Probabilité et Physique Statistique des Systèmes Désordonnés

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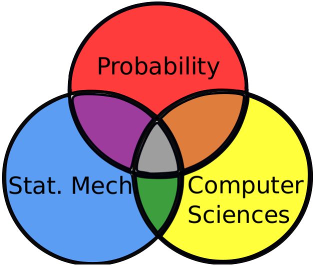
Outline



The Dean's Problem

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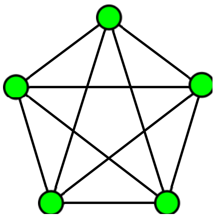

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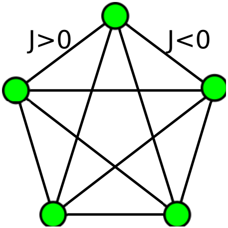

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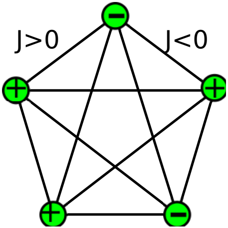

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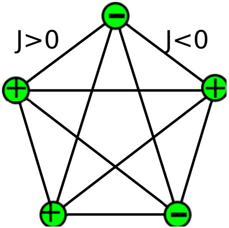
- Assignment: $\sigma(i) = +1$ or -1
- The pair (i, j) is satisfied iif $\sigma(i)\sigma(j) = \text{sgn } J_{ij}$.
- The functional to minimize is

$$H_{N,J}(\sigma) = -\sum_{i \neq j} J_{ij}\ \sigma(i)\sigma(j)$$

over configurations $\sigma \in \Sigma_N = \{-1, +1\}^N$:

$$\sigma : \{1, \ldots, N\} \to \{-1, +1\}$$
$$i \mapsto \sigma(i)\ .$$

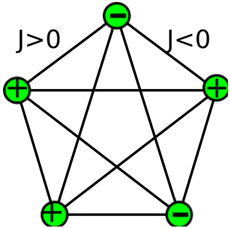


The Dean's Problem

Question

For N large,

- *Describe minimum*
  $\min_{\sigma \in \Sigma_N} \; H_{N,J}(\sigma)$ *?*
- *Describe minimizers*
  How different are two $\sim$ optimal σ and σ' ?



- The Dean's problem is a combinatorial optimization problem.
  Other examples: Traveling Salesman, K-SAT.
- Finding the minimizer of $H_{N,J}$ is NP complete.

The Deans's Problem

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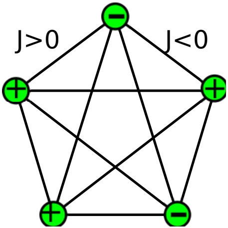

Why is it hard ?

The Deans's Problem

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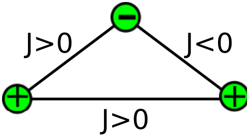

Why is it hard ?

- Frustrations:
  Odd number of $J_{ij} < 0$ in cycle $\Rightarrow$ Some constraints are unsatisfied.
- A "Typical" J has some $J_{ij} < 0$.

Random Constraints and SK model

- $\sim$ Typical:

$$(J_{ij};\ 1 \leq i < j \leq N) \text{ i.i.d. Gaussian r.v. mean 0 variance } \frac{1}{\sqrt{N}}.$$

  Disorder!

- This the Sherrington-Kirkpatrick model in statistical physics (1975):

$$H_{N,J}(\sigma) = -\sum_{i \neq j} J_{ij}\ \sigma(i)\sigma(j)$$

- It is a model of a spin glass on a complete graph.
  Other graphs, say $\mathbb{Z}^d$, much harder !

The Probability Perspective

Write $\mathbb{P}$ for the law of $J = (J_{ij})$ i.i.d. Gaussians.

$$J \mapsto (H_{N,J}(\sigma), \sigma \in \Sigma_N) \text{ is a Gaussian process}$$

- Covariance $\quad \mathbb{E}[H_{N,J}(\sigma)H_{N,J}(\sigma')] = N R_N(\sigma,\sigma')^2 - 1$
- Overlap $\quad R_N(\sigma,\sigma') = \frac{1}{N}\sum_{i=1}^N \sigma(i)\sigma'(i)$

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- The correlations are regular, strong, and high-dimensional.
- R_N is a symmetric, positive definite form on Σ_N ($\sim$ inner product).
- R_N is related to the Hamming distance on Σ_N

$$R_N(\sigma,\sigma') = 1 - \frac{2\#\{i : \sigma(i) \neq \sigma'(i)\}}{N}$$

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Question

For N large,

1. *Describe the random variable* $\min_{\sigma \in \Sigma_N} \; H_{N,J}(\sigma)$ *?*
2. *Describe the correlations* $R_N(\sigma, \sigma')$ *for* σ *and* $\sigma' \sim$ *minimizers.*

As probabilists, we are looking for universal structures.

The Statistical Physics Approach

Question

Does the limit $\frac{1}{N}\min_{\sigma\in\Sigma_N} H_{N,J}(\sigma)$ *exist ?*

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- Gibbs Measure: A probability measure on assignments Σ_N

$$G_{\beta,N,J}(\sigma) = \frac{\exp -\beta H_{N,J}(\sigma)}{Z_{N,J}(\beta)}, \ \beta > 0$$

β is the inverse temperature.

Gibbs measure for β large: information on the minima of $(H_{N,J}(\sigma), \sigma \in \Sigma_N)$ and their relative location.

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- Free Energy: $f_{N,J}(\beta) = \frac{1}{N}\log\sum_{\sigma\in\Sigma_N}\exp -\beta H_{N,J}(\sigma)$.

Note that

$$-\frac{\log 2}{\beta} + \frac{\min H_{N,J}(\sigma)}{N} \leq \frac{-1}{\beta} f_{N,J}(\beta) \leq \frac{\min H_{N,J}(\sigma)}{N}$$

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GOAL: Understand the free energy and the Gibbs measure

Concentration of Measure Phenomenon

Lemma (Self-Averaging)

For J i.i.d. Gaussians of law $\mathbb{P}$,

$$\left|f_{N,J}(\beta) - \mathbb{E}[f_{N,J}(\beta)]\right| \to 0 \ \mathbb{P}\text{-almost surely as } N \to \infty$$

- Common instance: the law of large number.
- Here $f_{N,J}$ is not a linear function of J, but it is Lipshitz.
- Concentration of measure is a pillar of today's probability (see e.g. Ledoux-Talagrand).

An approximation

- Jensen's Inequality:

$$f_N(\beta) \leq \frac{1}{N} \log \mathbb{E}\Big[\sum_{\sigma \in \Sigma_N} \exp -\beta H_{N,J}(\sigma) \Big] = \log 2 + \frac{\beta^2}{2}$$

Equality should hold for β small.

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Equality should hold for β small.

- Suppose that the variables $H_{N,J}(\sigma)$ are independent ! $R_N(\sigma, \sigma') = \delta_{\sigma\sigma'}$

$$\mathbb{P}\Big(\min H_{N,J}(\sigma) > x \Big) = \Big(1 - \int_{-\infty}^{x} \frac{e^{-y^2/2N}}{\sqrt{2\pi N}} \, dy \Big)^{2^N}$$

$$\min H_{N,J}(\sigma) = N\sqrt{2 \log 2} + \text{Fluct.}$$

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Theorem (Derrida '81)

In the case where $(H_{N,J}(\sigma), \sigma \in \Sigma_N)$ *are i.i.d. Gaussian variables* $R_N(\sigma, \sigma') = \delta_{\sigma\sigma'}$

$$f(\beta) = \lim_{N \to \infty} \mathbb{E}[f_{N,J}(\beta)] = \begin{cases} \log 2 + \frac{\beta^2}{2} & \text{if } \beta \leq \sqrt{2 \log 2} \\ \beta\sqrt{2 \log 2} & \text{if } \beta > \sqrt{2 \log 2} \end{cases}.$$

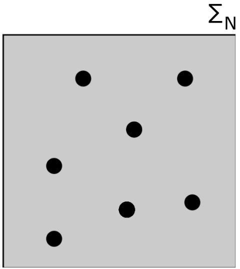
A Phase Transition

The approximation $R_N(\sigma,\sigma') = \delta_{\sigma\sigma'}$ is surprisingly complex:

1. Phase transition: $f(\beta)$ is singular at $\beta_c = \sqrt{2\log 2}$
2. Nature of the transition:

$$\frac{d}{d\beta} f(\beta) \overset{\text{Convexity}}{=} \lim_{N\to\infty} \frac{d}{d\beta} \mathbb{E} f_{N,J}(\beta)$$

$$\overset{\text{Gaussian}}{=} \lim_{N\to\infty} \beta \ \mathbb{E} \ G^{\times 2}_{\beta,N,J}[1 - R_N(\sigma,\sigma')]$$



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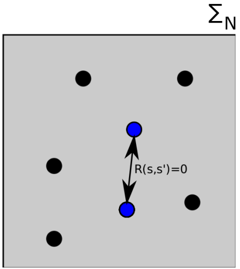

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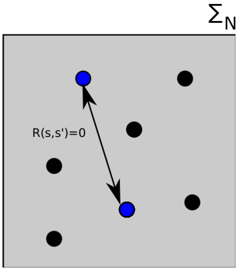

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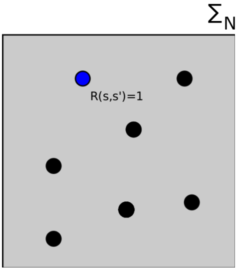

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2. Transition of what ?:

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$$\overset{\text{Gaussian}}{=} \lim_{N\to\infty} \beta\, \mathbb{E}\, G^{\times 2}_{\beta,N,J}[1 - R_N(\sigma,\sigma')]$$

$$\lim_{N\to\infty} \mathbb{E}\, G^{\times 2}_{\beta,N,J}[R_N(\sigma,\sigma')] = \lim_{N\to\infty} \mathbb{E}\, G^{\times 2}_{\beta,N,J}\{\sigma = \sigma'\}$$

$$= \begin{cases} 0 & \text{if } \beta \le \sqrt{2\log 2} \\ 1 - \frac{\sqrt{2\log 2}}{\beta} & \text{if } \beta > \sqrt{2\log 2} \end{cases}.$$

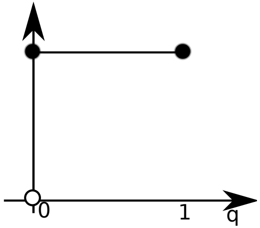
Some σ's have macroscopic weight ! More than one.

A Phase Transition

Order Parameter is a distribution function c.d.f.:

$$x_\beta(q) = \lim_{N\to\infty} \mathbb{E}\, G^{\times 2}_{\beta,N,J}\{R_N(\sigma,\sigma') \le q\}$$

$\beta \le \beta_c$



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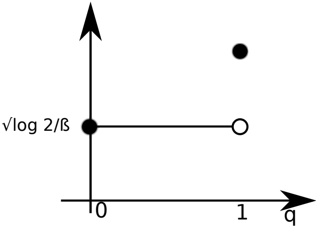


The Gibbs Measure

Universal results for $G_{\beta,N,J}$?

$(H_{N,\beta,J}(\sigma), \sigma \in \Sigma_N)$ Gaussian field with R_N generic.

Theorem (A & Chatterjee '10)

If $f(\beta)$ is differentiable at β, then the Gibbs measure is stochastically stable:

$$\left| \mathbb{E}\tilde{G}^{\times s}_{\beta,N,J} F(\sigma_1, \ldots, \sigma_s) - \mathbb{E}G^{\times s}_{\beta,N,J} F(\sigma_1, \ldots, \sigma_s) \right| \to 0$$

for

$$\tilde{G}_{\beta,N,J}(\sigma) = \frac{e^{g(\sigma)} \ \exp -\beta H_{N,\beta,J}(\sigma)}{Norm.}$$

g, a Gaussian field of covariance $R_N(\sigma, \sigma')$ independent of $H_{N,J}$.

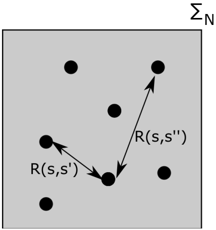
- Limit of $(G_{\beta,N,J})_N$ exists ? OK along subsequences

The Gibbs Measure

What do we learn from stochastic stability ?

Theorem (A & Chatterjee '10)

If the Gibbs measure is stochastically stable, then its limit measures are supported on an ∞-dim. space (or on only one point).



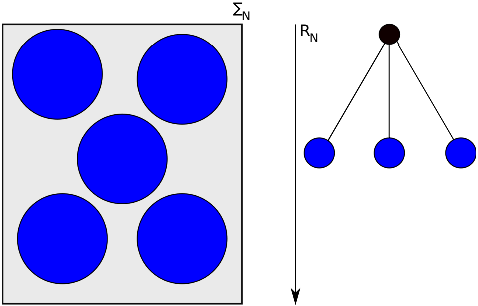
Theorem (Independent Case)

If $R_N(\sigma,\sigma') = \delta_{\sigma,\sigma'}$, then $(G_{\beta,N,J})_N$ converges to an atomic measure with Poisson-Dirichlet weights.

The Parisi Ultrametricity Conjecture

- The physicists (G. Parisi '80) proposed the following picture for the support of the Gibbs measure for high β:

- Solutions form equivalence classes of macroscopic weights.
- Distance is well defined and ultrametric.



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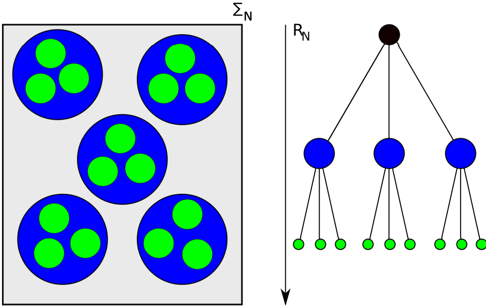

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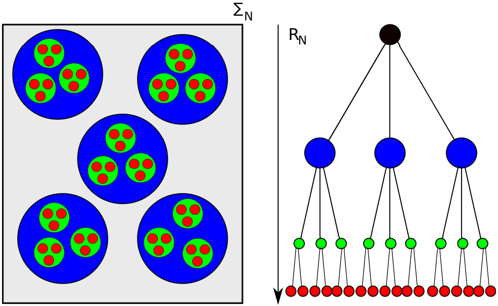
- Distance is well defined and ultrametric.
- This structure repeats itself within classes.
- Again and again...

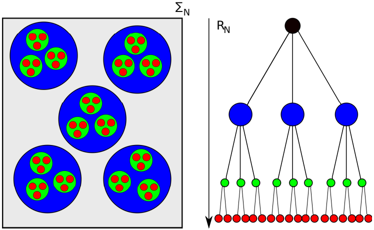


Optimal solutions are organized in a genealogical way. Why?

The Parisi Ultrametricity Conjecture

What is PROVED ?

- A & Aizenman '08:
  Stochastic stability (in a strong sense) ⇒ Parisi picture
  when $R_N(\sigma, \sigma')$ takes finite number of values (unif. in N).
- Panchenko ('09) SAME using strong moment identities.
- The whole structure is a Bolthausen-Sznitman coalescent.



- Strong: Holds for a perturbation of SK model but not SK.
- Enough to prove a formula for the free energy.

The Parisi Formula

Given $\psi \in C^2(\mathbb{R})$ and $x : [0,1] \to [0,1]$ a c.d.f.
Define the functional P_ψ by the evolution:

- $f(1,y) := \psi(y)$.
- $\partial_q f(q,y) + \frac{1}{2}\left(\partial_y^2 f(q,y) + x(q)(\partial_y f(q,y))^2\right) = 0$
- $P_\psi(x) = f(0,0)$.

Theorem (Parisi Formula for the SK model)

$$\lim_{N\to\infty} f_N(\beta) = \min_{x:[0,1]\to[0,1]\, c.d.f} \{P_{\psi_1}(x) - P_{\psi_2}(x)\}$$

for $\psi_1(y) = \log 2\cosh(\beta y)$ *and* $\psi_2(y) = \beta y$.

Order parameter

$$x_\beta(q) = \lim_{N\to\infty} \mathbb{E}\, G^{\times 2}_{\beta,N,J}\{R_N(\sigma,\sigma') \le q\}$$

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for $\psi_1(y) = \log 2\cosh(\beta y)$ *and* $\psi_2(y) = \beta y$.

- Phase transition at $\beta_c = 1$ (Toninelli '02)
- Below β_c same as IID (Aizenman Lebowitz Ruelle '87)
- Proved by Guerra and Talagrand '03 (Not constructive).
- Constructive proof IF Parisi picture holds: Panchenko '10, A&Chatterjee '10.

Open Questions

What is NOT PROVED ?

- What is $x(q)$ above $\beta > \beta_c$? Conjecture: It is a continuous function!
- Does the Parisi picture hold for pure SK (ok for perturbed model) ?
- What about SK model on a different graph, say $\mathbb{Z}^d$?
  Is the picture as complex ?!?!