Poisson-Dirichlet statistics for the extremes of log-correlated Gaussian fields

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Probability Seminar, Harvard University, 26 October 2012

Outline

1. Log-Correlated Fields
   - Hierarchical cases (Polymers on tree, BBM)
   - Non-Hierarchical cases (2D GFF, MRM)

2. Main Result: Poisson-Dirichlet Statistics of the Gibbs weights
   (Log-correlated Gaussian fields are 1-RSB)

3. Ideas of the Proof
   - Ghirlanda-Guerra Identities
   - Multiscale Decomposition
   - Bovier-Kurkova Technique
   - Bolthausen-Deuschel-Giacomin tree approximation

4. Beyond the Gibbs measure: the Extremal Process

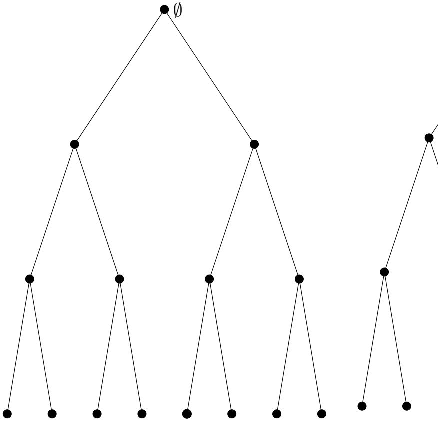
An example: Gaussian Field on a binary tree

- Consider a binary tree with n generations.
  Let $\mathcal{T}_n$ be the leaves $|\mathcal{T}_n| = 2^n$.
- Let (g_e) be i.i.d. $\mathcal{N}(0,1)$ on each edge e.
  Consider $X = (X_v, v \in \mathcal{T}_n)$

$$X_v = \sum_{e:\emptyset \to v} g_e$$

- $\mathbb{E}[X_v^2] = n$
  $\mathbb{E}[X_v X_{v'}] =$ time of branching
- For any $0 \leq r \leq 1$

$$\#\{v' : \mathbb{E}[X_v X_{v'}] \geq rn\} = \frac{2^n}{2^{rn}}$$



- Correlations are hierarchical:
  two correlations of a triplet v, v', v'' must be equal.

Log-correlated Gaussian fields

- Consider a Gaussian field

$$(X_v, v \in \mathcal{X}_n)$$

indexed by 2^n points in Euclidean space, say $[0,1]$.

- $\mathbb{E}[X_v^2] = n$ and for $c(v, v') := \mathbb{E}[X_v X_v']$

$$\#\{v' : c(v, v') \geq rn\} = \frac{2^n}{2^{rn}}$$

- Thus, $c(v, v')$ must be log-correlated

$$c(v, v') \sim -\log \|v - v'\|$$

where $\|v - v'\|$ is the Euclidean distance

Non-hierachical example: 2D discrete GFF

- Consider a box $\mathcal{V}_n \subset \mathbb{Z}^2$ with 2^n points.
- $(X_v, v \in \mathcal{V}_n)$ Gaussian field with

$$\mathbb{E}[X_v X_{v'}] = E^v \left[\sum_{k=0}^{\tau_{\partial V_n}} 1_{\{S_k = v'\}} \right] .$$

  $(S_k)_{k \geq 0}$ SRW starting at v.
- The field is log-correlated

$$\mathbb{E}[X_v^2] \sim n \log 2$$

$$\mathbb{E}[X_v X_{v'}] \sim \log \frac{2^n}{\|v - v'\|}$$

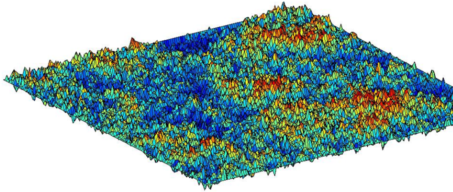
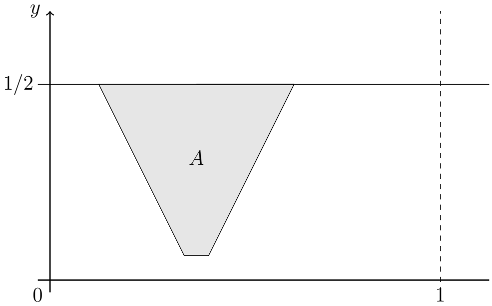


Figure by Samuel April

Non-hierachical example in 1D

We focus on a particular representation based on Bacry & Muzy '03
Multifractal Random Measure.
They consider a random measure μ on $[0,1]_\sim \times [0,1/2]$ such that:

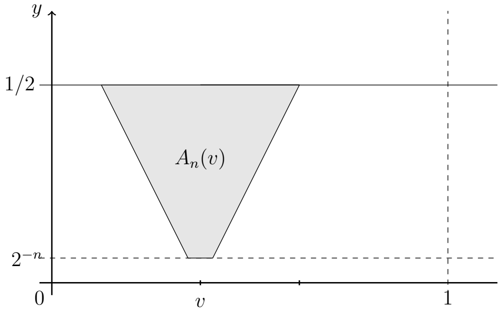
1. $\mu(A) \sim \mathcal{N}(0, \theta(A))$
2. $\theta(A) = \int_A y^{-2} dx dy$
3. $A \cap B = \emptyset \Leftrightarrow$ $\mu(A) \perp \mu(B)$.



Example in 1D

We construct a field $(X_v, v \in \mathcal{X}_n)$ on 2^n points.

- $\mathcal{X}_n$: 2^n equidistant points on $[0,1]_\sim$.
- $X_v = \mu(A_n(v))$ cones of slope $1/2$ truncated at 2^{-n}



$$\mathbb{E}[X_v^2] = \int_{A_n(v)} y^{-2} dx dy = n \log 2 + O(1)$$

$$\mathbb{E}[X_v X_{v'}] = \int_{A_n(v) \cap A_n(v')} y^{-2} dx dy = -\log |v - v'| + O(1)$$

Correlations are not hierarchical :
For a triplet v, v', v'', equality of two correlations is not ensured.

Correlations and Extremal Statistics

To which extent does correlations affect the statistics of high values (extremes) of $(X_v, v \in \mathcal{X}_n)$?

- Gaussian fields considered have strong correlations, of the order of the variance ($\sim$ spin glasses).

Objects of study

- Free energy: $\beta > 0$, $Z_n(\beta) = \sum_{v\in\mathcal{X}_n} e^{\beta X_v}$

$$f(\beta) := \lim_{n\to\infty} \frac{1}{\log 2^n} \log Z_n(\beta)$$

- Gibbs measure: Measure on $\mathcal{X}_n$ concentrating on extremes

$$\langle \ \cdot \ \rangle_{\beta,n} = \frac{\sum_{v\in\mathcal{X}_n} (\cdot) e^{\beta X_v}}{Z_n(\beta)}$$

- Overlap distribution: Normalized covariance $q(v,v') = \frac{\mathbb{E}[X_v X'_v]}{\mathbb{E}[X_v^2]}$

$$x_{\beta,n}(r) := \mathbb{E}\big[\langle \ 1_{\{q(v,v')\leq r\}} \ \rangle_{\beta,n}^{\times 2}\big]$$

Extremal Statistics for IID variables (REM model)

Suppose $(X_v, v \in \mathcal{X}_n)$ are IID of variance $n \log 2$.

- Free energy:

$$\lim_{n\to\infty} \frac{1}{\log 2^n} \log Z_n(\beta) := \begin{cases} \log 2 + \frac{\beta^2 \log 2}{2} & \beta \leq \beta_c := \sqrt{2} \\ \sqrt{2} \log 2\beta & \beta \geq \beta_c \end{cases}$$

- Overlap distribution: Normalized covariance $q(v,v') = \delta_{vv'}$

$$x_\beta(dr) := \lim_{n\to\infty} \mathbb{E}\big[\langle\ 1_{\{q(v,v')\in dr\}}\ \rangle_{\beta,n}^{\times 2}\big] = \frac{\beta_c}{\beta}\delta_0 + (1 - \frac{\beta_c}{\beta})\delta_1$$

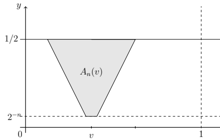
- Gibbs measure: For $\beta > \beta_c$, the Gibbs weights

$$\big(\ \langle\ \delta_v\ \rangle_{\beta,n}\ , v \in \mathcal{X}_n\big)_{\downarrow} \to \mathrm{PD}(\beta_c/\beta)$$

$\mathrm{PD}(\alpha) = (\eta_i / \sum_j \eta_j, i \in \mathbb{N})_\downarrow$ where (η_i) is Poisson$(s^{-\alpha-1}ds)$ on $\mathbb{R}^+$.

The REM is said to exhibit 1-RSB.

Main results: Free energy



The Gaussian field $(X_v, v \in \mathcal{X}_N)$ (cones) is 1-RSB:

Theorem (A-Zindy '12)

The free energy is the same as the REM:

$$\lim_{n\to\infty} \frac{1}{\log 2^n} \log Z_n(\beta) = \begin{cases} \log 2 + \frac{\beta^2 \log 2}{2} & \beta \leq \beta_c := \sqrt{2} \\ \sqrt{2} \log 2\beta & \beta \geq \beta_c \end{cases} \quad a.s.\ and\ in\ L^1$$

- This was shown for BBM (hierarchical) by Derrida & Spohn '88 based on the work of Bramson '78.
- The result for non-hierarchical field was conjectured by Carpentier & Ledoussal '00.
- The result follows from the works on 2DGFF of Bolthausen, Deuschel & Giacomin '01, and Daviaud '06.

Main results: 1-RSB and PD weights

Theorem (A-Zindy '12)

The joint distribution of overlaps is the same as the REM: for $\beta \geq \beta_c$

1.
$$x_\beta(dr) = \lim_{n\to\infty} \mathbb{E}\big[\langle\ 1_{\{q(v,v')\in dr\}}\ \rangle_{\beta,n}^{\times 2}\big] = \frac{\beta_c}{\beta}\delta_0 + (1-\frac{\beta_c}{\beta})\delta_1$$

2.
$$\mathbb{E}\Big[\Big\langle\ F(\{q(v_k,v_l)\})\ \Big\rangle_{\beta,n}^{\times s}\Big] \to E\Big[\sum_{i_1,,i_s} \xi_{i_1}\ldots\xi_{i_s} F(\{\delta_{kl}\})\Big]$$

where $(\xi_i, i\in\mathbb{N})_\downarrow$ are PD(β_c/β).

- This was shown in the hierarchical case by Bovier & Kurkova '04.
- This shows the Ultrametricity Conjecture for the field considered: Correlations not hierarchical for finite n, but are in the limit $n\to\infty$!
- Open questions: What about other test-functions ? Conjectured in Duplantier, Rhodes, Sheffield & Vargas '12

Ideas of the Proofs

The method of proof is robust and is applicable to other log-correlated fields

- Bacry & Muzy construction on $[0,1]^d$ (Multifractal Random Measure)
- 2D discrete Gaussian free field

We restrict to the 1D case for simplicity.

1. GG Identities and AC Stochastic Stability
2. Multi-scale decomposition
3. Bovier & Kurkova technique '04
4. Tree approximation (Bolthausen-Deuschel-Giacomin '01 and Daviaud '06)

1. Gibbs Measures of Gaussian Fields

Theorem (Panchenko '10)

If the free energy is differentiable at $\beta > 0$, then the field concentrates:

$$\frac{1}{\log 2^n}\mathbb{E}\langle\Big|X_v - \mathbb{E}[\langle X_v\rangle_{\beta,n}]\Big|\rangle_{\beta,n} \to 0\ .$$

In particular, by integration by parts, for any smooth F,

$$\mathbb{E}\left[\left\langle q(v_1, v_{s+1}) F(\{q(v_i,v_j)\}_{i,j\le s})\right\rangle_{\beta,n}^{\times s+1}\right] =$$

$$\frac{1}{s}\mathbb{E}\left[\left\langle q(v_1,v_2)\right\rangle_{\beta,n}^{\times 2}\right]\mathbb{E}\left[\left\langle F(\{q(v_i,v_j)\})\right\rangle_{\beta,n}^{\times s}\right]$$

$$+\frac{1}{s}\sum_{k=2}^{s}\mathbb{E}\left[\left\langle q(v_1,v_k)F(\{q(v_i,v_j)\})\right\rangle_{\beta,n}^{\times s}\right] + o(1)$$

Ghirlanda-Guerra Identities

1. Gibbs Measures of Gaussian Fields

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$$+\frac{1}{s}\sum_{k=2}^{s}\mathbb{E}\left[\Big\langle q(v_1,v_k) F(\{q(v_i,v_j)\})\Big\rangle_{\beta,n}^{\times s}\right] + o(1)$$

- Similar results obtain for Stochastic Stability (Aizenman-Contucci '98,A-Chatterjee '10).
- In the case where $q(v,v') \to \delta_{vv'}$, the GG identities characterizes PD distributions (Talagrand '03).
- GG identities and Stochastic stability are at the core of Ultrametricity (Aizenman-A '08, Panchenko '09 '12).

1. Gibbs Measures of Gaussian Fields

Theorem (Panchenko '10)

If the free energy is differentiable at $\beta > 0$, then the field concentrates:

$$\frac{1}{\log 2^n}\mathbb{E}\Big|\langle X_v\rangle_{\beta,n} - \mathbb{E}[\langle X_v\rangle_{\beta,n}]\Big| \to 0 \ .$$

In particular, by integration by parts, for any smooth F,

$$\mathbb{E}\left[\Big\langle q(v_1,v_{s+1})F(\{q(v_i,v_j)\})\Big\rangle_{\beta,n}^{\times s+1}\right] =$$

$$\frac{1}{s}\mathbb{E}\left[\Big\langle q(v_1,v_2)\Big\rangle_{\beta,n}^{\times 2}\right]\mathbb{E}\left[\Big\langle F(\{q(v_i,v_j)\})\Big\rangle_{\beta,n}^{\times s}\right]$$

$$+\frac{1}{s}\sum_{k=2}^{s}\mathbb{E}\left[\Big\langle q(v_1,v_k)F(\{q(v_i,v_j)\})\Big\rangle_{\beta,n}^{\times s}\right] + o(1)$$

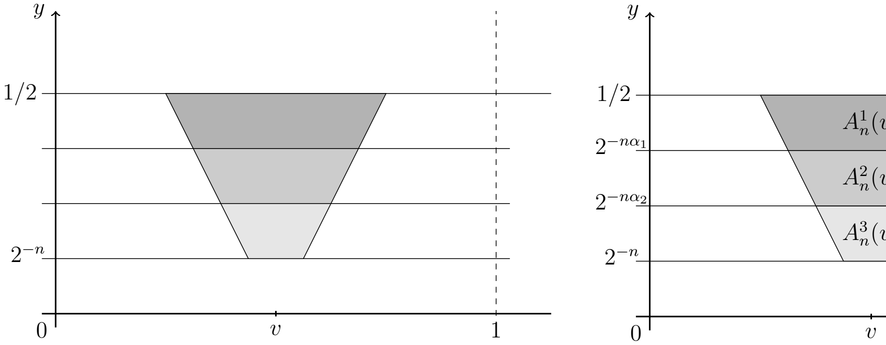
Reduces the problem to computing prove
$x_\beta(r) = \lim_{n\to\infty}\mathbb{E}\Big\langle 1_{\{q(v,v')\le r\}}\Big\rangle_{\beta,n}^{\times 2}$ is 1-RSB.

2. Multi-scale Decomposition

- Independence of disjoint sets $\rightarrow$ multiscale decomposition in strips
- Pick $\alpha = (\alpha_1, \alpha_2)$, $0 < \alpha_1 < \alpha_2 < 1$, and $\sigma = (\sigma_1, \sigma_2, \sigma_3)$.
- Write $Y^{(\sigma,\alpha)} = (Y_v^{(\sigma,\alpha)}, n \in \mathcal{X}_n)$ for the Gaussian field

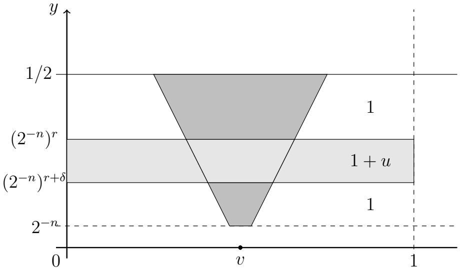
$$Y_v^{(\sigma,\alpha)} = \sigma_1\ \mu(A_n^1(v)) + \sigma_2\ \mu(A_n^2(v)) + \sigma_3\ \mu(A_n^3(v))$$

- This is similar to a GREM (Derrida '85).



3. The Bovier-Kurkova technique

- Bovier & Kurkova '04 obtained the overlap distribution of a continuous version of the GREM by considering perturbation of the model.
- For $0 < r < 1$, δ and u small.



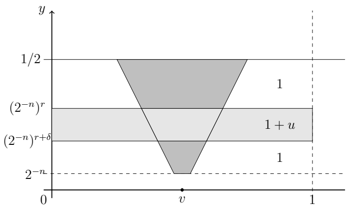
3. The Bovier-Kurkova technique

- For $0 < r < 1$, δ small and u close to 1

$$x_\beta(r) = \lim_{n\to\infty} \mathbb{E}\left\langle 1_{\{q(v,v')\le r\}}\right\rangle_{\beta,n}^{\times 2} \qquad Z_n^{(u,r,\delta)}(\beta) = \sum_{v\in\mathcal{X}_n} e^{\beta Y^{(u,r,\delta)(v)}}$$

Lemma (Bovier & Kurkova '04)

$$\beta^2 \int_r^{r+\delta} x_\beta(s)ds = \frac{d}{du}\left(\lim_{n\to\infty}\frac{1}{n\log 2}\mathbb{E}\log Z_n^{(u,r,\delta)}(\beta)\right)\Big|_{u=0}$$



4. BDG tree approximation

To compute the free energy, it suffices to compute the log-number of high points

$$\mathcal{E}^{(\sigma,\alpha)}(\gamma) = \lim_{n\to\infty} \frac{\log \#\{v \in \mathcal{X}_n : Y_v^{(\sigma,\alpha)} \geq \gamma\sqrt{2}\log 2n\}}{\log 2^n} \text{ in prob.}$$

Theorem

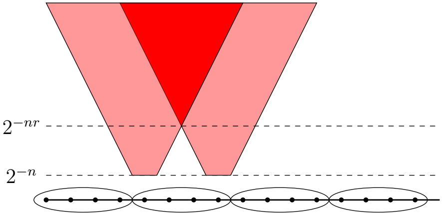
- *Daviaud '06: Case* $\sigma_1 = \sigma_2 = \sigma_3 = 1$

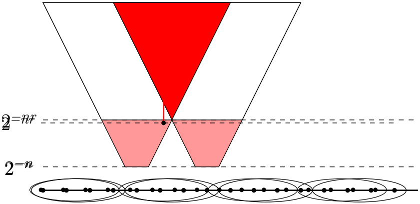
$$\mathcal{E}(\gamma) = 1 - \gamma^2 \text{ (like IID)}$$

- *A-Zindy '12: The number of high points* $\mathcal{E}^{(\sigma,\alpha)}(\gamma)$ *is the same as for the GREM*(σ,α).

4. BDG tree approximation

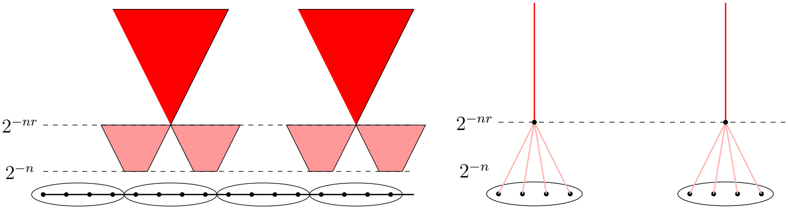
- Divide the 2^n points into 2^{nr} boxes with $2^{n(1-r)}$ points/box (offspring)
- Contribution at scale 2^{-nr} is not the same for the points in the box.
- Log-Miracle #1:
  Non-common part is smaller than the common part: $1 \ll rn \log 2$.





4. BDG tree approximation

- The offspring within a box are not independent.
- Log-Miracle #2:
  The offspring of two boxes are independent at scale below 2^{-nr}.
  Enough independent boxes for the offspring to reach a high value.



- Bolthausen, Deuschel & Giacomin '01 and Daviaud '06 uses this approximation to compute the first order of the maximum and the log-number of high points in the 2D GFF.

Beyond the Gibbs measure: the Extremal Process

The analysis of the extremal process

$$(X_v, v \in \mathcal{X}_n) \text{ close to } \max_v X_v$$

is much more delicate than the one of the Gibbs measure.

Not in the same universality class as the REM.

Hierarchical case

- Bramson '78: for BBM, $\max_v X_v - m(n)$ converges as $n \to \infty$ for an appropriate $m(n)$.
- The limit law is not Gumbel as in the REM.
- $(X_v - m(n), v \in \mathcal{T}_n)$ converges to a Poisson cluster process (A, Bovier & Kistler '11, Aïdekon. Berestycki, Brunet & Shi '11).

Beyond the Gibbs measure: the Extremal Process

The analysis of the extremal process

$$(X_v, v \in \mathcal{X}_n) \text{ close to } \max_v X_v$$

is much more delicate than the one of the Gibbs measure.

Universality class of log-correlated fields

Non-Hierarchical case (cones, 2DGFF, etc)

- The extremal process should be like the one of BBM: Carpentier & Ledoussal '00, Fyodorov & Bouchaud '08.
- Recent results: BDG '01, Bramson & Zeitouni '10, Ding & Zeitouni '12, Duplantier, Rhodes, Sheffield & Vargas '12
- Convergence of max not proved...