Extrema of Branching Brownian Motion

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joint work with
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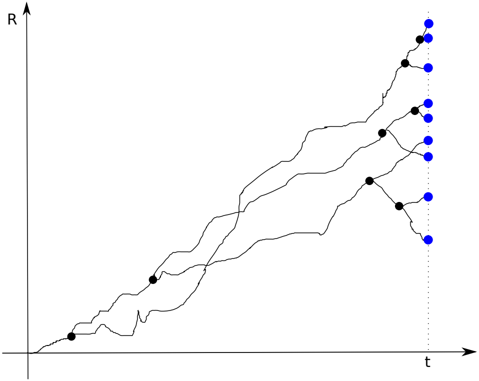
Branching Brownian Motion

1. Consider a Brownian motion $X(t)$ starting at 0 at time 0.
2. After random time T, $\mathbb{P}(T > t) = e^{-t}$, the particle splits into two.
3. The particles are iid BM's starting at $X(T)$ with same splitting rule.



Branching Brownian Motion

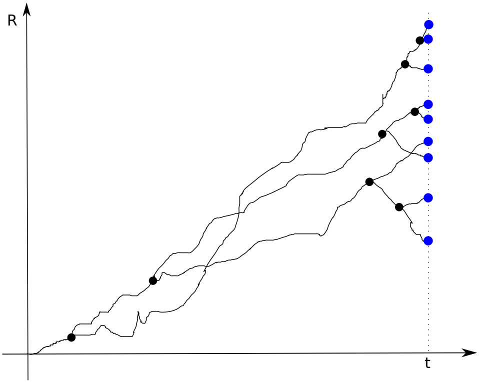
- The branching process produces a binary tree with leafs $\mathcal{T}_t$.
- Number of leafs $|\mathcal{T}_t|$ is independent of the positions and

$$\mathbb{E}|\mathcal{T}_t| = e^t \ .$$

- The process of positions is Gaussian, conditionally on $\mathcal{T}_t$

$$(X_\sigma(t), \sigma \in \mathcal{T}_t)$$

- $\mathbb{E}[X_\sigma(t) X_{\sigma'}(t) | \mathcal{T}_t]$ is the time $T_{\sigma\sigma'}$ where the ancestor of σ and σ' splits.



Statistics at the Edge ?

- We are interested in the process $(X_\sigma(t), \sigma \in \mathcal{T}_t)$ as a Gaussian process on $\sim e^t$ r.v. with high correlations, the branching times.

Questions of Interest in Extremal Statistics

1. Maximum
   Find $a(t)$ and $b(t)$ such that
   $$\mathbb{P}\left(\max_{\sigma \in \mathcal{T}_t} \frac{X_\sigma(t) - a(t)}{b(t)} \leq x\right) \to \omega(x) \text{ as } t \to \infty.$$

2. Extremal Process
   Show that under $\mathbb{P}$
   $$\mathcal{E}(t) = \sum_{\sigma \in \mathcal{T}_t} \delta_{\frac{X_\sigma(t) - a(t)}{b(t)}} \to \mathcal{E} \quad \text{as } t \to \infty$$

   SPACE average

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   SPACE average
3. Ergodicity
   Show that on a set of trajectories of $\mathbb{P}$-probability one,
   $$\frac{1}{T}\int_0^T 1_{\left\{\max \frac{X_\sigma(t) - a(t)}{b(t)} \leq x\right\}} \to \omega'_X(x)$$
   TIME average

1. The Maximum

The Law of the Maximum of BBM

Define $u(t,x) = \mathbb{P}\Big(\max_{\sigma \in \mathcal{T}_t} \; X_\sigma(t) \le x \Big)$

Theorem (McKean '76)

$u(t,x)$ is solution of the KPP equation:

$$\boxed{\partial_t u = \tfrac{1}{2}\partial_{xx} u + u^2 - u \ ,}$$

with initial condition $u(0,x) = \mathbb{1}_{[0,\infty)}(x)$.

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Theorem (KPP '37)

For $m(t) = \sqrt{2}t + o(t)$,

$$\boxed{u(t, x + m(t)) \to \omega(x)}$$

$\omega(x)$ is the unique solution to $\frac{1}{2}\omega'' + \sqrt{2}\omega' + \omega^2 - \omega = 0$ up to shift.

BBM right tail: $1 - \omega(x) \sim x e^{-\sqrt{2}x}$

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Theorem (Bramson '78-'83)

$$\boxed{m(t) = \sqrt{2}t - \frac{3}{2\sqrt{2}} \log t}.$$

The law of the maximum is tight around $m(t)$.
Convergence holds for the same $m(t)$ for initial conditions similar to $u(0,x) = \mathbb{1}_{[0,\infty)}(x)$.

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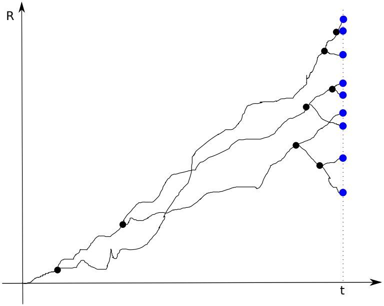
Theorem (Lalley & Sellke '87)

$$\boxed{\omega(x) = \mathbb{E}[e^{-CZe^{-\sqrt{2}x}}]}$$

$\omega(x)$ is a Random shift of a double-exp!
$Z(t) = \sum_{\sigma \in \mathcal{T}_t} \left(\sqrt{2}t - X_\sigma(t)\right) \, e^{-\sqrt{2}(\sqrt{2}t - X_\sigma(t))} \to Z$

2. The Extremal Process



The Extremal Process

Define the random measure (point process)

$$\mathcal{E}(t) = \sum_{\sigma} \delta_{X_\sigma(t) - m(t)}.$$

Consider the Laplace transform

$$u_\phi(t,x) = \mathbb{E}\left[\exp - \sum_{\sigma} \phi(x + X_\sigma(t))\right] .$$

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Lemma (Brunet & Derrida '10; ABK '11)

$u_\phi(t,x)$ satisfies KPP equation with $u(0,y) = \exp - \phi(y)$.

$$u_\phi(t, x + m(t)) \to \mathbb{E}\left[e^{-C_\phi Z e^{-\sqrt{2}x}}\right] .$$

- $\Rightarrow \mathcal{E}(t) \to \mathcal{E}$. What is $\mathcal{E}$?

Main Result

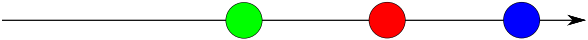
Theorem (ABK '10-'11)

The point process $\mathcal{E}$ is the random shift of a Poisson cluster process:

$$\mathcal{E} = \sum_{i,j} \delta_{\xi_i + \Delta_{i,j}}$$

where

1. ξ *is Poisson*$(CZ\ \sqrt{2}e^{-\sqrt{2}x}dx)$



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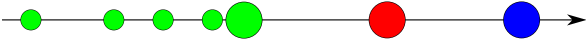
$$\Delta = \lim_{t\to\infty} \sum_{\sigma\in\mathcal{T}_t} \delta_{X_\sigma(t) - \max X_\sigma(t)} \ \textit{cond.}\ \left\{\max X_\sigma(t) > \sqrt{2}t\right\}$$



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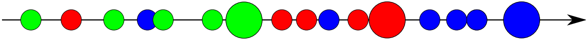
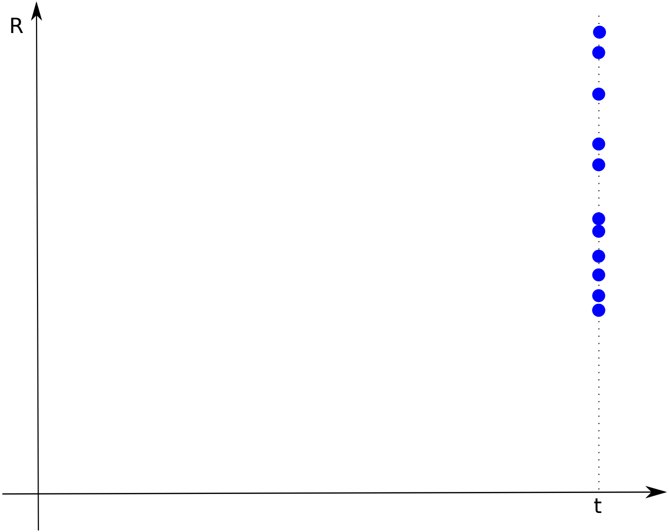


The picture of the proof

- Understand the correlations/branching times at the edge.



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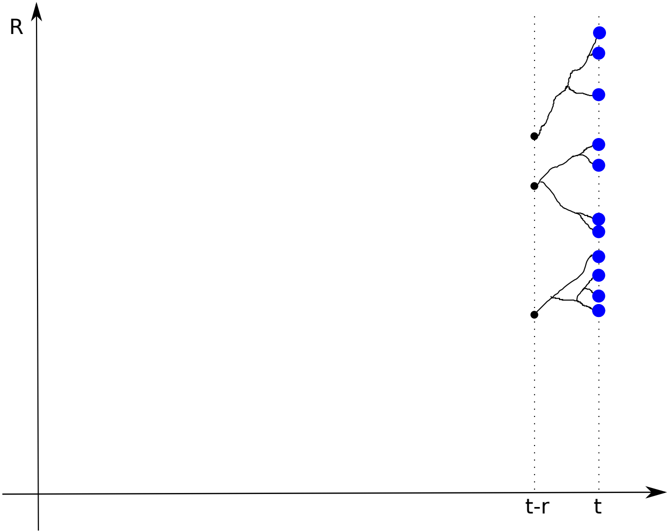
- For $1 < r \ll t$



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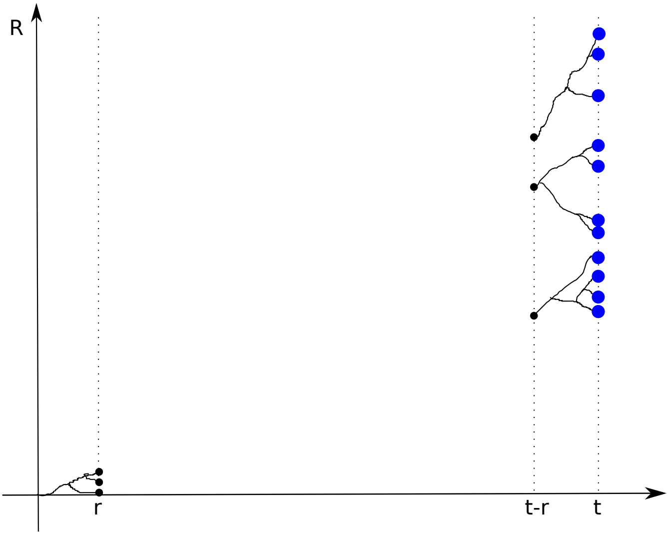
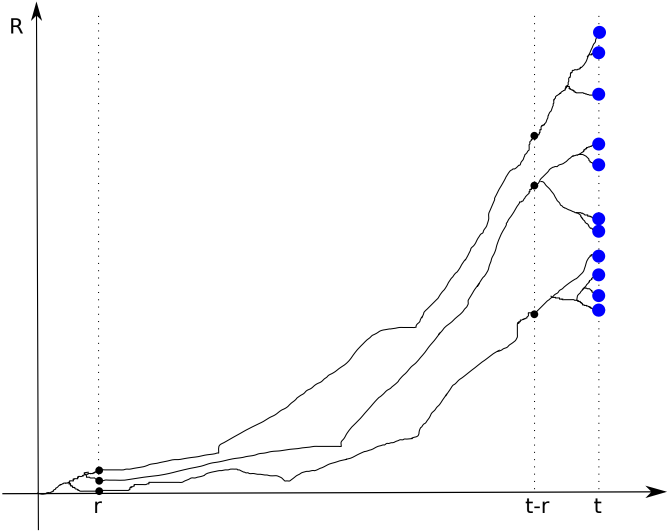


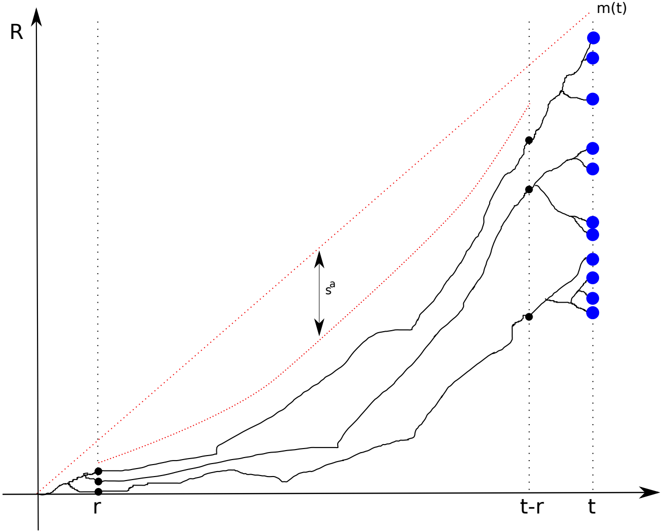
The picture of the proof

- For $1 < r \ll t$, with high probability, the typical picture is:



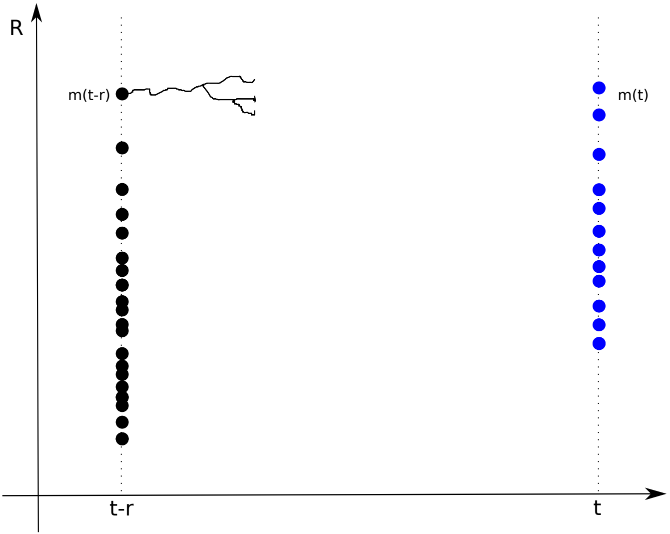
The picture of the proof

- Z is produced by the early times.
- Poissonian statistics comes from the independence of the paths.



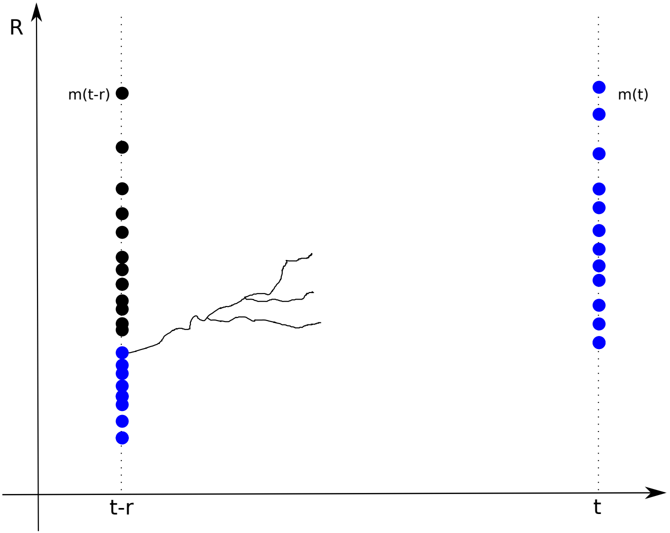
The picture of the proof

- $r \ll t$
- BBM arithmetic: $m(t) = m(t-r) + \sqrt{2}r + o_t(1)$
- BUT $m(r) \ll \sqrt{2}r$



The picture of the proof

- The maximal particles come from the tail!
  Occupy the Edge: The 1% of tomorrow comes from the 99% of today.



Spin Glass Heuristics

1. Find an auxiliary process that mimics the tail

$$\text{Poisson}\Big(\sqrt{\frac{2}{\pi}}(-x)e^{-x\sqrt{2}}dx\Big) \text{ on } (-\infty, 0).$$

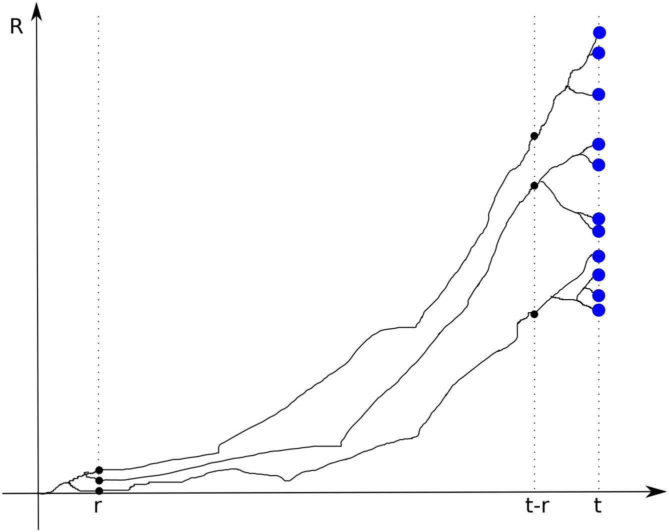
   Tail is simpler!

2. The process $(X_\sigma(r) - \sqrt{2}r)_\sigma$ conditioned on $\{\max_\sigma X_\sigma(r) > \sqrt{2}r\}$ exists in the limit $r \to \infty$.

Properties of the conditioned process:

- The law does not depend on the starting point $\Rightarrow$ IID Clusters.
- The law of the maximum is exponential.
- Open question: More properties.

The Extremal Process of BBM



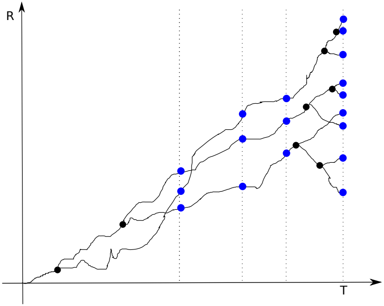
3. Ergodicity

An ergodic theorem

Conjecture (Lalley & Sellke '87)

On a set of trajectories of $\mathbb{P}$-probability one,

$$\frac{1}{T}\int_0^T 1_{\{\max X_\sigma(t)-m(t)\leq x\}}\, dt \to \exp -C{\color{red}Z}e^{-\sqrt{2}x} \quad \forall x \ .$$



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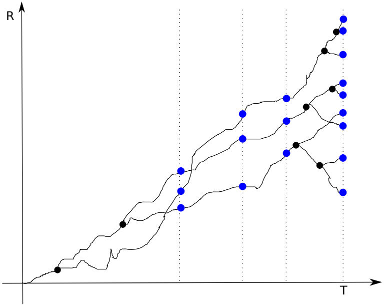

Theorem (ABK '11-'12)

True for the maximum and the particle system.

Idea of the Proof

There are two ingredients:

1. $$\lim_{r\to\infty}\lim_{t\to\infty}\mathbb{P}(\max X_\sigma(t)-m(t)\le x|\mathcal{F}_r)=\exp -CZe^{-\sqrt{2}x}$$

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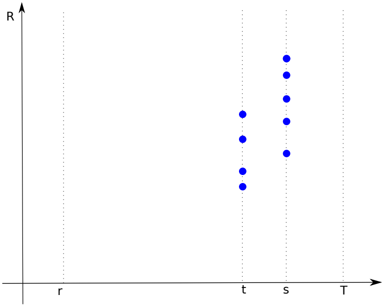
2. Correlations/branching times between the high points at two different times ?



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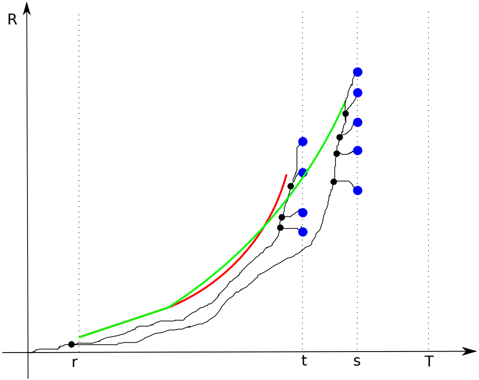

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2. Decorrelation at suitable time scales !
   For $Y(t)=1_{\{\max X_\sigma(t)-m(t)\le x\}}\ dt-\mathbb{P}(\max X_\sigma(t)-m(t)\le x|\mathcal{F}_r)$

$$\mathbb{E}\left[\left(\frac{1}{T}\int_{\varepsilon T}^{T}Y(t)\right)^2\Big|\mathcal{F}_r\right]$$
$$=\frac{1}{T^2}\int_{\varepsilon T}^{T}\int_{\varepsilon T}^{T}\mathbb{E}[Y(t)Y(s)\mid\mathcal{F}_r]\ dtds$$

- Decorrelation yields a weak law. Strong law if summability condition holds (Lyons '88).

Conclusion: A New Universality Class ?

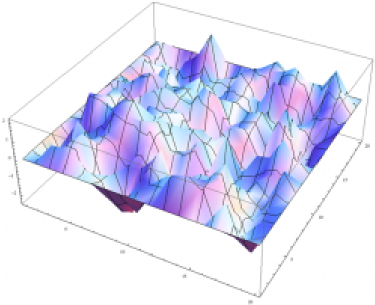
The University Class of BBM

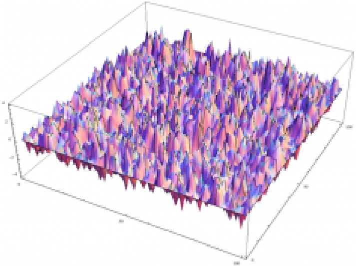
Conjecture

2D discrete GFF is in the same class as BBM for Extreme Value Statistics

$$\mathbb{E}x_i(t)x_j(t) = \mathbb{E}\sum_{n=0}^{\tau_V} 1_{S_i(n)=j}, \qquad V = \{1, .., e^{t/2}\} \times \{1, ..., e^{t/2}\} \ .$$

In particular, $1 - \omega_{\max}(x) \sim xe^{-\sqrt{2}x}$.





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- $\Big(\max x_i(t) - \mathbb{E}\max x_i(t)\Big)_t$ is tight (Bramson & Zeitouni '10)
- Other models falling in this class: log-correlated Gaussian fields (see Carpentier & Ledoussal '01, Fyodorov & Bouchaud '08).
- Work in progress...

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