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A Generalization of a Theorem of Boyd and Lawton

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Abstract. The Mahler measure of a nonzero n-variable polynomial P is the integral of $\log|P|$ on the unit n-torus. A result of Boyd and Lawton says that the Mahler measure of a multivariate polynomial is the limit of Mahler measures of univariate polynomials. We prove the analogous result for different extensions of Mahler measure such as generalized Mahler measure (integrating the maximum of $\log|P|$ for possibly different P's), multiple Mahler measure (involving products of $\log|P|$ for possibly different P's), and higher Mahler measure (involving $\log^k|P|$).

1 Introduction

The Mahler measure of a nonzero polynomial $P(x_1,\dots,x_n) \in \mathbb{C}[x_1,\dots,x_n]$ is defined by

$$\mathrm{m}(P) := \frac{1}{(2\pi i)^n}\int_{\mathbb{T}^n} \log|P(x_1,\dots,x_n)|\,\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n},$$

where $\mathbb{T}^n = \{(z_1,\dots,z_n) \in \mathbb{C}^n : |z_1| = \cdots = |z_n|\}$ is the unit torus in dimension n. This formula has a particularly simple expression for univariate polynomials. If $P(x) = a\prod_i (x-\alpha_i)$, then Jensen's formula implies that $\mathrm{m}(P) = \log|a| + \sum_i \max\{0, \log|\alpha_i|\}$. In fact, Lehmer [Le33] only considered the measure for univariate polynomials which was later extended to multivariate polynomials by Mahler [Ma62]. Lehmer's motivation for considering this object was finding a method to construct large prime numbers that generalizes Mersenne's sequence. Mahler, on the other hand, was interested in relating heights of products of polynomials with the heights of the factors. The Mahler measure is a height that is multiplicative, and therefore it was a natural object for Mahler to study.

Boyd and Lawton proved the following useful and interesting result.

Theorem 1.1 ([Bo81a, Bo81b, La83]) *Let $P(x_1,\dots,x_n) \in \mathbb{C}[x_1,\dots,x_n]$ and $\mathbf{r} = (r_1,\dots,r_n)$, $r_i \in \mathbb{Z}_{>0}$. Define $P_{\mathbf{r}}(x)$ as $P_{\mathbf{r}}(x) = P(x^{r_1},\dots,x^{r_n})$, and let*

$$q(\mathbf{r}) = \min\Big\{H(\mathbf{t}) : \mathbf{t} = (t_1,\dots,t_n) \in \mathbb{Z}^n,\ \mathbf{t} \neq (0,\dots,0),\ \sum_{j=1}^n t_j r_j = 0\Big\},$$

where $H(\mathbf{t}) = \max\{|t_j| : 1 \le j \le n\}$. Then

$$\lim_{q(\mathbf{r})\to\infty} \mathrm{m}(P_{\mathbf{r}}) = \mathrm{m}(P).$$

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This result implies that the multivariate Mahler measure is a limit of univariate Mahler measures. In particular, it gives evidence that the extension to multivariate polynomials is the right generalization.

The Mahler measure of multivariate polynomials often yields special values of the Riemann zeta function and L-functions; thus one can construct sequences of numbers that approach these special values in this way.

In addition, this theorem has consequences in terms of limit points of Mahler measure. The most famous open question in this area is the so-called Lehmer's question. *Is there a constant $c > 0$ such that for every polynomial $P \in \mathbb{Z}[x]$ with $m(P) > 0$, then $m(P) \geq c$?* Thus, Theorem 1.1 tells us that given a multivariate polynomial whose measure is smaller than a certain constant c, we can generate infinitely many univariate polynomials with the same property.

In this work, we are going to consider two extensions of Mahler measure.

Given $P_1, \dots, P_s \in \mathbb{C}[x_1, \dots, x_n]$, (not necessarily distinct) nonzero polynomials, the *generalized Mahler measure* is defined in [GO04] by

$$\mathrm{m}_{\max}(P_1, \dots, P_s) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \max\{\log|P_1(x_1, \dots, x_n)|, \dots, \log|P_s(x_1, \dots, x_n)|\} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}.$$

On the other hand, the *multiple Mahler measure* is defined in [KLO08] by

$$\mathrm{m}(P_1, \dots, P_s) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log|P_1(x_1, \dots, x_n)| \cdots \log|P_s(x_1, \dots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}.$$

For the particular case in which $P_1 = \cdots = P_s = P$, the multiple Mahler meausure is called *higher Mahler measure*

$$\mathrm{m}_s(P) := \frac{1}{(2\pi i)^n} \int_{\mathbb{T}^n} \log^s|P(x_1, \dots, x_n)| \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n}.$$

These objects have been related to special values of the Riemann zeta function and L-functions ([GO04, La08] for generalized Mahler measure, [KLO08, Sa10, BS, BBSW] for multiple Mahler measure), but the nature of this relationship is not as well understood than in the classical case.

Our goal in this note is to prove the equivalent for Theorem 1.1 for these generalizations.

Theorem 1.2 *Let $P_1, \dots, P_s \in \mathbb{C}[x_1, \dots, x_n]$, and let $\mathbf{r}$ be as before. Then*

(i) $\lim_{q(\mathbf{r})\to\infty} \mathrm{m}_{\max}(P_{1\mathbf{r}}, \dots, P_{s\mathbf{r}}) = \mathrm{m}_{\max}(P_1, \dots, P_s)$;
(ii) $\lim_{q(\mathbf{r})\to\infty} \mathrm{m}(P_{1\mathbf{r}}, \dots, P_{s\mathbf{r}}) = \mathrm{m}(P_1, \dots, P_s)$;
(iii) *if* $P_1 = \cdots = P_s = P$, $\lim_{q(\mathbf{r})\to\infty} \mathrm{m}_s(P_{\mathbf{r}}) = \mathrm{m}_s(P)$.

2 Some Preliminary Results

The difficulty in obtaining Theorem 1.2 lies in the case where (some of) the polynomials vanish in the domain of integration and the logarithm is not bounded. This problem already appears in the proof of Theorem 1.1. The key result for solving this is a theorem by Lawton [La83].

Let μ_n denote the Lebesgue measure in the torus $\mathbb{T}^n$.

Theorem 2.1 ([La83, Theorem 1]) *Let $P(x) \in \mathbb{C}[x]$ be a monic polynomial and let k be the number of nonzero coefficients of P. Then if $k \geq 2$, there is a positive constant C_k that depends only on k such that*

$$\mu_1\Big(\{z \in \mathbb{T} : |P(z)| < y\}\Big) \leq C_k y^{\frac{1}{k-1}},$$

for any real number $y > 0$.

The strength of this result lies in the fact that the constant is absolute and depends on the number of nonzero coefficients of P, but it does not depend on P.

Notice that we can always assume that the polynomials involved in multiple Mahler measure have at least two nonzero monomials, since $\log |ax^k|$ is a constant and can be easily extracted from the integral. It should be noted that the above theorem remains true for $k = 1$ if y is sufficiently small (*i.e.*, $y < |a|$) and $C_1 = 0$.

It is not hard to prove a result where the constant depends on P. For example,

Lemma 2.2 ([EW99, Lemma 3.8, p. 58]) *Let $P(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$. There there are constants C_P, δ_P that depend on P such that*

$$\mu_n\Big(\{(z_1, \dots, z_n) \in \mathbb{T}^n : |P(z_1, \dots, z_n)| < y\}\Big) \leq C_P y^{\delta_P}, \tag{2.1}$$

for small $y > 0$.

In what follows, we will let

$$S_n(P, y) = \{(z_1, \dots, z_n) \in \mathbb{T}^n : |P(z_1, \dots, z_n)| < y\},$$

where the n depends on the number of variables involved. Thus n is greater than or equal to the number of variables of P. We will write $S(P, y)$ for $S_1(P, y)$.

The following elementary lemma will be useful to bound integrals.

Lemma 2.3 *Let ℓ be a positive integer and $y, \delta > 0$. Then*

$$\begin{aligned} J_{\ell,\delta}(y) &:= (-1)^\ell \int_0^y \log^\ell z\, d\left(z^\delta\right) \\ &= y^\delta \Big((-1)^\ell \log^\ell y + \frac{\ell}{\delta}(-1)^{\ell-1} \log^{\ell-1} y + \frac{\ell(\ell-1)}{\delta^2}(-1)^{\ell-2} \log^{\ell-2} y + \cdots \\ &\qquad + \frac{\ell(\ell-1)\cdots 2}{\delta^{\ell-1}}(-1) \log y + \frac{\ell!}{\delta^\ell}\Big). \end{aligned}$$

Proof The proof is easily obtained by repeated integration by parts. ■

Corollary 2.4 *For $0 < y \le 1$ we have*

$$0 \le J_{\ell,\delta}(y) \le y^{\delta}(\ell+1)!\max\left\{\frac{1}{\delta}, (-\log y)\right\}^{\ell}.$$

In other words, $\lim_{y\to 0} J_{\ell,\delta}(y) = 0$.

For the remainder of this work, we will let

$$I_{\ell,k}(y) := J_{\ell,\frac{1}{k-1}}(y) = (-1)^{\ell}\int_0^y \log^{\ell} z\, d\big(z^{\frac{1}{k-1}}\big). \tag{2.2}$$

We finish this section by recalling the statement of the following extension of Hölder's inequality.

Lemma 2.5 *Let S be a measurable set of $\mathbb{R}^n$ or $\mathbb{C}^n$ and let $f_1, \dots, f_s$ be measurable complex or real valued functions. Then*

$$\int_S |f_1\cdots f_s|dx \le \left(\int_S |f_1|^s dx\right)^{\frac{1}{s}} \cdots \left(\int_S |f_s|^s dx\right)^{\frac{1}{s}}.$$

3 Integration over Combinations of $S(P, y)$

In this section, we consider the integration over sets resulting from combining the different $S(P, y)$'s.

Lemma 3.1 *Let $P(x) \in \mathbb{C}[x]$ be a polynomial having $k \ge 2$ non-zero complex coefficients each having modulus ≥ 1. Let $0 < y \le 1$. Then*

$$0 \le (-1)^{\ell}\int_{S(P,y)} \log^{\ell}|P(x)|\frac{dx}{x} \le C_k I_{\ell,k}(y).$$

Analogously, if $P(x_1, \dots, x_n) \in \mathbb{C}[x_1, \dots, x_n]$ and $0 < y$ small enough to satisfy equation (2.1),

$$0 \le (-1)^{\ell}\int_{S_n(P,y)} \log^{\ell}|P(x_1,\dots,x_n)|\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n} \le C_P J_{\ell,\delta_P}(y).$$

Proof The case $\ell = 1$ is [La83, Lemma 4]. The general proof starts in the same way. Define for $0 < z \le 1$

$$h(z) := \mu_1(S(P,z)),$$

where we recall that μ_1 stands for the Lebesgue measure of the set. Let the leading coefficient of $P(x)$ be a with $|a| \ge 1$. Then $a^{-1}P$ is monic, and so Theorem 2.1 implies that

$$h(z) \le C_k\left(\frac{z}{|a|}\right)^{\frac{1}{k-1}} \le C_k z^{\frac{1}{k-1}}.$$

Now we compute the desired integral:

$$(-1)^\ell \int_{S(P,y)} \log^\ell |P(x)| \frac{dx}{x} = (-1)^\ell \int_{z=0}^{z=y} \int_{\substack{|x|=1 \\ |P(x)|=z}} \log^\ell z \frac{dx}{x} dz$$

$$= (-1)^\ell \log^\ell y h(y) - \int_0^y \frac{d}{dz}\left[(-\log z)^\ell\right] h(z) dz$$

$$\leq (-1)^\ell \log^\ell y C_k y^{\frac{1}{k-1}} - \int_0^y \frac{d}{dz}\left[(-\log z)^\ell\right] C_k z^{\frac{1}{k-1}} dz,$$

where the last inequality is a consequence of the fact that $(-\log z)^\ell$ is a positive decreasing function and its derivative is negative. By applying integration by parts again we obtain

$$\leq (-1)^\ell C_k \int_0^y \log^\ell z d\left(z^{\frac{1}{k-1}}\right),$$

which finishes the proof of the first statement by Lemma 2.3 and equation (2.2)

The proof of the second statement follows along the same lines. ■

Lemma 3.2 *Let $P_1(x), \ldots, P_s(x) \in \mathbb{C}[x]$ be polynomials having $k_1, \ldots, k_s$ nonzero complex coefficients with absolute value greater than 1 and $0 < y_1, \ldots, y_s \leq 1$. Let $1 \leq n \leq s$. Then*

$$0 \leq (-1)^s \int_{\bigcap_{i=1}^n S(P_i,y_i) \setminus \bigcup_{i=n+1}^s S(P_i,y_i)} \log |P_1(x)| \cdots \log |P_s(x)| \frac{dx}{x}$$

$$\leq \left(C_{k_1} I_{n,k_1}(y_1) \cdots C_{k_n} I_{n,k_n}(y_n)\right)^{\frac{1}{n}} (-1)^{s-n} \log y_{n+1} \cdots \log y_s.$$

Proof Notice that $0 \leq -\log |P(x)| \leq -\log y$ for $x \notin S(P,y)$ for $0 < y \leq 1$. Therefore,

$$(-1)^s \int_{\bigcap_{i=1}^n S(P_i,y_i) \setminus \bigcup_{i=n+1}^s S(P_i,y_i)} \log |P_1(x)| \cdots \log |P_s(x)| \frac{dx}{x}$$

$$\leq (-1)^s \log y_{n+1} \cdots \log y_s \int_{\bigcap_{i=1}^n S(P_i,y_i) \setminus \bigcup_{i=n+1}^s S(P_i,y_i)} \log |P_1(x)| \cdots \log |P_n(x)| \frac{dx}{x}$$

$$\leq (-1)^s \log y_{n+1} \cdots \log y_s \int_{\bigcap_{i=1}^n S(P_i,y_i)} \log |P_1(x)| \cdots \log |P_n(x)| \frac{dx}{x}$$

$$\leq (-1)^{s-n} \log y_{n+1} \cdots \log y_s \left(C_{k_1} I_{n,k_1}(y_1) \cdots C_{k_n} I_{n,k_n}(y_n)\right)^{\frac{1}{n}}$$

by Lemmas 2.5 and 3.1. ■

Lemma 3.3 *Let $P_1(x), \ldots, P_s(x) \in \mathbb{C}[x]$ be polynomials having $k_1, \ldots, k_s$ nonzero complex coefficients with absolute value greater than 1 and $0 < y_1, \ldots, y_s \leq 0$. Then*

$$0 \leq (-1)^s \int_{S(P_1,y_1) \cup \cdots \cup S(P_s,y_s)} \log |P_1(x)| \cdots \log |P_s(x)| \frac{dx}{x}$$

$$\leq \sum_{A \subset \{1,\ldots,s\}} \prod_{i \in A} \left(C_{k_i} I_{|A|,k_i}(y_i)\right)^{\frac{1}{|A|}} \prod_{i \in \{1,\ldots,s\} \setminus A} (-\log y_i).$$

Proof We start with the observation that

$$\bigcup_{i=1}^{s} S(P_i, y_i) = \bigcup_{A\subset\{1,\dots,s\}} \left(\bigcap_{i\in A} S(P_i, y_i) \setminus \bigcup_{i\in\{1,\dots,s\}\setminus A} S(P_i, y_i) \right).$$

By applying Lemma 3.2, we get

$$\begin{aligned}
&(-1)^s \int_{S(P_1,y_1)\cup\cdots\cup S(P_s,y_s)} \log|P_1(x)|\cdots\log|P_s(x)| \frac{dx}{x} \\
&\leq \sum_{A\subset\{1,\dots,s\}} (-1)^s \int_{\bigcap_{i\in A} S(P_i,y_i)\setminus\bigcup_{i\in\{1,\dots,s\}\setminus A} S(P_i,y_i)} \log|P_1(x)|\cdots\log|P_s(x)| \frac{dx}{x} \\
&\leq \sum_{A\subset\{1,\dots,s\}} \prod_{i\in A} \left(C_{k_i} I_{|A|,k_i}(y_i) \right)^{\frac{1}{|A|}} \prod_{i\in\{1,\dots,s\}\setminus A} (-\log y_i).
\end{aligned}$$

■

Setting $y_1 = \cdots = y_s = y$ and letting $y \to 0$, we get the following result by Corollary 2.4.

Corollary 3.4 *Let $P_1(x), \dots, P_s(x) \in \mathbb{C}[x]$ be polynomials having $k_1, \dots, k_s$ nonzero complex coefficients. Let $0 < y < 1$. As y approaches 0, we obtain*

$$\lim_{y\to 0} \int_{S(P_1,y)\cup\cdots\cup S(P_s,y)} \log|P_1(x)|\cdots\log|P_s(x)| \frac{dx}{x} = 0,$$

where the speed of convergence is independent of the polynomials $P_1(x), \dots, P_s(x)$.

Lemma 3.5 *Let $P_1(x), \dots, P_s(x) \in \mathbb{C}[x]$ be polynomials having $k_1, \dots, k_s$ nonzero complex coefficients with absolute value greater than 1 and $0 < y_1, \dots, y_s \leq 1$. Then*

$$\begin{aligned}
0 &\leq (-1)^s \int_{S(P_1,y_1)\cap\cdots\cap S(P_s,y_s)} \log|P_1(x)|\cdots\log|P_s(x)| \frac{dx}{x} \\
&\leq \left(C_{k_1} I_{s,k_1}(y_1) \cdots C_{k_s} I_{s,k_s}(y_s) \right)^{\frac{1}{s}}.
\end{aligned}$$

Proof This is a consequence of Lemma 3.2 with $n = s$. ■

Lemma 3.6 *Let $P_1(x), \dots, P_s(x) \in \mathbb{C}[x]$ be polynomials having $k_1, \dots, k_s$ nonzero complex coefficients with absolute value greater than 1 and $0 < y_1, \dots, y_s \leq 1$. Then*

$$\begin{aligned}
0 &\leq \int_{S(P_1,y_1)\cap\cdots\cap S(P_s,y_s)} \max_{1\leq i\leq s} \{\log|P_i(x)|\} \frac{dx}{x} \\
&\leq (2\pi)^{1-\frac{1}{s}} \left(C_{k_1} I_{s,k_1}(y_1) \cdots C_{k_s} I_{s,k_s}(y_s) \right)^{\frac{1}{s}}.
\end{aligned}$$

Proof Notice that

$$\max_{1\le i\le s}\{\log|P_i(x)|\} = -\min_{1\le i\le s}\{-\log|P_i(x)|\}.$$

In $S(P_1, y_1)\cap\cdots\cap S(P_s, y_s)$, we have $0 \le \min_{1\le i\le s}\{-\log|P_i(x)|\} \le -\log|P_i(x)|$ for any $i = 1,\dots,s$. Thus,

$$\begin{aligned}\Big(-\max_{1\le i\le s}\Big\{\log|P_i(x)|\Big\}\Big)^s &= \Big(\min_{1\le i\le s}\{-\log|P_i(x)|\}\Big)^s\\ &\le (-1)^s\log|P_1(x)|\cdots\log|P_s(x)|.\end{aligned}$$

By applying Hölder's inequality, and taking into account that the measure of $S(P_1, y_1)\cap\cdots\cap S(P_s, y_s)$ is bounded by 2π, we get

$$\begin{aligned}0 &\le \int_{S(P_1,y_1)\cap\cdots\cap S(P_s,y_s)} -\max_{1\le i\le s}\{\log|P_i(x)|\}\frac{dx}{x}\\ &\le (2\pi)^{1-\frac{1}{s}}\left(\int_{S(P_1,y_1)\cap\cdots\cap S(P_s,y_s)}\Big(-\max_{1\le i\le s}\Big\{\log|P_i(x)|\Big\}\Big)^s\frac{dx}{x}\right)^{\frac{1}{s}}\\ &\le (2\pi)^{1-\frac{1}{s}}\big(C_{k_1}I_{s,k_1}(y_1)\cdots C_{k_s}I_{s,k_s}(y_s)\big)^{\frac{1}{s^2}}.\end{aligned}$$

■

Again, we let $y_1 = \cdots = y_s = y$ and $y \to 0$, and we obtain the following result.

Corollary 3.7 *Let $P_1(x),\dots,P_s(x) \in \mathbb{C}[x]$ be polynomials having $k_1,\dots,k_s$ nonzero complex coefficients. Let $0 < y \le 1$. As y approaches 0, we obtain*

$$\lim_{y\to 0}\int_{S(P_1,y)\cap\cdots\cap S(P_s,y)}\max_{1\le i\le s}\{\log|P_i(x)|\}\frac{dx}{x} = 0,$$

where the speed of convergence is independent of the polynomials $P_1(x),\dots,P_s(x)$.

Observe that when $k_i = 1$, the previous result is trivially true since the set $S(P_i, y)$ becomes empty for y sufficiently small.

Remark 3.8 Results analogous to Corollaries 3.4 and 3.7 can be proved for the case where $P_1(x_1,\dots,x_n),\dots,P_s(x_1,\dots,x_n)$ are *fixed* polynomials in $\mathbb{C}[x_1,\dots,x_n]$.

4 Proof of Theorem 1.2

We begin by proving that the extended versions of Mahler measures always exist (*i.e.,* that the integrals always converge). This has been used repeatedly in previous works, but the details have never been published, so we include them here for completeness.

Theorem 4.1 *Let $P_1(x_1,\dots,x_n),\dots,P_s(x_1,\dots,x_n) \in \mathbb{C}[x_1,\dots,x_n]$ be nonzero polynomials. Then the integrals giving the generalized Mahler measure and the multiple Mahler measure converge,* i.e.,

(i) $|\mathrm{m}_{\max}(P_1, \dots, P_s)| < \infty$;
(ii) $|\mathrm{m}(P_1, \dots, P_s)| < \infty$;
(iii) *if* $P_1 = \cdots = P_s = P$, $|\mathrm{m}_s(P)| < \infty$.

Proof (i) Let $y > 0$. We write

$$\begin{aligned}
&\int_{\mathbb{T}^n} \max_{1\le i\le s}\{\log|P_i(x_1,\dots,x_n)|\}\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n} \\
&\quad = \int_{S_n(P_1,y)\cap\cdots\cap S_n(P_s,y)} \max_{1\le i\le s}\Big\{\log|P_i(x_1,\dots,x_n)|\Big\}\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n} \\
&\qquad + \int_{S(P_1,y)^c\cup\cdots\cup S(P_s,y)^c} \max_{1\le i\le s}\Big\{\log|P_i(x_1,\dots,x_n)|\Big\}\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}.
\end{aligned}$$

The second integral converges, while the first integral approaches 0 as $y \to 0$ by Corollary 3.7 and Remark 3.8. Therefore, the integral on the left converges.

(ii) For $y > 0$, we consider

$$\begin{aligned}
&\int_{\mathbb{T}^n} \log|P_1(x_1,\dots,x_n)|\dots\log|P_s(x_1,\dots,x_n)|\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n} \\
&\quad = \int_{S_n(P_1,y)\cup\cdots\cup S_n(P_s,y)} \log|P_1(x_1,\dots,x_n)|\dots\log|P_s(x_1,\dots,x_n)|\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n} \\
&\qquad + \int_{S_n(P_1,y)^c\cap\cdots\cap S_n(P_s,y)^c} \log|P_1(x_1,\dots,x_n)|\dots\log|P_s(x_1,\dots,x_n)|\frac{dx_1}{x_1}\cdots\frac{dx_n}{x_n}.
\end{aligned}$$

As before, the second integral converges, while the first integral approaches 0 as $y \to 0$ by Corollary 3.4 and Remark 3.8. Thus, the first integral converges.

(iii) This statement is a particular case of (ii). ∎

Proof of Theorem 1.2 (i) Following [La83], we define $F\colon \mathbb{T}^n \to \mathbb{R}$ by $F(\omega) = -\max_{1\le i\le s}\{\log|P_i(\omega)|\}$ for $\omega \in \mathbb{T}^n$. It suffices to prove that

$$\lim_{q(\mathbf{r})\to\infty}\left|\int_{\mathbb{T}} F_{\mathbf{r}} - \int_{\mathbb{T}^n} F\right| = 0.$$

Without loss of generality, we may assume that each coefficient of P_i has modulus greater than or equal to 1, and therefore the same is true for $P_{i,\mathbf{r}}$ for $q(\mathbf{r})$ sufficiently large. For any $0 \le y \le 1$ we construct a continuous function $g_y\colon \mathbb{T}^n \to \mathbb{R}$ such that $0 \le g_y(\omega) \le 1$ for all $\omega \in \mathbb{T}^n$, $g_y(\omega) = 1$ for $\max_{1\le i\le s}\{|P_i(\omega)|\} \ge y$, and $g_y(\omega) = 0$ for $\max_{1\le i\le s}\{|P_i(\omega)|\} \le \frac{1}{2}y$. Therefore, $g_yF_{\mathbf{r}}$ is a continuous function on $\mathbb{T}^n$ for $0 \le y \le 1$. Since $F = g_yF + (1-g_y)F$, the triangle inequality implies that

$$\begin{aligned}
\limsup_{q(\mathbf{r})\to\infty}\left|\int_{\mathbb{T}} F_{\mathbf{r}} - \int_{\mathbb{T}^n} F\right| &\le \limsup_{q(\mathbf{r})\to\infty}\left|\int_{\mathbb{T}} [g_yF]_{\mathbf{r}} - \int_{\mathbb{T}^n} g_yF\right| \\
&\quad + \limsup_{q(\mathbf{r})\to\infty}\left|\int_{\mathbb{T}} \big[(1-g_y)F\big]_{\mathbf{r}}\right| + \limsup_{q(\mathbf{r})\to\infty}\left|\int_{\mathbb{T}^n}(1-g_y)F\right|.
\end{aligned} \tag{4.1}$$

Now, by the Weierstrass approximation theorem, the first term goes to zero, since $g_y F$ is continuous on $\mathbb{T}^n$. The function $[(1-g_y)F]_{\mathbf{r}} = (1-g_{y,\mathbf{r}})F_{\mathbf{r}}$ vanishes in the set $\bigcup S(P_{i,\mathbf{r}}, y)^c = (\bigcap S(P_{i,\mathbf{r}}, y))^c$, and it is bounded below by 0 and above by $F_{\mathbf{r}}$ in $\bigcap S(P_{i,\mathbf{r}}, y)$. This implies

$$0 \le \limsup_{q(\mathbf{r})\to\infty} \left| \int_{\mathbb{T}} \big[(1-g_y)F\big]_{\mathbf{r}} \right| \le \limsup_{q(\mathbf{r})\to\infty} \left| \int_{\bigcap S(P_{i,\mathbf{r}},y)} F_{\mathbf{r}} \right|,$$

which goes to zero as $y \to 0$ by Corollary 3.7. Finally, the third term in (4.1) tends to 0 as $y \to 0$, since F is integrable over $\mathbb{T}^n$ by Theorem 4.1(i).

Thus, $\limsup_{q(\mathbf{r})\to\infty} |\int_{\mathbb{T}} F_{\mathbf{r}} - \int_{\mathbb{T}^n} F| = 0$ since it is independent of y and tends to zero as $y \to 0$.

(ii) We proceed as before. We define $F\colon \mathbb{T}^n \to \mathbb{R}$ by $F(\omega) = \prod_{i=1}^{s}(-\log|P_i(\omega)|)$ for $\omega \in \mathbb{T}^n$. Without loss of generality, we may assume that each coefficient of P_i has modulus greater than or equal to 1, and therefore the same is true for $P_{i,\mathbf{r}}$ for $q(\mathbf{r})$ sufficiently large. For any $0 \le y \le 1$ we construct a continuous function $g_y\colon \mathbb{T}^n \to \mathbb{R}$ such that $0 \le g_y(\omega) \le 1$ for all $\omega \in \mathbb{T}^n$, $g_y(\omega) = 1$ if $|P_i(\omega)| \ge y$ for all i, and $g_y(\omega) = 0$ if there is an i such that $|P_i(\omega)| \le \frac{1}{2}y$. Therefore, $g_y F$ is a continuous function on $\mathbb{T}^n$ for $0 \le y \le 1$. The triangle inequality implies that

$$\begin{aligned}(4.2)\quad \limsup_{q(\mathbf{r})\to\infty} \left| \int_{\mathbb{T}} F_{\mathbf{r}} - \int_{\mathbb{T}^n} F \right| &\le \limsup_{q(\mathbf{r})\to\infty} \left| \int_{\mathbb{T}} [g_y F]_{\mathbf{r}} - \int_{\mathbb{T}^n} g_y F \right| \\ &\quad + \limsup_{q(\mathbf{r})\to\infty} \left| \int_{\mathbb{T}} \big[(1-g_y)F\big]_{\mathbf{r}} \right| + \limsup_{q(\mathbf{r})\to\infty} \left| \int_{\mathbb{T}^n} (1-g_y)F \right|\end{aligned}$$

The Weierstrass approximation theorem implies that the first term goes to zero, since $g_y F$ is continuous on $\mathbb{T}^n$. Now the function $[(1-g_y)F]_{\mathbf{r}} = (1-g_{y,\mathbf{r}})F_{\mathbf{r}}$ vanishes in the set $\bigcap S(P_{i,\mathbf{r}}, y)^c = (\bigcup S(P_{i,\mathbf{r}}, y))^c$, and it is bounded below by 0 and above by $F_{\mathbf{r}}$ in $\bigcup S(P_{i,\mathbf{r}}, y)$. Combining all of this,

$$0 \le \limsup_{q(\mathbf{r})\to\infty} \left| \int_{\mathbb{T}} \big[(1-g_y)F\big]_{\mathbf{r}} \right| \le \limsup_{q(\mathbf{r})\to\infty} \left| \int_{\bigcup S(P_{i,\mathbf{r}},y)} F_{\mathbf{r}} \right|.$$

The term on the right goes to zero as $y \to 0$ by Corollary 3.4. The third term in (4.2) tends to 0 as $y \to 0$, since F is integrable over $\mathbb{T}^n$ by Theorem 4.1(ii).

Finally, $\limsup_{q(\mathbf{r})\to\infty} |\int_{\mathbb{T}} F_{\mathbf{r}} - \int_{\mathbb{T}^n} F| = 0$, since it is independent of y and tends to zero as $y \to 0$.

(iii) This case follows from (ii) by setting $P_1 = \cdots = P_s = P$. This concludes the proof of the theorem. ∎

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References

[BBSW] D. Borwein, J. Borwein, A. Straub, and J. Wan, *Log-sine evaluations of Mahler measures. II.* Integers, to appear.

[BS] J. Borwein and A. Straub, *Log-sine evaluations of Mahler measures.* J. Aust. Math. Soc., to appear.

[Bo81a] D. W. Boyd, *Speculations concerning the range of Mahler's measure.* Canad. Math. Bull. **24**(1981), no. 4, 453–469. http://dx.doi.org/10.4153/CMB-1981-069-5 http://dx.doi.org/10.4153/CMB-1981-069-5

[Bo81b] ———, *Kronecker's theorem and Lehmer's problem for polynomials in several variables.* J. Number Theory **13**(1981), no. 1, 116–121. http://dx.doi.org/10.1016/0022-314X(81)90033-0 http://dx.doi.org/10.1016/0022-314X(81)90033-0

[EW99] G. Everest and T. Ward, *Heights of polynomials and entropy in algebraic dynamics.* Universitext, Springer-Verlag London, Ltd., London, 1999.

[GO04] Y. Gon and H. Oyanagi, *Generalized Mahler measures and multiple sine functions.* Internat. J. Math. **15**(2004), no. 5, 425–442. http://dx.doi.org/10.1142/S0129167X04002363 http://dx.doi.org/10.1142/S0129167X04002363

[KLO08] N. Kurokawa, M. Lalín, and H. Ochiai, *Higher Mahler measures and zeta functions.* Acta Arith. **135**(2008), no. 3, 269–297. http://dx.doi.org/10.4064/aa135-3-5 http://dx.doi.org/10.4064/aa135-3-5

[La08] M. N. Lalín, *Mahler measures and computations with regulators.* J. Number Theory **128**(2008), no. 5, 1231–1271. http://dx.doi.org/10.1016/j.jnt.2007.03.002 http://dx.doi.org/10.1016/j.jnt.2007.03.002

[La83] W. M. Lawton, *A problem of Boyd concerning geometric means of polynomials.* J. Number Theory **16**(1983), no. 3, 356–362. http://dx.doi.org/10.1016/0022-314X(83)90063-X http://dx.doi.org/10.1016/0022-314X(83)90063-X

[Le33] D. H. Lehmer, *Factorization of certain cyclotomic functions.* Ann. of Math. (2) **34**(1933), no. 3, 461–479. http://dx.doi.org/10.2307/1968172 http://dx.doi.org/10.2307/1968172

[Ma62] K. Mahler, *On some inequalities for polynomials in several variables.* J. London Math. Soc. **37**(1962), 341–344. http://dx.doi.org/10.1112/jlms/s1-37.1.341

[Sa10] Y. Sasaki, *On multiple higher Mahler measures and multiple L values.* Acta Arith. **144**(2010), no. 2, 159–165. http://dx.doi.org/10.4064/aa144-2-5 http://dx.doi.org/10.4064/aa144-2-5

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