9. Tensor product and Hom

Starting from two R-modules we can define two other R-modules, namely $M \otimes_R N$ and $\mathrm{Hom}_R(M, N)$, that are very much related. The defining properties of these modules are simple, but those same defining properties induce many, many different constructions in the theory of R-modules. Understanding these constructions gave rise to a whole new theory about theories: the theory of categories and functors. In any theory with similar constructions, we get similar consequences. In this section we give a timid introduction. The defining property of the tensor product, gives rise to the idea of adjointness, which implies preservation of exactness of certain sequences. And indeed, everything becomes more and more abstract (giving a feeling of unease in the beginning), but the proofs become more simple and easy to generalize to other theories (giving a good reason to persist).

Getting used to it is a bit like getting used to the "first isomorphism theorem" in group theory, and then in ring theory, and then in module theory, and then in sheaf theory, and then in category theory.... The first time was hard, the second time it was easier, by now it is (starting to become) "trivial".

9.1. **Defining property of tensor product.** Let R be a ring and M, N, P three R-modules. An *R-bilinear map* $\beta : M \times N \to P$ is a map which is bilinear in the two variables, i.e.,

$$\beta(r_1 m_1 + r_2 m_2, n) = r_1 \beta(m_1, n) + r_2 \beta(m_2, n)$$

$$\beta(m, r_1 n_1 + r_2 n_2) = r_1 \beta(m, n_1) + r_2 \beta(m, n_2)$$

for all $r_1, r_2 \in R$, $m, m_1, m_2 \in M$ and $n, n_1, n_2 \in N$. We shall prove that all R-bilinear maps on $M \times N$ are (uniquely) produced by one universal R-blinear map.

Theorem 9.1. *Let R be a ring and M, N two R-modules. There is an R-module, denoted $M \otimes_R N$, and an R-bilinear map $\tau : M \times N \to M \otimes N$ with the following universal property.*

For any R-module P and any R-bilinear map $\beta : M \times N \to P$, there exists a unique *R-module homomorphism $f : M \otimes_R N \to P$ such that $f \circ \tau = \beta$.*

We shall give a construction of the module $M \otimes_R N$ (the tensor product of M and N over R) and the R-bilinear map τ in the proof of the theorem. The precise construction is not so important, but rather the fact that the pair exists and has the stated universal property. The pair is unique up to a unique isomorphism in any case, as stated in the corollary.

Corollary 9.1. *Suppose the module T and bilinear map $\sigma : M \times N \to T$ also has the universal property. Then there is a* unique *R-module isomorphism $f : M \otimes_R N \to T$ such that $f \circ \tau = \sigma$.*

Proof. By the universal property of the pair $(M \otimes_R N, \tau)$ there is a unique R-module homomorphism $f : M \otimes_R N \to T$ such that $f \circ \tau = \sigma$. By the universal property of (T, σ) there is a unique R-module homomorphism $g : T \to M \otimes_R N$ such that $g \circ \sigma = \tau$. So $f \circ g \circ \sigma = f \circ \tau = \sigma$. But also $\mathrm{Id}_T \circ \sigma = \sigma$, so by the uniqueness property $f \circ g = \mathrm{Id}_T$. And $g \circ f \circ \tau = \tau = \mathbf{1}_{M \otimes N} \circ \tau$ implies by the uniqueness property, that $g \circ f = \mathbf{1}_{M \otimes N}$. □

We shall write $m \otimes n := \tau(m,n) \in M \otimes_R N$, called a *pure tensor*. Any element of $M \otimes_R N$, sometimes called a *tensor* is a finite sum of pure tensors, which is what we prove next. Since τ is bilinear we have at least some relations among the pure tensors:

$$(m_1 + m_2) \otimes n = m_1 \otimes n + m_2 \otimes n$$

$$m \otimes (n_1 + n_2) = m \otimes n_1 + m \otimes n_2$$

$$r(m \otimes n) = rm \otimes n = m \otimes rn$$

for all $r \in R$, $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$.

Corollary 9.2. *The R-module $M \otimes_R N$ is generated by the image of τ, i.e., any element of $M \otimes_R N$ can be written in the form $\sum_{i=1}^n m_i \otimes n_i$.*

Proof. Write T for the R-submodule of $M \otimes_R N$ generated by the image of τ. So if we compose τ with the quotient map $\nu : M \to M/T$ we get the zero R-bilinear map. But since $\nu \circ \tau = 0 \circ \tau$, the unicity property gives that $\nu = 0$, i.e. $M = T$. So any tensor can be written as

$$\sum_{i=1}^n r_i(m_i \otimes n_i) = \sum_{i=1}^n (r_i m_i \otimes n_i) = \sum_{i=1}^n (m_i \otimes r_i n_i).$$

□

We now can at least write down a typical element of the tensor product, and manipulate somewhat the expressions using bilinearity. Suppose now we want to define a homomorphism $f : M \otimes_R N \to P$. At least we need to give the value on the generators, i.e., what $f(m \otimes n) \in P$ is. But there are many relations among the generators, so we have a problem of well-definedness. So suppose we have a formula depending on (m,n), say $\beta(m,n)$, then the key in the definition of tensor products is, that if the map $(m,n) \to \beta(m,n)$ is R-bilinear then indeed the formula $f(m \otimes n) = \beta(m,n)$ induces a well defined R- module homomorphism $f : M \otimes_R N \to P$.

Example 9.1. We show that $R \otimes_R N \simeq N$, for any R-module N. The map $R \times N \to N$ given by $r, n \mapsto rn$ is R-bilinear, so there is a unique R-homomorphism $f : R \otimes_R N \to N$ such that $f(r \otimes n) = rn$. On the other hand, define the map $g : N \to R \otimes N$ by $g(n) := 1 \otimes n$, which is R-linear by the bilinear relations among the pure tensors. We check $(g \circ f)(r \otimes n) = g(rn) = 1 \otimes rn = r \otimes n$, so $g \circ f$ is the identity on the generators, hence $g \circ f$ is the identity map. Likewise $(f \circ g)(n) = f(1 \otimes n) = 1 \cdot n = n$, and so $f \circ g$ is the identity also. We conclude that f is an isomorphism.

The two previous corollaries give a suggestion of the proof, which we will give now.

Proof of Theorem 9.1. Let L be the free R-module on the *set* $M \times N$, i.e., F is the collection of functions from $M \times N$ to R that only finitely many times attains a non-zero value

$$L = \{f : M \times N \to R;\ \#\{(m,n) \in M \times N; f(m,n) \neq 0\} < \infty\}.$$

Write $[m,n]$ for the function that takes value 1 on (m,n) and value 0 on any other couple $(m', n') \in M \times N$. Than we can express $f \in L$ as a finite sum $f = \sum_{(m,n) \in M \times N} f(m,n) \cdot [m,n]$.

We define $K \subseteq L$ to be the R-submodule generated by the elements

$$[m_1 + m_2, n] - [m_1, n] - [m_2, n];$$

$$[m, n_1 + n_2] - [m, n_1] - [m, n_2];$$

$$r \cdot [m, n] - [rm, n];$$

$$r \cdot [m, n] - [m, rn],$$

for all possible $m, m_1, m_2 \in M$, $n, n_1, n_2 \in N$ and $r \in R$.

We define $M \otimes_R N := L/K$ and $\tau : M \times N \to L/K$ by $\tau(m, n) = [m, n] + K =: m \otimes n$. The form of the generators of K force the map τ to be R-bilinear.

Now let $Q : M \times N \to P$ be any R-bilinear map. We first can extend to the unique R-linear map $F : L \to P$ defined by $F(f) = \sum_{(m,n) \in M \times N} f(m, n)\beta(m, n)$, in particular $[m, n] \mapsto \beta(m, n)$. For example $[m_1 + m_2, n] - [m_1, n] - [m_2, n] \mapsto \beta(m_1 + m_2, n) - \beta(m_1, n) - \beta(m_2, n) = 0$, and more generally any generator of K is mapped to zero. So there is a unique R-module homomorphism $f : L/K \to P$ such that $F = f \circ \nu_K$, where $\nu_K : L \to L/K$ is the quotient map. So $f(m \otimes n) = \beta(m, n)$ indeed, and f is unique since f is totally determined by the value of the generators. □

9.2. **Tensor product as functor.** We shall say that F is a *functor* from R-modules to R-modules, if for any module M we are given a module $F(M)$ and for any homomorphism $f : M' \to M$ we are given a homomorphism $F(f) : F(M') \to F(M)$ and where we require that $F(g \circ f) = F(g) \circ F(f)$ for $g : M \to M''$.

Taking tensor product with a fixed module N is such a functor, according to the next result.

Corollary 9.3. *Let $f : M' \to M$ be an R-module homomorphism, and N an R-module. There is a unique R-module homomorphism, noted $f \otimes 1 : M' \otimes_R N \to M \otimes_R N$, such that $f(m' \otimes n) = f(m') \otimes n$. If $g : M \to M''$ is a second homomorphism, then $(g \otimes 1) \circ (f \otimes 1) = (g \circ f) \otimes 1$.*

Proof. The map $\beta : M' \times N \to M \otimes_R N : m', n \mapsto f(m') \otimes n$ is bilinear, so by the defining property of the tensor product there is a unique R-homomorpism such that $m \otimes n \mapsto f(m') \otimes n$. Hence $f \otimes 1$ exists.

The homomorphism $(g \otimes 1) \circ (f \otimes 1)$ and $(g \circ f) \otimes 1$ both map the generators $m \otimes n$ to $g(f(m)) \otimes n$, so they are equal. □

Fix a module N then we define $F(M) = M \otimes_R N$ and $F(f) = f \otimes 1$, where $f : M' \to M$. Then F is a functor.

Let D be a multiplicatively closed subset of R, then localization ($M \mapsto D^{-1}M$ and $f \mapsto D^{-1}f$) is also a functor. This functor transforms exact sequences to exact sequences ("localization is an exact functor").

This last property is no longer true for $- \otimes_R N$. We will prove a little bit later (right after Proposition 9.1) that " the functor $- \otimes_R N$ is right exact", but in general does not preserve injections.

Theorem 9.2. *Let*

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

be exact and N an R-module. Then

$$M' \otimes_R N \xrightarrow{u\otimes 1} M \otimes_R N \xrightarrow{v\otimes 1} M'' \otimes_R N \longrightarrow 0$$

is also exact.

Example 9.2. Let us give an example where injectivity is not preserved. Let $f : \mathbb{Z} \to \mathbb{Z}$ be multiplication by 2, i.e., $f(n) = 2n$. Take for N the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$. Then $\mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z} \xrightarrow{f\otimes 1} \mathbb{Z}\otimes_{\mathbb{Z}}\mathbb{Z}/2\mathbb{Z}$ maps $n \otimes \overline{m}$ to

$$2n \otimes \overline{m} = n \otimes 2\overline{m} = n \otimes \overline{2m} = n \otimes \overline{0} = n \otimes 0\overline{0} = 0n \otimes \overline{0} = 0 \otimes \overline{0} = 0.$$

So $f \otimes 1$ is the zero-map, which is non-injective, since $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z} \simeq \mathbb{Z}/2\mathbb{Z} \neq 0$.

9.3. **The** $\mathrm{Hom}_R(,)$ **functor.** There are other important functors. We recall that for two R-modules N and M the collection of R-module homomorphisms from N to M, denoted by $\mathrm{Hom}_R(N, M)$, is an R-module with external multiplication $(rf)(n) := r \cdot f(n) = f(r \cdot n)$. If $f : M' \to M$ and $\phi \in \mathrm{Hom}_R(N, M)$ then $f \circ \phi \in \mathrm{Hom}_R(N, M')$, or we get a map

$$\mathrm{Hom}_R(N, M') \xrightarrow{f\circ -} \mathrm{Hom}_R(N, M)$$

which is an R-module homomorphism. If $g : M \to M''$ is another homomorphism then

$$((g \circ f) \circ -) = (g \circ -) \circ (f \circ -).$$

In particular, for fixed N we get another functor $G(M) := \mathrm{Hom}_R(N, M)$ and $G(f) = (f \circ -)$. Which is "left exact", but we will prove more in the next two theorems.

Similarly if $f : N' \to N$ and $g : N \to N''$ are R-module homomorphisms, we get homomorphism

$$\mathrm{Hom}_R(N'', M) \xrightarrow{-\circ g} \mathrm{Hom}_R(N, M) \xrightarrow{-\circ f} \mathrm{Hom}_R(N', M)$$

and

$$(- \circ (g \circ f)) = (- \circ f) \circ (- \circ g).$$

Theorem 9.3. *The sequence of R-modules*

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

is exact if and only if for any R-module N the sequence

$$0 \longrightarrow \mathrm{Hom}_R(M'', N) \xrightarrow{-\circ v} \mathrm{Hom}_R(M, N) \xrightarrow{-\circ u} \mathrm{Hom}_R(M', N)$$

is exact.

Proof. (i) We first prove in one direction. Suppose

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

is exact. And let N be any R-module.

In particular $\mathrm{Im}(u) \subseteq \mathrm{Ker}(v)$, i.e., $v \circ u = 0$. So for every $\phi \in \mathrm{Hom}_R(M'', N)$ it holds that

$$[(- \circ u) \circ (- \circ v)](\phi) = \phi \circ v \circ u = 0,$$

i.e., $\mathrm{Im}(- \circ v) \subseteq \mathrm{Ker}(- \circ u)$.

Suppose $\psi \in \operatorname{Ker}(- \circ u)$, i.e., $\psi : M \to N$ such that the composition $\psi \circ u : M' \to N$ is the zero map. Or $\operatorname{Im} u \subset \operatorname{Ker} \psi$ and we get a factorization of ψ. With $\nu : M \to M/\operatorname{Im} u$ the quotient map, there is a homomorphism $f : M/\operatorname{Im} u \to N$ such that $\psi = f \circ \nu$. By assumption, we have $\operatorname{Im} u = \operatorname{Ker} v$ and v is surjective, so there is an isomorphism $h : M/\operatorname{Im} u \simeq M''$ such that $v = h \circ \nu$. Then

$$\psi = f \circ \nu = (f \circ h^{-1}) \circ (h \circ \nu) = (f \circ h^{-1}) \circ v$$

and $\psi \in \operatorname{Im}(- \circ v)$. We conclude that $\operatorname{Im}(- \circ v) \supseteq \operatorname{Ker}(- \circ u)$. Note that we needed surjectivity of v to prove this inclusion. It remains to prove the injectivity of $(- \circ v)$. Let $\phi \in \operatorname{Ker}(- \circ v)$. This means that $\phi : M'' \to N$ such that the composition $\phi \circ v : M \to N$ is the zero map. Let $m'' \in M''$ be any element. By the surjectivity of v there exists an $m \in M$ such that $v(m) = m''$. We get that $\phi(m'') = (\phi \circ v)(m) = 0$, or that ϕ is the zero map. We showed that $(- \circ v)$ is injective.

(ii) Now suppose for any R-module N the sequence

$$0 \longrightarrow \operatorname{Hom}_R(M'', N) \xrightarrow{-\circ v} \operatorname{Hom}_R(M, N) \xrightarrow{-\circ u} \operatorname{Hom}_R(M', N)$$

is exact. We want to show that the sequence

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

is exact.

We start proving that v necessarily surjective. Consider the quotient map $\nu : M'' \to M''/\operatorname{Im} v$. The composition $\nu \circ v : M \to M''/\operatorname{Im} v$ is the zero map. When we take $N = M''/\operatorname{Im} v$ we get an exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(M'', M''/\operatorname{Im} v) \xrightarrow{-\circ v} \operatorname{Hom}_R(M, M''/\operatorname{Im} v)$$

and we reinterprete the statement above as saying that $\nu \in \operatorname{Ker}(-\circ v)$. We conclude from exactness that $\nu = 0$. This means that for any m'' we have $\nu(m'') = 0$, i.e., $m'' \in \operatorname{Im} v$, or that v is surjective.

To show that $\operatorname{Im} u \subseteq \operatorname{Ker} v$ or that $v \circ u = 0$ consider the identity map $\operatorname{Id} : M'' \to M''$ and take the exact sequence corresponding to the choice $N = M$. Then for $\operatorname{Id} \in \operatorname{Hom}_R(M'', M'')$ we get from the exactness that

$$[(- \circ u) \circ (- \circ v)](\operatorname{Id}) = 0 \in \operatorname{Hom}_R(M', M)$$

or $v \circ u = \operatorname{Id} \circ v \circ u = 0$, indeed.

Finally, we must show that $\operatorname{Ker} v \subseteq \operatorname{Im} u$. This time we consider the quotient map $\nu : M \to M/\operatorname{Im} u$ having composition $\nu \circ u = 0$ or $\nu \in \operatorname{Ker}(-\circ u) = \operatorname{Im}(-\circ v$. So there is a $\phi : M'' \to M/\operatorname{Im} u$ such that $\nu = \phi \circ v$. So $\nu(\operatorname{Ker} v) = \phi(v(\operatorname{Ker} v)) = 0$ or $\operatorname{Ker} v \subseteq \operatorname{Ker} \nu = \operatorname{Im} u$. □

Theorem 9.4. *The sequence of R-modules*

$$0 \longrightarrow N' \xrightarrow{u} N \xrightarrow{v} N''$$

is exact if and only if for any R-module N the sequence

$$0 \longrightarrow \operatorname{Hom}_R(M, N') \xrightarrow{u\circ -} \operatorname{Hom}_R(M, N) \xrightarrow{v\circ -} \operatorname{Hom}_R(M, N)$$

is exact.

Proof. Exercise. □

9.4. **Adjoint functors.** For a fixed module N the functors $-\otimes_R N$ and $\mathrm{Hom}_R(N,-)$ are in some sense dual, or rather adjoint, to each other.

Let F and G be two functors from R-modules to R-modules. We shall say that (F,G) is an *adjoint couple* if for any pair of modules (M,P) we are given an isomorphism

$$\alpha_{M,P}\,\mathrm{Hom}_R(F(M),P)\simeq \mathrm{Hom}_R(M,G(P))$$

that is *natural* in the following technical sense.

For any $f: M'\to M$ and $\phi\in \mathrm{Hom}_R(F(M),P)$ we have

$$\alpha_{M,P}(\phi)\circ f=\alpha_{M',P}(\phi\circ F(f))$$

and for any $g: P'\to P$ and $\psi\in \mathrm{Hom}_R(F(M),P')$ we have

$$\alpha_{M,P}(g\circ\psi)=G(g)\circ\alpha_{M,P'}(\psi).$$

The following result says that $(-\otimes_R N, \mathrm{Hom}_R(N,-))$ is indeed an adjoint couple. This follows almost directly from the defining property of a tensor product in terms of bilinear maps.

Lemma 9.1. *Let R be a ring and M,N,P three R-modules.*

(i) The map

$$\alpha_{M,N,P}:\mathrm{Hom}_R(M\otimes_R N,P)\simeq \mathrm{Hom}_R(M,\mathrm{Hom}_R(N,P))$$

defined by $\alpha_{M,N,P}(\phi)(m)(n)=\phi(m\otimes n)$ is an isomorphism. The inverse is given by

$$\beta_{M,N,P}:\psi\mapsto(m\otimes n\mapsto\psi(m)(n)).$$

(ii) Let $f: M'\to M$ and $\phi\in \mathrm{Hom}_R(M\otimes_R N,P)$, then

$$\alpha_{M,N,P}(\phi)\circ f=\alpha_{M',N,P}(\phi\circ(f\otimes 1))$$

(iii) Let $g: P'\to P$ and $\psi\in \mathrm{Hom}_R(M\otimes_R N,P')$, then

$$\alpha_{M,N,P}(g\circ\psi)=(g\circ -)\circ\alpha_{M,N,P'}(\psi).$$

Proof. (i) We compose the two maps and obtain identities:

$$(\beta\circ\alpha)(\phi)=\beta(\alpha(\phi))=(m\otimes n\mapsto\alpha(\phi)(m)(n))=(m\otimes n\mapsto\phi(m\otimes n))=\phi.$$

$$(\alpha\circ\beta)(\psi)=\alpha(\beta(\psi))=(m\mapsto(n\mapsto\beta(\psi)(m\otimes n)))=(m\mapsto(n\mapsto\psi(m)(n)))=\psi.$$

(ii) Let $f: M'\to M$ and $\phi\in \mathrm{Hom}_R(M\otimes_R N,P)$. Then $\alpha_{M,N,P}(\phi)\circ f$ and $\alpha_{M',N,P}(\phi\circ(f\otimes 1))$ both are homomorphism from M' to $\mathrm{Hom}_R(N,P)$. Let $m'\in M'$ and $n\in N$. Then

$$(\alpha_{M,N,P}(\phi)\circ f)(m')(n)=(\alpha_{M,N,P}(\phi))(f(m'))(n)=\phi(f(m')\otimes n),$$

and

$$\alpha_{M',N,P}(\phi\circ(f\otimes 1))(m')(n)=(\phi\circ(f\otimes 1))(m'\otimes n)=\phi(f(m')\otimes n).$$

We get equality.

(iii) Exercise. □

Proposition 9.1. *Let (F, G) be an adjoint couple of functors from R-modules to R-modules.*

(i) Suppose

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

is an exact sequence. Then the sequence

$$F(M') \xrightarrow{F(u)} F(M) \xrightarrow{F(v)} F(M'') \longrightarrow 0$$

is also exact.

(ii) Suppose

$$0 \longrightarrow P' \xrightarrow{u} P \xrightarrow{v} P''$$

is an exact sequence. Then the sequence

$$0 \longrightarrow G(P') \xrightarrow{G(u)} G(P) \xrightarrow{G(v)} G(P'')$$

is also exact.

Proof. (i) Since the sequence of R-modules

$$M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

is exact, by Theorem 9.3 for all R-module P the sequence

$$0 \longrightarrow \mathrm{Hom}_R(M'', G(P)) \xrightarrow{-\circ v} \mathrm{Hom}_R(M, G(P)) \xrightarrow{-\circ u} \mathrm{Hom}_R(M', G(P))$$

is exact. Now applying the isomorphism α we get for all P an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(F(M''), P) \xrightarrow{-\circ F(v)} \mathrm{Hom}_R(F(M), P) \xrightarrow{-\circ F(u)} \mathrm{Hom}_R(F(M'), P)$$

The naturality conditions ensure that the maps are indeed $-\circ F(v)$ and $-\circ F(u)$. We conclude by Theorem 9.3 again that

$$F(M') \xrightarrow{F(u)} F(M) \xrightarrow{F(v)} F(M'') \longrightarrow 0$$

is also exact.

(ii) Exercise. □

Proof of Theorem 9.2. The theorem follows since it is the left part of an adjoint couple. □

9.5. Tor **and** Ext**.** Although $-\otimes_R N$ is not quite exact, we still have a handle.

In homological algebra it is then shown that there are other functors L^iF (the left-derived functors for all non-negative integers $i \geq 0$ such that $L^0F = F$ with the property that if

$$0 \longrightarrow M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

is a short exact sequence, then there is a long exact sequence (with "natural" δ's)

$$\ldots \longrightarrow L^3F(M'') \xrightarrow{\delta_2} L^2F(M') \xrightarrow{L^2F(u)} L^2F(M) \xrightarrow{L^2F(v)} L^2F(M'') \xrightarrow{\delta_1}$$

$$\xrightarrow{\delta_1} L^1F(M') \xrightarrow{L^1F(u)} L^1F(M) \xrightarrow{L^1F(v)} L^1F(M'') \xrightarrow{\delta_0} F(M') \xrightarrow{F(u)} F(M) \xrightarrow{F(v)} F(M'') \longrightarrow 0$$

And for all non-negative integers $i \geq 0$ we get "right-derived functors" R^iF such that $R^0F = F$ with the property that if

$$0 \longrightarrow M' \xrightarrow{u} M \xrightarrow{v} M'' \longrightarrow 0$$

is a short exact sequence, then there is a long exact sequence (with ”natural” δ’s)

$$0 \longrightarrow G(M') \xrightarrow{G(u)} G(M) \xrightarrow{G(v)} G(M'') \xrightarrow{\delta_0} R^1G(M') \xrightarrow{R^1G(u)} R^1G(M) \xrightarrow{R^1G(v)} R^1G(M'') \xrightarrow{\delta_1}$$
$$\xrightarrow{\delta_1} R^2G(M') \xrightarrow{R^2G(u)} R^2G(M) \xrightarrow{R^2G(v)} R^2G(M'') \xrightarrow{\delta_2} R^3G(M') \cdots$$

For our couple special names were given. For $F = - \otimes_R N$ we have $L^iF(M) = \mathrm{Tor}_i^R(M, N)$ and for $G = \mathrm{Hom}_R(N, -)$ we have $R^iG(P) = \mathrm{Ext}_R^i(N, P)$.

For special modules N the functor $- \otimes_R N$ can still be exact. In that case N is called *flat*. If $\mathrm{Hom}_R(N, -)$ is still exact, N is called *projective*. If $\mathrm{Hom}_R(-, N)$ preserves exactness, then N is called *injective*. As you can guess, there is a whole theory of projective, injective and flat modules.

Example 9.3. If D is a multiplicatively closed subset then $D^{-1}R$ is a flat R-module. This follows from the exactness of localization and the following easy lemma.

Lemma 9.2. *Let $D \subset R$ be a multiplicatively closed subset and M an R-module. Then $\alpha_M : D^{-1}R \otimes_R M \to D^{-1}M : \frac{r}{d} \otimes m \mapsto \frac{rm}{d}$ is an isomorphism of $D^{-1}R$-modules. If $f : M' \to M$ is a homomorphism, then $D^{-1}f \circ \alpha_{M'} = \alpha_M \circ (1 \otimes f)$. So localization functor is up to an isomorphism the same thing as $- \otimes_R D^{-1}R$, and so $D^{-1}R$ is a flat R-module.*

There are many tensor product identities that hold and should be verified, like $M \otimes N \simeq N \otimes N$ of $(N \otimes M) \otimes P \simeq N \otimes (M \otimes P)$, but we will stop here, and skip to a different topic.

9.6. Coming back to Hilbert’s theorems from the 1890’s. For R a polynomial ring over over a field with n variables. Hilbert’s deep syzygy theorem is a forerunner of homological algebra, implying among others that we have vanishing theorems $\mathrm{Tor}_{>n}^R(M, N) = 0 = \mathrm{Ext}_R^{>n}(M, N)$. This sounds very technical, but there are many unexpected consequences and motivations to look for similar things in other mathematical theories. Hilbert’s theorem was a very deep first exploration in homological algebra.

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