+1}.$$

Let $\bar\lambda = t/\gcd\{\Delta^r_{p+1,\,p+r+1}, t\}$. We have the sequence

$$\cdots \longleftarrow \mathbb{Z}_v[\sigma_{k-1}^{(p-r+1),r}] \xleftarrow{\delta^r_p} \mathbb{Z}_t[\sigma_k^{(p+1),r}] \xleftarrow{\delta^r_{p+r}} E^r_{p+r} \longleftarrow \cdots. \tag{13}$$

Either $E^r_{p+r} = \mathbb{Z}[\sigma_k^{(p+r),r}]$ or $E^r_{p+r} = \mathbb{Z}_w[\sigma_k^{(p+r),r}]$. However, we know from Remark 5.4 and Proposition 5.2 that

$$\lambda = u \leq t \quad \text{and} \quad \bar\lambda = c^{p+r}_{p+r,r+1}/c^{p+r}_{p+r,r+1} \leq w.$$

It follows from Lemma 5.6 that

$$\mathrm{Ker}\ \delta_p^r = \frac{\lambda\mathbb{Z}[\sigma_k^{(p+1),r}]}{t\mathbb{Z}[\sigma_k^{(p+1),r}]}$$

and

$$\mathrm{Im}\ \delta_{p+r}^r = \frac{\mathbb{Z}[\sigma_k^{(p+1),r}]}{\bar{\lambda}\mathbb{Z}[\sigma_k^{(p+1),r}]} = \frac{\gcd\{\Delta_{p+1,p+r+1}^r, t\}\mathbb{Z}[\sigma_k^{(p+1),r}]}{t\mathbb{Z}[\sigma_k^{(p+1),r}]}.$$

Then

$$\frac{\mathrm{Ker}\ \delta_p^r}{\mathrm{Im}\ \delta_{p+r}^r} = \frac{\lambda\mathbb{Z}[\sigma_k^{(p+1),r}]}{\gcd\{\Delta_{p+1,p+r+1}^r, t\}\mathbb{Z}[\sigma_k^{(p+1),r}]}.$$

On the other hand, since $\Delta_{p+1,p+r+1}^r$ is a change-of-basis pivot, by Lemma 5.3 we have that

$$E_p^{r+1} = \frac{u\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{\gcd\{\Delta_{p+1,p+r+1}^r, t\}\mathbb{Z}[\sigma_k^{(p+1),r+1}]},$$

where $u = \lambda$ by Proposition 5.2.

(d) If $\Delta_{p+1,p+r+1}^r$ is an entry above a primary pivot, then there exists a primary pivot in the $(p+r+1)$st column below $\Delta_{p+1,p+r+1}^r$. Hence $\mu^{(p+r+1),r} = 0$ and $\sigma_{k+1}^{(p+r+1),r} \notin Z_{p+r}^r$. It follows that $E_{p+r}^r = 0$ and hence $\mathrm{Im}\ \delta_{p+r}^r = 0$. Then

$$\frac{\mathrm{Ker}\ \delta_p^r}{\mathrm{Im}\ \delta_{p+r}^r} = \mathrm{Ker}\ \delta_p^r.$$

(i) If $E_{p,k-p}^r = \mathbb{Z}[\sigma_k^{(p+1),r}]$, we have the sequence

$$\cdots \longleftarrow \mathbb{Z}_v[\sigma_{k-1}^{(p-r+1),r}] \xleftarrow{\delta_p^r} \mathbb{Z}[\sigma_k^{(p+1),r}] \xleftarrow{\delta_{p+r}^r} 0 \longleftarrow \cdots \tag{14}$$

and, by Lemma 5.6,

$$\mathrm{Ker}\ \delta_p^r = \lambda\mathbb{Z}[\sigma_k^{(p+1),r}] = u\mathbb{Z}[\sigma_k^{(p+1),r}].$$

(ii) If $E_{p,k-p}^r = \mathbb{Z}_t[\sigma_k^{(p+1),r}]$, we have the sequence

$$\cdots \longleftarrow \mathbb{Z}_v[\sigma_{k-1}^{(p-r+1),r}] \xleftarrow{\delta_p^r} \mathbb{Z}_t[\sigma_k^{(p+1),r}] \xleftarrow{\delta_{p+r}^r} 0 \longleftarrow \cdots \tag{15}$$

and, by Lemma 5.6,

$$\mathrm{Ker}\ \delta_p^r = \frac{\lambda\mathbb{Z}[\sigma_k^{(p+1),r}]}{t\mathbb{Z}[\sigma_k^{(p+1),r}]} = u\mathbb{Z}_t[\sigma_k^{(p+1),r}].$$

On the other hand, we know from Lemma 4.3 that

$$E_{p,k-p}^{r+1} = \frac{\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{s\mathbb{Z}[\sigma_k^{(p+1),r+1}]},$$

where

$$s = \gcd\{\mu^{(p+r+1),r} c_{p+1}^{p+1,r} \Delta^r_{p+1,p+r+1}, \mu^{(r+p),r-1} c_{p+1}^{p+1,r-1} \Delta^{r-1}_{p+1,r+p}, \dots, \mu^{(\overline{\kappa}),\overline{\kappa}-p-1} c_{p+1}^{p+1,\overline{\kappa}-p-1} \Delta^{\overline{\kappa}-p-1}_{p+1,\overline{\kappa}}\}/c_{p+1}^{p+1,r+1}$$

$$= \gcd\left\{\frac{\mu^{(p+r+1),r} c_{p+1}^{p+1,r} \Delta^r_{p+1,p+r+1}}{c_{p+1}^{p+1,r}}, \frac{\gcd\{\mu^{(r+p),r-1} c_{p+1}^{p+1,r-1} \Delta^{r-1}_{p+1,r+p}, \dots, \mu^{(\overline{\kappa}),\overline{\kappa}-p-1} c_{p+1}^{p+1,\overline{k}-p-1} \Delta^{\overline{\kappa}-p-1}_{p+1,\overline{\kappa}}\}}{c_{p+1}^{p+1,r}}\right\} \times c_{p+1}^{p+1,r}/c_{p+1}^{p+1,r+1}.$$

Since $\mu^{(p+r+1),r} = 0$, we have $s = t/u$. When $E^r_{p,k-p} = \mathbb{Z}_t[\sigma_k^{(p+1),r}]$ as in (ii),

$$E^{r+1}_{p,k-p} = \frac{u\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{t\mathbb{Z}[\sigma_k^{(p+1),r+1}]}.$$

When $E^r_{p,k-p} = \mathbb{Z}[\sigma_k^{(p+1),r}]$ as in (i), we take $t = 0$ and obtain

$$E^{r+1}_p = u\mathbb{Z}[\sigma_k^{(p+1),r+1}].$$

(3) $\Delta^r_{p-r+1,p+1} = 0$ with a column of zeros below it. In this case, $\operatorname{Ker}\delta^r_p = E^r_p$. Moreover, $\sigma_k^{(p+1),r} = \sigma_k^{(p+1),r+1}$ and hence $u = 1$.

(a) If $\Delta^r_{p+1,p+r+1}$ is an entry above a primary pivot, then, as in (2)(d), we have $\mu^{(p+r+1),r} = 0$ and $E^r_{p+r} = 0$. Hence $\operatorname{Im}\delta^r_{p+r} = 0$ and thus

$$\frac{\operatorname{Ker}\delta^r_p}{\operatorname{Im}\delta^r_{p+r}} = E^r_p.$$

On the other hand, since $\mu^{(p+r+1),r} = 0$, we have $E^{r+1}_p = E^r_p$.

(b) If $\Delta^r_{p+1,p+r+1} = 0$ with a column of zero entries below it, then $\operatorname{Im}\delta^r_{p+r} = 0$ and

$$\frac{\operatorname{Ker}\delta^r_p}{\operatorname{Im}\delta^r_{p+r}} = E^r_p.$$

On the other hand, it follows from Lemmas 5.3 and 5.5 that $E^{r+1}_p = E^r_p$.

(c) If $\Delta^r_{p+1,p+r+1} \neq 0$ is a primary pivot, then there is neither a primary pivot in the $(p+1)$st row nor a primary pivot in the $(p+r+1)$st column on a diagonal below the rth auxiliary diagonal. Hence $E^r_p = \mathbb{Z}[\sigma_k^{(p+1),r}]$ and $E^r_{p+r} = \mathbb{Z}[\sigma_k^{(p+r+1),r}]$. We have

$$\cdots \longleftarrow E^r_{p-r} \xleftarrow{\delta^r_p} \mathbb{Z}[\sigma_k^{(p+1),r}] \xleftarrow{\delta^r_{p+r}} \mathbb{Z}[\sigma_{k+1}^{(p+r+1),r}] \longleftarrow \cdots \tag{16}$$

and therefore

$$\frac{\operatorname{Ker}\delta^r_p}{\operatorname{Im}\delta^r_{p+r}} = \frac{\mathbb{Z}[\sigma_k^{(p+1),r}]}{\Delta^r_{p+1,p+r+1}\mathbb{Z}[\sigma_k^{(p+1),r}]}.$$

On the other hand, since $\Delta^r_{p+1,p+r+1}$ is a primary pivot, Lemma 5.5 gives

$$E^{r+1}_{p,k-p} = \frac{\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{\Delta^r_{p+1,p+r+1}\mathbb{Z}[\sigma_k^{(p+1),r+1}]}.$$

(d) If $\Delta^r_{p+1,p+r+1}$ is a change-of-basis pivot, then there is a primary pivot in the $(p+1)$st row on a diagonal below the rth auxiliary diagonal. Hence $E^r_p = \mathbb{Z}_t[\sigma_k^{(p+1),r}]$. We have

$$\cdots \longleftarrow E^r_{p-r} \xleftarrow{\delta^r_p} \mathbb{Z}_t[\sigma_k^{(p+1),r}] \xleftarrow{\delta^r_{p+r}} E^r_{p+r} \longleftarrow \cdots \tag{17}$$

and E^r_{p+r} can be either $\mathbb{Z}[\sigma_{k+1}^{(p+r+1),r}]$ or $\mathbb{Z}_w[\sigma_{k+1}^{(p+r+1),r}]$. Let

$$\overline{\lambda} = \frac{t}{\gcd\{\Delta^r_{p+1,p+r+1}, t\}} \quad \text{and} \quad \widetilde{u} = \frac{c^{p+r+1,r}_{p+r+1}}{c^{p+r+1,r+1}_{p+r+1}}.$$

Since $\Delta^r_{p+1,p+r+1}$ is a change-of-basis pivot, by Proposition 5.2 and Remark 5.4 for $(p+r)$ we have

$$\overline{\lambda} = \widetilde{u} \le \gcd\{\Delta^r_{p+1,p+r+1}, w\} \le w.$$

By Lemma 5.6,

$$\operatorname{Im}\delta^r_{p+r} = \frac{\gcd\{\Delta^r_{p+1,p+r+1}, t\}\mathbb{Z}[\sigma_k^{(p+1),r}]}{t\mathbb{Z}[\sigma_k^{(p+1),r}]}.$$

Then

$$\frac{\operatorname{Ker}\delta^r_p}{\operatorname{Im}\delta^r_{p+r}} = \frac{\mathbb{Z}[\sigma_k^{(p+1),r}]}{\gcd\{\Delta^r_{p+1,p+r+1}, t\}\mathbb{Z}[\sigma_k^{(p+1),r}]}.$$

On the other hand, since $\Delta^r_{p+1,p-r+1}$ is a zero entry with only zero entries below it, we have by Lemma 5.3 that

$$E^{r+1}_{p,k-p} = \frac{\mathbb{Z}[\sigma_k^{(p+1),r+1}]}{\gcd\{\Delta^r_{p+1,p+r+1}, t\}\mathbb{Z}[\sigma_k^{(p+1),r+1}]}.$$

Thus we have seen that, in all cases,

$$\frac{\operatorname{Ker} d^r_p}{\operatorname{Im} d^r_{p+r}} = E^{r+1}_{p,k-p} = \frac{\operatorname{Ker}\delta^r_p}{\operatorname{Im}\delta^r_{p+r}}. \qquad \square$$

6. *Spectral sequence analysis for the existence of connecting orbits*

In the next theorem, we analyze the non-zero differentials d^r in a spectral sequence associated to a Morse flow φ. We show that, although we may not always have a connecting orbit in the flow φ associated to d^r, there is always a path formed by connecting orbits of φ which is determined by d^r.

THEOREM 6.1. *Let (E^r, d^r) be a spectral sequence induced by a Morse Conley chain complex $(C\Delta, \Delta)$ of a flow φ, where Δ is a connection matrix over $\mathbb{Z}$. Given a non-zero $d^r : E^r_{p,q} \to E^r_{p-r,q+r-1}$, there exists a path of connecting orbits of φ joining the singularity $h_k^{(p+1)}$ which generates $E^1_{p,q}$ to the singularity $h_{k-1}^{(p-r+1)}$ which generates $E^1_{p-r,q+r-1}$.*

We adopt a loose definition of a path in a flow.

Definition 6.2. A *path associated to d^r* is a juxtaposition of connecting orbits where the orbits that are represented in the matrices by primary pivots or change-of-basis pivots $\Delta^{\xi}_{i,j}$ for $\xi < r$ may be considered as having reverse orientation.

More precisely, let $\gamma_{i,j}$ be a path between the singularities $h_k^{(j)}$ and $h_{k-1}^{(i)}$. If $\gamma_{i,j}$ corresponds to a connecting orbit in the flow, we will say that $\gamma_{i,j}$ is an elementary path and define the length of $\gamma_{i,j}$ as $\ell(\gamma_{i,j}) = (j - i)$. However, when $\gamma_{i,j}$ does not correspond to a connecting orbit in the flow, $\gamma_{i,j}$ can be written as a sequence of elementary paths. The construction of this sequence is done recursively by defining

$$\gamma_{i,j} = [\gamma_{\bar{i},j}, -\gamma_{\bar{i},\bar{j}}, \gamma_{i,\bar{j}}]$$

where $\bar{j} < j$ and $\bar{i} > i$, i.e. $h_k^{(\bar{j})}$ is associated to a column of Δ to the left of $h_k^{(j)}$ and $h_{k-1}^{(\bar{i})}$ is associated to a row of Δ below $h_{k-1}^{(i)}$.

The negative sign indicates that $\gamma_{\bar{i},\bar{j}}$ is considered with reverse orientation. If $\gamma_{\bar{i},\bar{j}}$ is an elementary path, the corresponding connecting orbit is considered to be in reverse orientation. If $\gamma_{\bar{i},\bar{j}}$ does not correspond to a connecting orbit, then it is a path

$$\gamma_{\bar{i},\bar{j}} = [\gamma_{\bar{\bar{i}},\bar{j}}, -\gamma_{\bar{\bar{i}},\bar{\bar{j}}}, \gamma_{\bar{i},\bar{\bar{j}}}]$$

where $\bar{\bar{j}} < \bar{j}$ and $\bar{\bar{i}} > \bar{i}$, and we define

$$-\gamma_{\bar{i},\bar{j}} = -[\gamma_{\bar{\bar{i}},\bar{j}}, -\gamma_{\bar{\bar{i}},\bar{\bar{j}}}, \gamma_{\bar{i},\bar{\bar{j}}}] = [-\gamma_{\bar{i},\bar{\bar{j}}}, \gamma_{\bar{\bar{i}},\bar{\bar{j}}}, -\gamma_{\bar{\bar{i}},\bar{j}}].$$

The *length* of $\gamma_{i,j} = [\gamma_{\bar{i},j}, -\gamma_{\bar{i},\bar{j}}, \gamma_{i,\bar{j}}]$ is defined as

$$\ell(\gamma_{i,j}) = \ell(\gamma_{\bar{i},j}) + \ell(\gamma_{\bar{i},\bar{j}}) + \ell(\gamma_{i,\bar{j}}).$$

In the next lemma we prove that certain columns need not be considered when changing basis in the sweeping method.

LEMMA 6.3. *Let $\Delta^r_{p-r+1,p+1}$ be a change-of-basis pivot. In the sweeping method, the choice of columns associated to $\sigma_k^{(p+1-\xi),r-\xi}$ that will zero out $\Delta^r_{p-r+1,p+1}$ in Δ^{r+1} need not take into consideration columns which have non-zero entries above the $(p-r)$th row.*

Proof. We show that if there is a linear combination in the sweeping method that uses columns with non-zero entries above the $(p-r)$th row, then there exists another linear combination that does not use these columns.

We know that

$$E^{r+1}_{p,k-p} = \frac{Z^{r+1}_{p,k-p}}{Z^r_{p-1,k-(p-1)} + \partial Z^r_{p+r,(k+1)-(p+r)}},$$

where

$$Z^{r+1}_{p,k-p} = \mathbb{Z}[\mu^{(p+1),r+1}\sigma_k^{(p+1),r+1}, \mu^{(p),r}\sigma_k^{(p),r}, \dots, \mu^{(\kappa),r-p+\kappa}\sigma_k^{(\kappa),r-p+\kappa}],$$
$$Z^{r}_{p-1,k-p+1} = \mathbb{Z}[\mu^{(p),r}\sigma_k^{(p),r}, \mu^{(p-1),r-1}\sigma_k^{(p-1),r-1}, \dots, \mu^{(\kappa),r-p+\kappa}\sigma_k^{(\kappa),r-p+\kappa}].$$

Moreover, from Proposition 5.1 we have that

$$\sigma_k^{(p+1),r+1} = u\mu^{(p+1),r}\sigma_k^{(p+1),r} + b_p\mu^{(p),r-1}\sigma_k^{(p),r-1} + \cdots + b_\kappa\mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}.$$

Suppose that for some $\xi \in \{1, 2, \dots, p+1-\kappa\}$, $\sigma_k^{(p+1-\xi),r-\xi}$ is such that $\partial\sigma_k^{(p+1-\xi),r-\xi}$ is zero in the $(p-r+1)$th row and $\mu^{(p+1-\xi),r-\xi} = 1$, that is,

$$\Delta^{r-\xi}_{p-r+1,p+1-\xi} = 0 \quad \text{and} \quad \Delta^{r-\xi}_{s,p+1-\xi} = 0 \quad \text{for all } s > p+r-1.$$

In this case, $\partial\sigma_k^{(p+1-\xi),r-\xi}$ is above the $(p-r)$th row and hence

$$\sigma_k^{(p+1-\xi),r-\xi} = \sigma_k^{(p+1-\xi),r-\xi+1} \in Z^r_{p-1,k-(p-1)}.$$

By the formula we have that

$$\begin{aligned}
&E^{r+1}_{p,k-p}\\
&= \frac{\mathbb{Z}[\mu^{(p+1),r+1}\sigma_k^{(p+1),r+1}, \dots, \sigma_k^{(p+1-\xi),r+1-\xi}, \dots, \mu^{(\kappa),r-p+\kappa}\sigma_k^{(\kappa),r-p+\kappa}]}{\mathbb{Z}[\mu^{(p),r}\sigma_k^{(p),r}, \dots, \sigma_k^{(p+1-\xi),r+1-\xi}, \dots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}] + \partial Z^r_{p+r,(k+1)-(p+r)}}\\
&= \frac{\mathbb{Z}[\mu^{(p+1),r+1}\sigma_k^{(p+1),r+1} - \sigma_k^{(p+1-\xi),r+1-\xi}, \dots, \sigma_k^{(p+1-\xi),r+1-\xi}, \dots, \mu^{(\kappa),r-p+\kappa}\sigma_k^{(\kappa),r-p+\kappa}]}{\mathbb{Z}[\mu^{(p),r}\sigma_k^{(p),r}, \dots, \sigma_k^{(p+1-\xi),r+1-\xi}, \dots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}] + \partial Z^r_{p+r,(k+1)-(p+r)}}\\
&= \frac{\mathbb{Z}[\mu^{(p+1),r+1}\sigma_k^{(p+1),r+1} - \sigma_k^{(p+1-\xi),r-\xi}, \dots, \mu^{(\kappa),r-p+\kappa}\sigma_k^{(\kappa),r-p+\kappa}]}{\mathbb{Z}[\mu^{(p),r}\sigma_k^{(p),r}, \dots, \mu^{(\kappa),r-p-1+\kappa}\sigma_k^{(\kappa),r-p-1+\kappa}] + \partial Z^r_{p+r,(k+1)-(p+r)}}.
\end{aligned}$$

The last equality above holds because the generator $\sigma_k^{(p+1-\xi),r-\xi+1}$ can be replaced by the generator $\sigma_k^{(p+1-\xi),r-\xi}$.

Consequently, there is no loss of generality in choosing a change of basis that does not sum the columns which have a zero entry in the $(p-r+1)$st row and zeros below it. □

Let $\Delta^0 = \Delta$. We have shown that the sweeping method produces a sequence of matrices Δ^r in which the matrix Δ^{r+1} is obtained from a change of basis of Δ^r; in other words, there exists a sequence of change-of-basis matrices $M_0, \dots, M_{m-1}$ such that

$$\Delta^{r+1} = M_r^{-1}\Delta^r M_r = M_r^{-1}M_{r-1}^{-1}\cdots M_0^{-1}\Delta M_0\cdots M_{r-1}M_r$$

for $r = 0, \dots, m-1$.

For each $r \in \{0, \dots, m-1\}$, we define $\overline{\Delta^r}$ to be the matrix $\Delta M_0 \dots M_{r-1}M_r$. Hence, if κ^* is the first h_{k-1} column and $\widetilde{\kappa}$ is the last h_{k-1} column, then we can write

$$\partial\sigma^{(j),r} = \overline{\Delta}^r_{\widetilde{\kappa},j}h^{(\widetilde{\kappa})}_{k-1} + \cdots + \overline{\Delta}^r_{\kappa^*,j}h^{(\kappa^*)}_{k-1}$$

where $\overline{\Delta}^r_{s,j} \in \mathbb{Z}$ for $s = \kappa^*, \dots, \widetilde{\kappa}$.

PROPOSITION 6.4. $\overline{\Delta}^r_{s,j} = 0$ *for all* $s > i$ *if and only if* $\Delta^r_{s,j} = 0$ *for all* $s > i$.

Proof. We know that

$$\partial\sigma^{(j),r} = \overline{\Delta}^r_{\widetilde{\kappa},j} h^{(\widetilde{\kappa})}_{k-1} + \cdots + \overline{\Delta}^r_{\kappa^*,j} h^{(\kappa^*)}_{k-1}$$

and

$$\partial\sigma^{(j),r} = \Delta^r_{\widetilde{\kappa},j} \sigma^{(\widetilde{\kappa}),r}_{k-1} + \cdots + \Delta^r_{\kappa^*,j} \sigma^{(\kappa^*),r}_{k-1}.$$

Suppose that $\overline{\Delta}^r_{s,j} = 0$ for all $s > i$, that is,

$$\partial\sigma^{(j),r} = \overline{\Delta}^r_{i,j} h^{(i)}_{k-1} + \cdots + \overline{\Delta}^r_{\kappa^*,j} h^{(\kappa^*)}_{k-1}.$$

Since the coefficient of $h^{(s)}_{k-1}$ is always non-zero in $\sigma^{(s),r}_{k-1}$, we have $\Delta^r_{s,j} = 0$ for all $s > i$, that is,

$$\partial\sigma^{(j),r} = \Delta^r_{i,j} \sigma^{(i),r}_{k-1} + \cdots + \Delta^r_{\kappa^*,j} \sigma^{(\kappa^*),r}_{k-1}.$$

The proof of the converse is completely analogous. □

As a direct consequence of Proposition 6.4, we have that $\Delta^r_{p-r,p}$ is a pivot if and only if $\overline{\Delta}^r_{p-r,p} \neq 0$ and $\overline{\Delta}^r_{s,p} = 0$ for all $s > p - r$.

It is clear that the square of $\overline{\Delta}^r$ is not necessarily equal to zero; however, it will be used as an auxiliary matrix to prove the main result in §6.

The proof of Theorem 6.1 is a direct consequence of the following lemma.

LEMMA 6.5. *Let* Δ *be a connection matrix. Applying the sweeping method to* Δ, *let* Δ^r *be the matrix obtained after the* rth *diagonal has been swept. If* $\overline{\Delta}^r_{j-\xi,j} \neq 0$ *for some* ξ, *then there is a path* $\gamma_{j-\xi,j} = [\gamma_{j-\bar{r},j}, -\gamma_{j-\bar{r},j-\zeta}, \gamma_{j-\xi,j-\zeta}]$, *for some* $\bar{r}$ *and* ζ *less than* r, *in the flow* φ *formed by connecting orbits joining the singularity* $h^{(j)}_k$ *to the singularity* $h^{(j-\xi)}_{k-1}$.

Proof. We will prove this result by induction on r and ξ.

(1) Consider the $r = 1$ case. Since $\sigma^{(j),1}_k = h^{(j)}_k$, we have $\overline{\Delta}^1_{s,j} = \Delta_{s,j}$ for $s = \kappa^*, \ldots, \widetilde{\kappa}$, where κ^* and $\widetilde{\kappa}$ are the first and last columns associated to a $(k-1)$-chain. Hence non-zero entries $\overline{\Delta}^1_{j-\xi,j}$ for all ξ represent the existence of connecting orbits between $h^{(j)}_k$ and $h^{(j-\xi)}_{k-1}$. For each ξ, we have a path in the flow φ which is a connecting orbit.

(2) Let ξ be the first auxiliary diagonal which intersects Δ_k such that $\overline{\Delta}^r_{j-\xi,j} \neq 0$. Then, for all r, $\overline{\Delta}^r_{s,j} = 0$ for all $s < j - \xi$ and $\overline{\Delta}^r_{j-\xi,\ell} = 0$ for $\ell < j$. Since $\overline{\Delta}^r_{j-\xi,\ell} = 0$ for all $\ell < j$, the jth column has not altered via a change of basis, hence $\overline{\Delta}^r_{j-\xi,j} = \overline{\Delta}^1_{j-\xi,j}$. Since $\overline{\Delta}^r_{s,j} = 0$ for all $s < j - \xi$, we have $\overline{\Delta}^r_{j-\xi,j} = c^{j-\xi,r}_{j-\xi} \Delta^r_{j-\xi,j}$ for r. Therefore $\Delta_{j-\xi,j} \neq 0$ and hence there is a connecting orbit in the flow φ.

(3) Suppose that the lemma holds for all $r' < r$ and $\xi' < \xi$, and let $\overline{\Delta}^r_{j-\xi,j} \neq 0$. If there is a connecting orbit between $h^{(j)}_k$ and $h^{(j-\xi)}_{k-1}$, then nothing needs to be shown. In particular, this would be the case when $\overline{\Delta}^1_{j-\xi,j} \neq 0$, since $\overline{\Delta}^1_{j-\xi,j} = \Delta_{j-\xi,j}$ and in this situation there is a connecting orbit between $h^{(j)}_k$ and $h^{(j-\xi)}_{k-1}$. Therefore, let us

suppose that $\overline{\Delta}^{1}_{j-\xi,j} = 0$ and that there are no connecting orbits between $h_k^{(j)}$ and $h_{k-1}^{(j-\xi)}$. We will show that if $\overline{\Delta}^{r}_{j-\xi,j} \neq 0$, then there is a 'path' of connecting orbits that joins $h_k^{(j)}$ and $h_{k-1}^{(j-\xi)}$.

Since $\overline{\Delta}^{r}_{j-\xi,j} \neq 0$ and $\overline{\Delta}^{1}_{j-\xi,j} = 0$, there exists an $\overline{r}$ with $\overline{r} < r$ and $\overline{r} < \xi$ such that $\overline{\Delta}^{\overline{r}}_{j-\xi,j} = 0$ and $\overline{\Delta}^{\overline{r}+1}_{j-\xi,j} \neq 0$, i.e. $\sigma_k^{(j),\overline{r}} \neq \sigma_k^{(j),\overline{r}+1}$.

The sweeping method asserts that a change of basis will only be prompted in the jth column of a matrix when a change-of-basis pivot is present in that column, which in this case will happen precisely when the sweeping method is going through the $\overline{r}$th auxiliary diagonal of $\Delta^{\overline{r}}$.

Hence there exists a change-of-basis pivot in the jth column on the $\overline{r}$th auxiliary diagonal of $\Delta^{\overline{r}}$. This change-of-basis pivot is $\Delta^{\overline{r}}_{j-\overline{r},j}$, and it is on the $(j-\overline{r})$th row of $\Delta^{\overline{r}}$. By Proposition 6.4, $\overline{\Delta}^{\overline{r}}_{j-\overline{r},j} \neq 0$ and $\Delta^{\overline{r}}_{j-\overline{r},j}$ has a column of zeros below it, that is,

$$\overline{\Delta}^{\overline{r}}_{j-\overline{r},j} = c^{j-\overline{r},\overline{r}}_{j-\overline{r}} \Delta^{\overline{r}}_{j-\overline{r},j} \neq 0.$$

By Proposition 5.1,

$$\begin{aligned} \partial\sigma_k^{(j),\overline{r}+1} &= u\mu^{(j),\overline{r}}\partial\sigma_k^{(j),\overline{r}} + b_{j-1}\mu^{(j-1),\overline{r}-1}\partial\sigma_k^{(j-1),\overline{r}-1} \\ &\quad + \cdots + b_\kappa\mu^{(\kappa),\overline{r}-j+\kappa}\partial\sigma_k^{(\kappa),\overline{r}-j+\kappa}. \end{aligned} \tag{18}$$

Upon equating the coefficients of $h_{k-1}^{(j-\overline{r})}$ on both sides of equation (18) (i.e. restricting to the $(j-\overline{r})$th row of $\overline{\Delta}$), we obtain

$$\begin{aligned} 0 = \overline{\Delta}^{\overline{r}+1}_{j-\overline{r},j} &= u\mu^{(j),\overline{r}}\overline{\Delta}^{\overline{r}}_{j-\overline{r},j} + b_{j-1}\mu^{(j-1),\overline{r}-1}\overline{\Delta}^{\overline{r}-1}_{j-\overline{r},j-1} \\ &\quad + \cdots + \mu^{(j-\zeta),\overline{r}-\zeta} b_{j-\zeta}\overline{\Delta}^{\overline{r}-\zeta}_{j-\overline{r},j-\zeta} \\ &\quad + \cdots + b_\kappa\mu^{(\kappa),\overline{r}-j+\kappa}\overline{\Delta}^{\overline{r}-j+\kappa}_{j-\overline{r},\kappa}. \end{aligned}$$

We know that if the primary pivot of a $\sigma^{(j-\zeta),\overline{r}-\zeta}$ is below the $(j-\overline{r})$th row, then $\mu^{(j-\zeta),\overline{r}-\zeta} = 0$. Hence, $\mu^{(j-\zeta),\overline{r}-\zeta} = 1$ only when there is either a primary pivot, a change-of-basis pivot or a zero entry on the $(j-\overline{r})$th row of $\Delta^{\overline{r}-\zeta}$ with a column of zeros below it. However, Lemma 6.3 says that we can assume without loss of generality that in a change of basis, columns having a zero entry in the $(j-\overline{r})$th row and zeros below it need not be considered. Hence $\mu^{(j-\zeta),\overline{r}-\zeta} = 1$ and $b_{j-\zeta} \neq 0$ only when $\Delta^{\overline{r}-\zeta}_{j-\overline{r},j-\zeta}$ is a change-of-basis pivot or a primary pivot. By Proposition 6.4, $\overline{\Delta}^{\overline{r}-\zeta}_{j-\overline{r},j-\zeta} \neq 0$ and it has a column of zeros below it, that is,

$$\overline{\Delta}^{\overline{r}-\zeta}_{j-\overline{r},j-\zeta} = c^{j-\overline{r},\overline{r}-\zeta}_{j-\overline{r}} \Delta^{\overline{r}-\zeta}_{j-\overline{r},j-\zeta} \neq 0.$$

Upon equating the coefficients of $h_{k-1}^{(j-\xi)}$ on both sides of equation (18) (i.e. restricting the equation to the $(j-\xi)$th row of $\overline{\Delta}$), we obtain

$$\begin{aligned} \overline{\Delta}^{\overline{r}+1}_{j-\xi,j} &= u\mu^{(j),\overline{r}}\overline{\Delta}^{\overline{r}}_{j-\xi,j} + b_{j-1}\mu^{(j-1),\overline{r}-1}\overline{\Delta}^{\overline{r}-1}_{j-\xi,j-1} + \cdots \\ &\quad + \mu^{(j-\zeta),\overline{r}-\zeta} b_{j-\zeta}\overline{\Delta}^{\overline{r}-\zeta}_{j-\xi,j-\zeta} + \cdots + b_\kappa\mu^{(\kappa),\overline{r}-j+\kappa}\overline{\Delta}^{\overline{r}-j+\kappa}_{j-\xi,\kappa}. \end{aligned}$$

Since $\overline{\Delta}^{\bar{r}+1}_{j-\xi,j} \neq 0$ and $\overline{\Delta}^{\bar{r}}_{j-\xi,j} = 0$, there exists $\zeta \in \{1, j-\kappa\}$ such that $\mu^{(j-\zeta),\bar{r}-\zeta} = 1$, $b_{j-\zeta} \neq 0$ and $\overline{\Delta}^{\bar{r}-\zeta}_{j-\xi,j-\zeta} \neq 0$.

- Since $\overline{\Delta}^{\bar{r}-\zeta}_{j-\xi,j-\zeta} \neq 0$ is such that $\xi - \zeta < \xi$ and $\bar{r} - \zeta < r$, it follows from the induction hypothesis that there is a path $\gamma_{j-\xi,j-\zeta}$ of connecting orbits joining $h_k^{(j-\zeta)}$ to $h_{k-1}^{(j-\xi)}$.
- Since $\overline{\Delta}^{\bar{r}-\zeta}_{j-\bar{r},j-\zeta} \neq 0$ is such that $\bar{r} - \zeta < \xi$ and $\bar{r} - \zeta < r$, it follows from the induction hypothesis that there is a path $\gamma_{j-\bar{r},j-\zeta}$ of connecting orbits joining $\overline{h}_k^{(j-\zeta)}$ to $h_{k-1}^{(j-\bar{r})}$.
- Since $\overline{\Delta}^{\bar{r}}_{j-\bar{r},j} \neq 0$ is such that $\bar{r} < \xi$ and $\bar{r} < r$, it follows from the induction hypothesis that there is a path $\gamma_{j-\bar{r},j}$ of connecting orbits joining $h_k^{(j)}$ to $h_{k-1}^{(j-\bar{r})}$.

Hence

$$\gamma_{j-\xi,j} = [\gamma_{j-\bar{r},j}, -\gamma_{j-\bar{r},j-\zeta}, \gamma_{j-\xi,j-\zeta}]$$

is a path joining $h_k^{(j)}$ to $h_{k-1}^{(j-\xi)}$.

Thus we have shown that $\overline{\Delta}^{r}_{j-\xi,j} \neq 0$ corresponds to a path in the flow φ. □

Proof of Theorem 6.1. Let $d_p^r \neq 0$. It follows from Theorem 5.7 that every $d^r \neq 0$ is induced by multiplication by a $\Delta^r_{p-r+1,p+1}$ which is either a primary pivot or a change-of-basis pivot. By Proposition 6.4, $\overline{\Delta}^{r}_{p-r+1,p+1} \neq 0$ and all entries in the $(p+1)$st column below the $(p-r+1)$st row are zero, that is,

$$\overline{\Delta}^{r}_{p-r+1,p+1} = c^{p-r+1,r}_{p-r+1} \Delta^r_{p-r+1,p+1} \neq 0.$$

By Lemma 6.5, there is a path in the flow formed by connecting orbits joining the singularity $h_k^{(p+1)}$ to the singularity $h_{k-1}^{(p-r+1)}$. □

Example 6.6. Consider Example 2.1. Note that the entry $\Delta^8_{5,13} = 3$ is a primary pivot in Δ^8 whose corresponding original entry in Δ was equal to zero, i.e. $\Delta_{5,13} = 0$. Hence there does not necessarily exist a connecting orbit between $h_{k+1}^{(13)}$ and $h_k^{(5)}$. However, we will now determine a path of connecting orbits between these two singularities.

Note that

$$\partial\sigma_{k+1}^{(13),4} = -h_k^{(9)} + h_k^{(8)} + 4h_k^{(7)} - 3h_k^{(6)} + h_k^{(4)},$$
$$\partial\sigma_{k+1}^{(13),5} = -h_k^{(7)} + h_k^{(6)} + h_k^{(5)} - 2h_k^{(4)},$$

and hence $\overline{\Delta}^4_{5,13} = 0$ and $\overline{\Delta}^5_{5,13} = 1 \neq 0$. Thus, consider $\bar{r} = 4$. We represent the path schematically by using the matrix-type representation in Figure 21. Computing the entries within the proof of Lemma 6.5, we get:

- $\overline{\Delta}^{\bar{r}}_{j-\bar{r},j} = \overline{\Delta}^4_{13-4,13} = \overline{\Delta}^4_{9,13} \neq 0$;
- $\overline{\Delta}^{\bar{r}-\zeta}_{j-\bar{r},j-\zeta} = \overline{\Delta}^{4-3}_{13-4,13-3} = \overline{\Delta}^1_{9,10} \neq 0$;
- $\overline{\Delta}^{\bar{r}-\zeta}_{j-\xi,j-\zeta} = \overline{\Delta}^{4-3}_{13-8,13-3} = \overline{\Delta}^1_{5,10} \neq 0$.

Hence, by Lemma 6.5, a path between $h_{k+1}^{(13)}$ and $h_k^{(5)}$ is $\gamma_{5,13} = [\gamma_{9,13}, -\gamma_{9,10}, \gamma_{5,10}]$. See Figure 22.

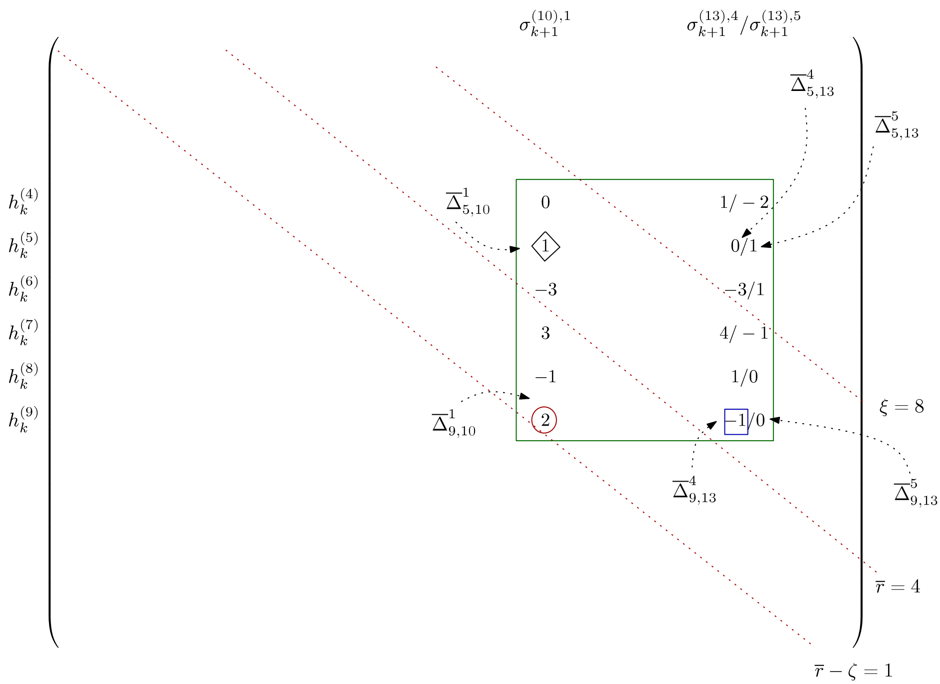

FIGURE 21. Schematic representation of the path $\gamma_{5,13}$.

The length of $\gamma_{5,13}$ is

$$\ell(\gamma_{5,13}) = \ell(\gamma_{9,13}) + \ell(\gamma_{9,10}) + \ell(\gamma_{5,10}) = 4 + 1 + 5 = 10.$$

Note that we could choose the path to be composed of connections which correspond to the entries $\overline{\Delta}_{7,13}^{6} \neq 0$, $\overline{\Delta}_{7,11}^{4} \neq 0$ and $\overline{\Delta}_{5,11}^{4} \neq 0$, i.e. $\gamma'_{5,13} = [\gamma'_{7,13}, -\gamma'_{7,11}, \gamma'_{5,11}]$; see Figure 23.

The entries $\overline{\Delta}_{7,13}^{6}$ and $\overline{\Delta}_{7,11}^{4}$ correspond to connecting orbits in φ, since $\overline{\Delta}_{7,13}^{1} \neq 0$ and $\overline{\Delta}_{7,11}^{1} \neq 0$. On the other hand, $\overline{\Delta}_{5,11}^{1} = 0$, i.e. there is not necessarily a connecting orbit between $h_{k+1}^{(11)}$ and $h_k^{(5)}$. However, there is a path $\gamma'_{5,11}$ between $h_{k+1}^{(11)}$ and $h_k^{(5)}$ made up of connecting orbits corresponding to the entries $\overline{\Delta}_{9,11}^{2} \neq 0$, $\overline{\Delta}_{9,10}^{1} \neq 0$ and $\overline{\Delta}_{5,10}^{1} \neq 0$, that is,

$$\gamma'_{5,13} = [\gamma'_{7,13}, -\gamma'_{7,11}, [\gamma'_{9,11}, -\gamma'_{9,10}, \gamma'_{5,10}]].$$

See Figure 24.

The length of $\gamma'_{5,13}$ is

$$\begin{aligned}\ell(\gamma'_{5,13}) &= \ell(\gamma'_{7,13}) + \ell(\gamma'_{7,11}) + \ell(\gamma'_{9,11}) + \ell(\gamma'_{9,10}) + \ell(\gamma'_{5,10}) \\ &= 6 + 4 + 2 + 1 + 5 = 18.\end{aligned}$$

This shows that the path between two singularities is often not unique. Even for a fixed length, the path need not be unique.

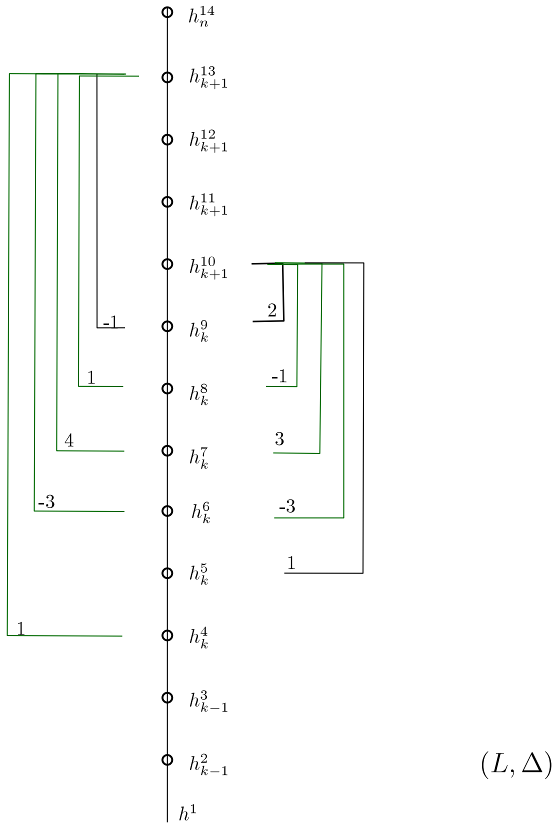

FIGURE 22. The path $\gamma_{5,13}$.

7. *Conclusion*

This work marks the beginning of a systematic study of the dynamical implications associated with the algebraic behavior of a spectral sequence. We have shown that as r increases, the $\mathbb{Z}$-modules E_p^r undergo a change of generators. In Theorems 4.4 and 5.7, the sweeping method relates this change in generators of E_p^r to a change of basis over $\mathbb{Q}$ of the connection matrix Δ. As we apply the sweeping method, important entries on the rth auxiliary diagonal of Δ^r are singled out in order to determine Δ^{r+1}. These entries are the primary and change-of-basis pivots, and it is worth noting that they remain integers throughout the sweeping process, as shown in Proposition 4.2. The dynamical interpretation of the intermediate matrices in this process is, as yet, not well understood since many of the entries are non-integers.

A question that remains unanswered is what the relationship is between the initial flow associated with Δ and the flow corresponding to the final matrix obtained from the sweeping method. Several examples suggest that we may have a continuation.

Another open question is how to interpret the appearance of torsion in the spectral sequence which may cancel algebraically before stabilization.

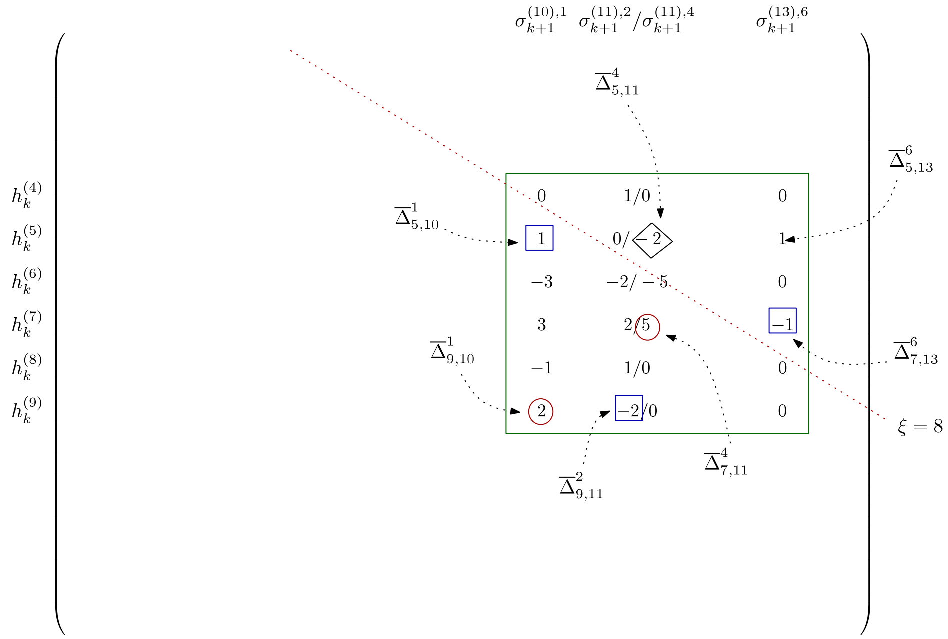

FIGURE 23. Schematic representation of the path $\gamma'_{5,13}$.

In proving the zig-zag theorem, we drew a parallel between 'long flow lines' which connect consecutive singularities $h_k \in F_p$ and $h_{k-1} \in F_{p-r}$ that are far apart and higher-order non-zero differentials d^r in the spectral sequence. These long flow lines are paths made up of connecting orbits, with some orbits being considered in the time-reversed flow.

In Theorem 6.1 we proved the existence of long flow lines φ. Some open problems that remain are to minimize the time spent in the reverse flow and to characterize the connecting orbits in which time-reversal is allowed.

The difficulty in determining minimal paths lies in the fact that zero entries $\Delta_{i,j}$ may have connecting orbits joining $h_k^{(j)}$ and $h_{k-1}^{(i)}$; this is because each entry is an intersection number (of attaching and belt spheres). Our interest is to determine, in this context, minimal paths in the absence of connecting orbits for zero entries.

Let $F(\gamma_{i,j})$ and $R(\gamma_{i,j})$ be the sets of all elementary paths which correspond to a flow line of φ_t and $-\varphi_t$, respectively, and which make up $\gamma_{i,j}$. Define

$$\ell^+(\gamma_{i,j}) = \sum_{\gamma \in F(\gamma_{i,j})} \ell(\gamma) \quad \text{and} \quad \ell^-(\gamma_{i,j}) = \sum_{\gamma \in R(\gamma_{i,j})} \ell(\gamma).$$

It is clear that $\ell(\gamma_{i,j}) = \ell^+(\gamma_{i,j}) + \ell^-(\gamma_{i,j})$ and $\ell^+(\gamma_{i,j}) - \ell^-(\gamma_{i,j}) = j - i$.

In the presence of several paths between $h_k^{(j)}$ and $h_{k-1}^{(i)}$, we choose one whose $\ell^-(\gamma_{i,j})$ is minimal. We define $\mathcal{L}_{ij}$ as the set of all paths between $h_k^{(j)}$ and $h_{k-1}^{(i)}$. Note that a path $\gamma_{i,j} \in \mathcal{L}_{ij}$ has minimum length if and only if $\ell^-(\gamma_{i,j})$ is minimal. In fact, $\gamma_{i,j}$ has minimum length in $\mathcal{L}_{ij}$, i.e. $\ell(\gamma_{i,j}) < \ell(\theta_{ij})$ for all $\theta_{ij} \in \mathcal{L}_{ij}$, if and only if

$$\ell^+(\gamma_{i,j}) + \ell^-(\gamma_{i,j}) < \ell^+(\theta_{ij}) + \ell^-(\theta_{ij}) \quad \text{for all } \theta_{ij} \in \mathcal{L}_{ij}. \tag{19}$$

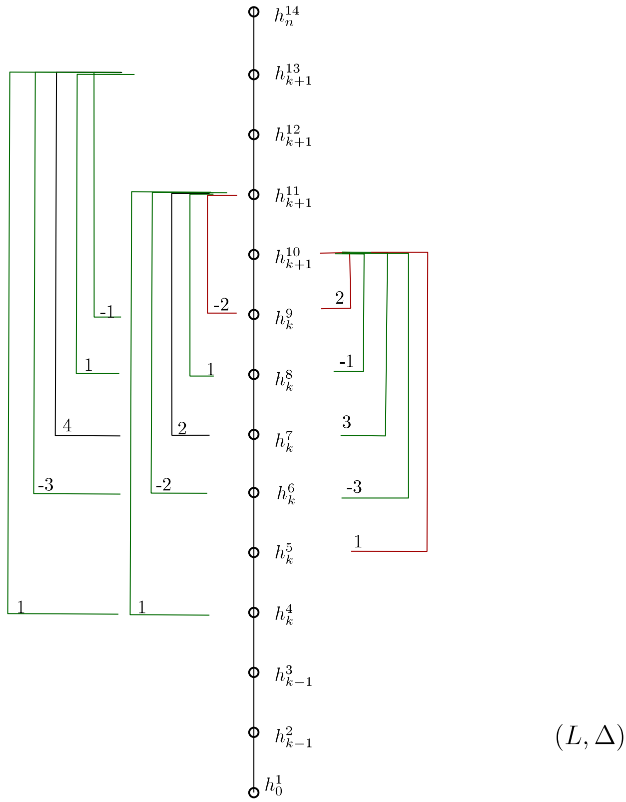

FIGURE 24. The path $\gamma'_{5,13}$.

Upon substituting $\ell^+(\gamma_{i,j}) = \ell^-(\gamma_{i,j}) + j - i$ and $\ell^+(\theta_{ij}) = \ell^-(\theta_{ij}) + j - i$ in (19), we obtain $\ell^-(\gamma_{i,j}) < \ell^-(\theta_{ij})$ for all $\theta_{ij} \in \mathcal{L}_{ij}$.

A natural extension of this work is to generalize the sweeping method in Theorems 4.4 and 5.7 to connection matrices associated with more general Morse decompositions.

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REFERENCES

[BaC] J. F. Barraud and O. Cornea. Lagrangian intersections and the Serre spectral sequence. *Ann. of Math.* (2) **166** (2007), 657–722.

[B] G. E. Bredon. *Topology and Geometry (Graduate Texts in Mathematics, 139)*. Springer, New York, 1993.

[Co] C. Conley. *Isolated Invariant Sets and the Morse Index (CBMS Regional Conference Series in Mathematics, 38)*. American Mathematical Society, Providence, RI, 1978.

[C1] O. Cornea. Homotopical dynamics: suspension and duality. *Ergod. Th. & Dynam. Sys.* **20** (2000), 379–391.

[C2] O. Cornea. Homotopical dynamics II: Hopf invariants, smoothing and the Morse complex. *Ann. Sci. École. Norm. Sup.* (4) **35** (2002), 549–573.

[C3] O. Cornea. Homotopical dynamics IV: Hopf invariants and Hamiltonian flows. *Comm. Pure Appl. Math.* **55** (2002), 1033–1088.

[CdRM] R. N. Cruz, K. A. de Rezende and M. Mello. Realizability of the Morse polytope. *Qual. Theory Dyn. Syst.* **6** (2007), 59–86.

[D] J. F. Davis and P. Kirk. *Lecture Notes in Algebraic Topology (Graduate Studies in Mathematics, 35)*. American Mathematical Society, Providence, RI, 2001.

[F1] J. Franks. Morse–Smale flows and homotopy theory. *Topology* **18** (1979), 199–215.

[F2] J. Franks. *Homology and Dynamical Systems (CBMS Regional Conference Series in Mathematics, 49)*. American Mathematical Society, Providence, RI, 1982.

[Fr1] R. Franzosa. Index filtrations and the homology index braid for partially ordered Morse decompositions. *Trans. Amer. Math. Soc.* **298** (1986), 193–213.

[Fr2] R. Franzosa. The continuation theory for Morse decompositions and connection matrices. *Trans. Amer. Math. Soc.* **310** (1988), 781–803.

[Fr3] R. Franzosa. The connection matrix theory for Morse decompositions. *Trans. Amer. Math. Soc.* **311** (1989), 561–592.

[Fr4] R. Franzosa and K. Mischaikow. Algebraic transition matrices in the Conley index theory. *Trans. Amer. Math. Soc.* **350** (1998), 889–912.

[K] H. L. Kurland. Homotopy invariants of repeller–attractor pairs I: The Puppe sequence of an R–A pair. *J. Differential Equations* **46** (1982), 1–31.

[L] R. Leclercq. Spectral invariants in Lagrangian Floer theory. *Preprint*, 2006. Available at arXiv:math/0612325.

[MC] C. McCord. The connection map for attractor–repeller pairs. *Trans. Amer. Math. Soc.* **307** (1988), 195–203.

[MCR] C. McCord and J. F. Reineck. Connection matrices and transition matrices. *Conley Index Theory (Banach Center Publications, 47)*. Polish Academy of Sciences, Warsaw, 1999, pp. 41–55.

[M1] J. W. Milnor. *Topology from the Differentiable Viewpoint*. University Press of Virginia, Charlottesville, VA, 1965.

[M2] J. W. Milnor. *Lectures on the h-Cobordism Theorem*. Princeton University Press, Princeton, NJ, 1965.

[Mo] R. Moeckel. Morse decompositions and connection matrices. *Ergod. Th. & Dynam. Sys.* **8** (1988), 227–249.

[R1] J. F. Reineck. The connection matrix in Morse–Smale flows. *Trans. Amer. Math. Soc.* **322** (1990), 523–545.

[R2] J. F. Reineck. The connection matrix in Morse–Smale flows II. *Trans. Amer. Math. Soc.* **347** (1995), 2097–2110.

[R3] J. F. Reineck. Continuation to the minimal number of critical points in gradient flows. *Duke Math. J.* **68** (1992), 185–194.

[Sa1] D. Salamon. Connected simple systems and the Conley index of invariant sets. *Trans. Amer. Math. Soc.* **291** (1985), 1–41.

[Sa2] D. Salamon. Morse theory, Conley index and Floer homology. *Bull. London Math. Soc.* **22** (1990), 113–140.

[S1] S. Smale. The generalized Poincaré conjecture in higher dimensions. *Bull. Amer. Math. Soc.* **66** (1960), 373–375.

[S2] S. Smale. On the structure of manifolds. *Amer. J. Math.* **84** (1962), 387–399.

[Sp] E. Spanier. *Algebraic Topology*. McGraw-Hill, New York, 1966.