Slides of lectures to the SMS2004.

Octav Cornea

Talks structured around two main ideas:

- using high dimensional moduli spaces of flow lines in Morse-Floer theory is possible and quite efficient (joint work with Jean-François Barraud).

- in the presence of bubbling, describe a method to manage algebraically pseudoholomorphic disks (joint work with François Lalonde - still in progress !).

<u>Plan:</u>

I. Some Morse theory.

II. First applications: Lagrangian non-squeezing in the absence of bubbling.

III. Bubbling of disks: the cluster complex.

IV. Invariance of cluster homology and applications.

I. Morse Theory. L closed, finite-dimensional n-manifold. A smooth

$$f : L \to \mathbf{R}$$

is Morse if

$$df(x) = 0 \Rightarrow det(Hess_f(x)) \neq 0 \ ,$$

In this case, for $x \in Crit(f)$ we let $ind_f(x) = \#$ of negative eigenvalues of $Hess_f(x)$ and $Crit_k(f) = \{x \in Crit(f) : \ ind_f(x) = k\}$.

α riemannian metric on L, ∇f gradient vector field of f w.r. to α, negative gradient flow

$$\gamma : \mathbf{R} \times L \to L, \ \gamma_t(x) = \gamma(t, x) \ .$$

Stable manifold of $x \in Crit(f)$:

$$W^s(x) = \{y \in L : \lim_{t \to \infty} \gamma_t(y) = x\} \ .$$

Unstable manifold of x:

$$W^u(x) = \{y \in L : \lim_{t \to -\infty} \gamma_t(y) = x\} \ .$$

Fact: $W^u(x) \approx Int(D^{n-ind(x)})$, $W^s(x) \approx Int(D^{ind(x)})$.

Assume (f, α) Morse-Smale $\Leftrightarrow$ $W^u(x) \cap W^s(y)$ transversally $\forall x, y \in Crit(f)$.

$x, y \in Crit(f)$,

$$M_y^x = \{p \in L : \lim_{t \to -\infty} \gamma_t(p) = x, \lim_{t \to +\infty} \gamma_t(p) = y\}/\mathbf{R}$$

manifold of dimension $ind(x) - ind(y) - 1$, non-compact.

Compactify, $M_y^x \to \overline{M}_y^x$.

Fact:

$$\partial \overline{M}_y^x = \bigcup_{z \in Crit(f)} \overline{M}_z^x \times \overline{M}_y^z \tag{1}$$

$C_*(X)$ cubical complex of X. Pick $m_y^x \in C_*(\overline{M}_y^x)$ so that:

i. $[m_y^x]$ represents the fundamental class in $H_{ind(x)-ind(y)-1}(\overline{M}_y^x, \partial \overline{M}_y^x)$

ii. $dm_y^x = \Sigma_z \, m_z^x \times m_y^z$.

Such a family $\{m_y^x\}$ is called a *representing chain system for the moduli spaces* $\{M_y^x\}$.

Fix path $\gamma \in L$ homeomorphic to $[0, 1]$ which joins all elements in $Critf$. Shall identify L and L/γ.

There are continous maps

$$i_b^a : M_b^a \to \Omega L$$

compatible with (1).

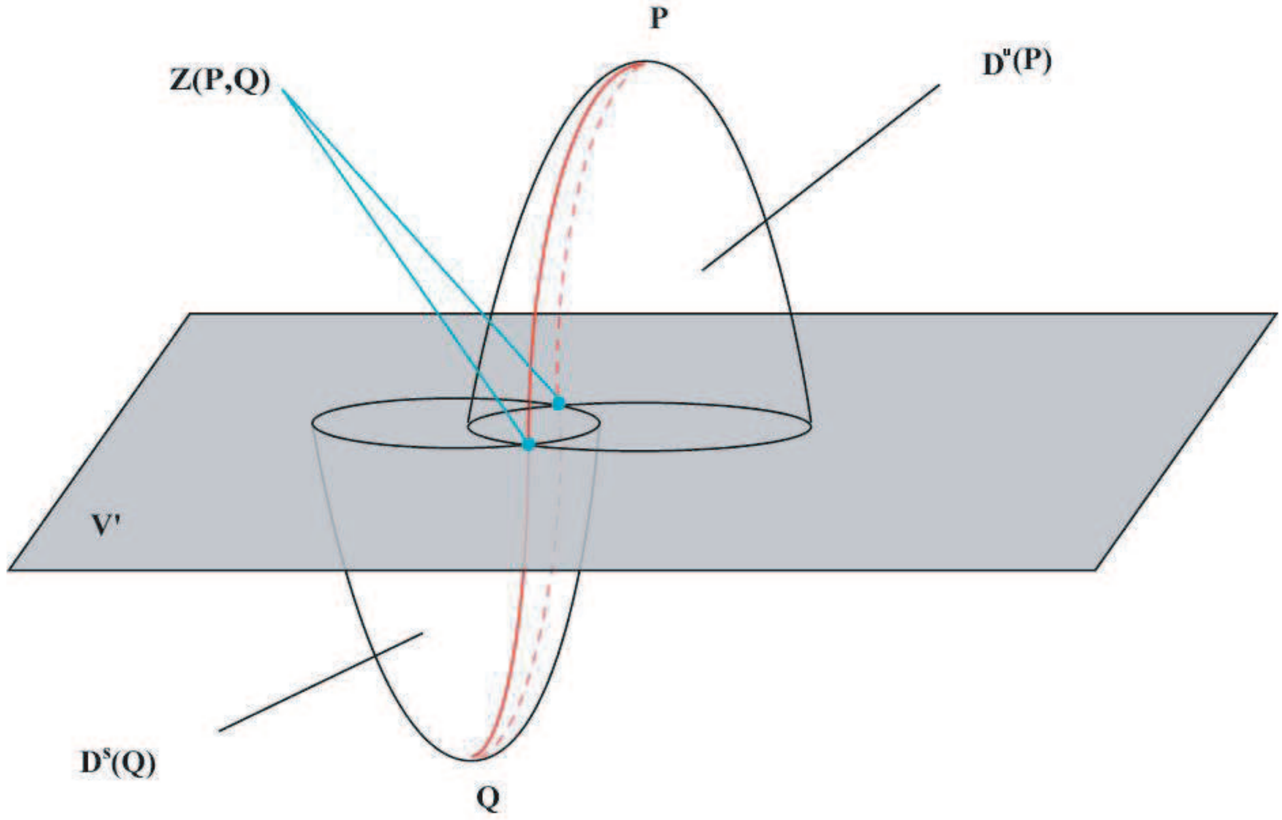


Figure 1: Stable and unstable manifolds: $W^s(Q)$, $W^u(P)$ and the connecting manifold M_Q^P.

Define:

$$a_y^x = i_y^x(m_y^x)\ ,\ \ A = (a_y^x) \in GL_{|Crit(f)|}(C_*(\Omega L))\ .$$

$$\text{equation (1)} \Rightarrow \mathrm{dA} = \mathrm{A}^2 \tag{2}$$

Define the *extended Morse complex*:

$$C(f) = (\ C_*(\Omega L) \otimes \mathbf{Z}/2 < Crit(f) >\ ,\ \ d\ )\ ,\ \ dx = \sum_y a_y^x y$$

Immediate (2) $\Rightarrow d^2 = 0$.

Remark: The coefficients of the differential represent the higher dimensional moduli spaces of connecting flow lines.

Filtration: $F^k C(f) = C_*(\Omega L) \otimes \mathbf{Z}/2 < Crit_{\leq k}(f) >$. Resulting spectral sequence $(E^r(f), d^r)$.

Theorem A.[J.-F. Barraud, O.C. 2003] *For $r \geq 2$, $(E^r(f), d^r)$ coincides with the Serre spectral sequence of the fibration $\Omega L \to PL \to L$.*

Remark: As the Serre spectral sequence is a homotopical invariant this means that the differentials in this spectral sequence provide an algebraic invariant measure of high dim. moduli spaces.

Proof: Assume f has a single minimum denoted by m and f is self-indexed $(ind(x) = k \Rightarrow f(x) = k)$.

Key idea in the proof: for $x \in Crit(f)$

$$\widehat{W}(x) = (\overline{M}_m^x \times [0, f(x)])/\sim$$

Where:

$$(\gamma, t) \sim (\gamma', t) \Leftrightarrow \gamma(\tau) = \gamma'(\tau) \ , \forall \tau \geq t$$

$\widehat{W}(x)$ is a compactification of $W^u(x)$ with some remarkable properties:

Lemma. *$\widehat{W}(x)$ is a compact manifold with boundary homeomorphic to a disk whose boundary is*

$$\partial\widehat{W}(x) = \bigcup_y (M_y^x \times \{f(y)\}) \times \widehat{W}(y) \ .$$

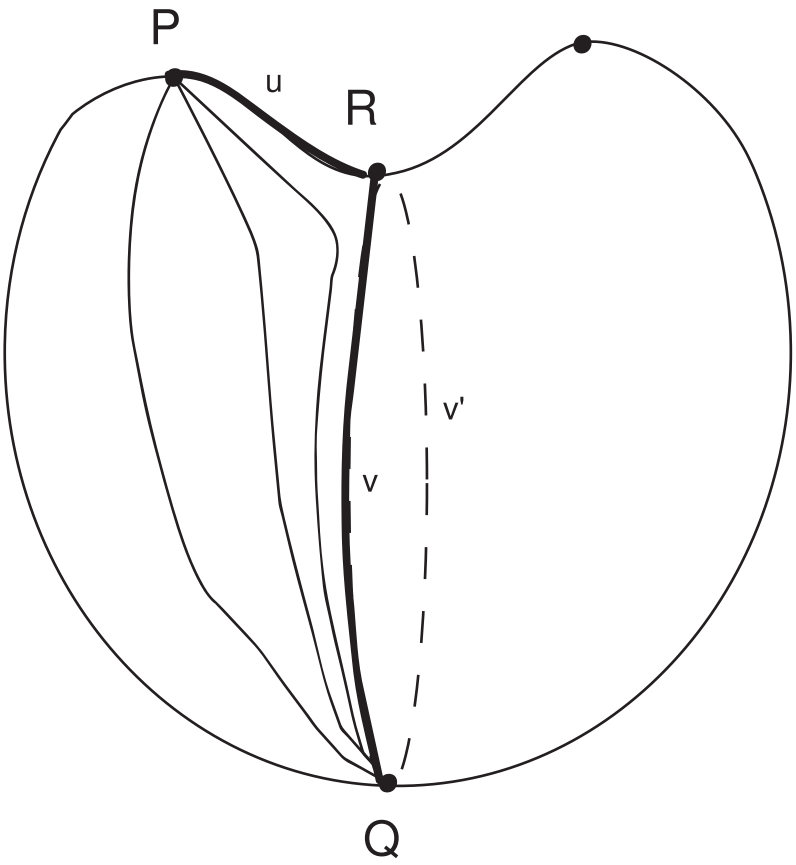


Figure 2: $M_Q^P \approx (0,1)$, $\overline{M}_Q^P \approx [0,1] = I$ where
$0 = \{u\} \times \{v\}$, $1 = \{u\} \times \{v'\}$; $\overline{M}_Q^P \times [0, f(P)] = [0,1] \times [0, f(P)]$

and $\sim$ is: $\{0\} \times \{\tau\} \sim \{1\} \times \{\tau\}, \tau \geq f(R))$

$(x, f(P)) \sim (y, f(P)), \forall y \in I; \widehat{W}(P) \approx D^2.$

May find representing chain system $c(x) \in C_{ind(x)}(\widehat{W}(x))$ so that $[c(x)]$ represents fund. class of $(\widehat{W}(x), \partial\widehat{W}(x))$ and

$$\partial c(x) = m^x_y \times c(y) \ . \tag{3}$$

Recall the Serre spectral sequence:

$L^{k-1} = f^{-1}([0, k-\epsilon])$; $i_{k-1} : L^{k-1} \to L$ inclusion; there is a pull back fibration:

$$\begin{tikzcd} P^kL \arrow[r] \arrow[d] & PL \arrow[d] \\ L^{k-1} \arrow[r, "i_{k-1}"] & L \end{tikzcd}$$

Spectral sequence is associated to the filtration of $C_*(PL)$:

$$F^kPL = Im(C_*(P^kL)) \ .$$

For $x \in Crit_{k-1}(f)$, there is a continous map $i_x : \widehat{W}(x) \to PL$.

Let $h(x) = i_x(c(x)) \in C_*(PL)$. and $\phi : C(f) \to C_*(PL)$ be defined by

$$\phi(x) = h(x), \forall x \in Crit(f) \ .$$

ϕ is a filtered chain map and it is easy to see that induces an isomorphism at the E^2 level of the respective s.s's. $\surd$

<u>Remark.</u> It follows from the Lemma that $L(j) = \cup_{ind(x)\leq j} W^u(x)$ is a CW-decomposition of L.

Shall apply this result to symplectic topology.

<u>II. Symplectic applications in the absence of bubbling.</u>

Symplectic setup.

L, L' lagrangian submanifolds of a symplectic manifold (M, ω) (compact or convex at infinity); $dim(M) = 2n$.

Will assume for now the *connectivity condition*:

$$\pi_1(L) = \pi_1(L') = 0 = c_1|_{\pi_2(M)} = \omega|_{\pi_2(M)} \tag{4}$$

Let

$$\mathcal{P}_\eta(L, L') = \{\gamma \in C^\infty([0,1], M) : \gamma(0) \in L, \gamma(1) \in L'\}$$

fixed path component; J almost complex structure taming ω; $H : [0,1] \times M \to \mathbf{R}$ smooth periodic Hamiltonian with associated Hamiltonian vector field X_H and hamiltonian isotopy ϕ^H.

Assume L transversal to L' and $\phi_1^H(L)$ transverse to L'.

Consider solutions of:

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H(t,u) = 0 \qquad (5)$$

$$u(s,t) : \mathbf{R} \times [0,1] \to M \ , u(\mathbf{R},0) \subset L \ , \ u(\mathbf{R},1) \subset L'$$

(where J may also be time dependent).

When H is constant, the solutions of (5) are *pseudoholomorphic strips*. They coincide with the zeros of the operator $\overline{\partial}_J = \frac{1}{2}(d + J \circ d \circ \mathbf{i})$.

$$\mathcal{M}' = \{u \text{ verifying (5)} : \ E_{L,L',H}(u) < \infty\}$$

were the energy is:

$$E_{L,L',H}(u) = \frac{1}{2}\int_{\mathbf{R}\times[0,1]} ||\frac{\partial u}{\partial s}||^2 + ||\frac{\partial u}{\partial t} - X_H^t(u)||^2 \ ds \ dt \ . \qquad (6)$$

Fact:

$$\mathcal{M}'/\mathbf{R} = \bigcup_{x,y \in Per_H(L,L')} \mathcal{M}_y^x$$

where $Per_H(L,L') = \{\text{orbits } \gamma \text{ of } \phi^H : \gamma(0) \in L, \gamma(1) \in L'\}$

$$\mathcal{M}_y^x = \{u \in \mathcal{M}'/\mathbf{R} : \lim_{s \to -\infty} u(s,t) = x(t), \lim_{s \to +\infty} u(s,t) = y(t)\}$$

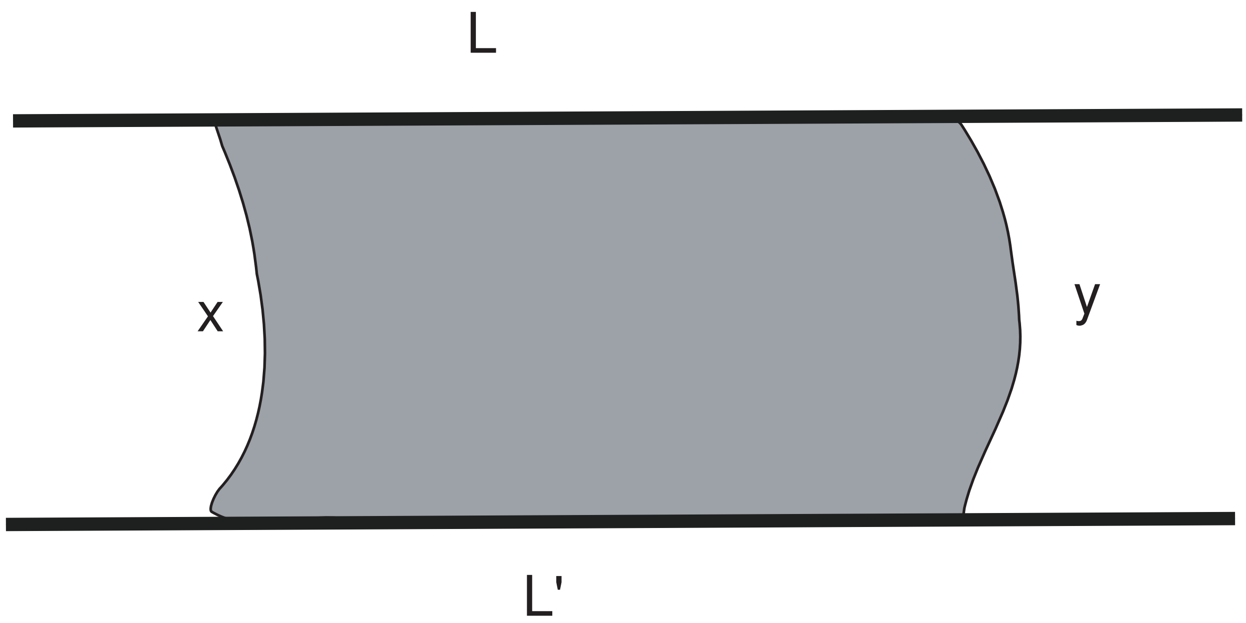


Figure 3: a semi-tube $u \in \mathcal{M}_y^x(J,H)$ = moduli space of finite energy solutions of the Floer equation which rest on L and L' and join $x \in Per_H(L,L')$ to $y \in Per_H(L,L')$.

We shall use the moduli spaces $\mathcal{M}_y^x$ to construct a chain complex $\mathcal{C}(H,J)$ similar to $C(f)$.

$\mathcal{M}_y^x$ are manifolds of dimension $\mu(x,y)-1$; $\mu(-)=$ Maslov index. Fix $x_0 \in Per_H(L,L')$ and assign $\mu(x_0)=0$. Due to the connectivity assumption (4) may put:

$$\mu(y) = \mu(x_0) + \mu(x,y) \ .$$

Fact (!): For generic J, $\mathcal{M}_y^x$ admit compactifications $\overline{\mathcal{M}}_y^x$ which are compact topological manifolds with boundary so that:

$$\partial \overline{\mathcal{M}}_y^x = \bigcup_{y \in Per} \overline{\mathcal{M}}_z^x \times \overline{\mathcal{M}}_y^z \tag{7}$$

There are continous maps $i_y^x : \mathcal{M}_y^x \to \Omega L$ induced by $u \to u|_{\mathbf{R}\times\{0\}}$ (clearly, the origins of the elements in L have been identified as has been done with critical points before).

i_y^x are compatible with (7); may again construct a representing chain system for the $\overline{\mathcal{M}}_y^x$'s $\Rightarrow \mathcal{A} \in GL_{|Per|}(C_*(\Omega L))$ with $\mathcal{A} = (a_y^x)$, a_y^x representing $\mathcal{M}_y^x$ and

$$d\mathcal{A} = \mathcal{A}^2 \ .$$

Resulting chain complex:

$$\mathcal{C}(H,J) = (C_*(\Omega L) \otimes \mathbf{Z}/2 < Per_H(L,L') >, d)$$

<u>Remark.</u> Representing chain systems avoid the use of triangulations.

Grading given by $|x| = \mu(x)$. Filtration:

$$F^k\mathcal{C} = C_*(\Omega L) \otimes \mathbf{Z}/2 < x \in Per_H(L, L') : \mu(x) \leq k >$$

Resulting spectral sequence: $(\mathcal{E}^r(L, L'; H), d^r)$. Denote $\mathcal{E}^r(L, L') = \mathcal{E}^r(L, L'; 0)$.

<u>Theorem B.</u>[J.-F. Barraud, O.C 2003] *For $r \geq 2$ we have:*
i. The isomorphism type of $(\mathcal{E}^r(L, L'; H), d^r)$ is independent of J, H.
ii. If G is a hamiltonian, then $\mathcal{E}^r(L, \phi_1^G(L')) \approx \mathcal{E}^r(L, L')$
iii. If $L' = \phi_1^G$, then $\mathcal{E}^r(L, L') \approx E^r(f)$.

<u>Remark.</u> Isomorphism $\approx$ above is up to horizontal translation.

<u>Proof.</u> Follows roughly the usual method in Floer theory:

For generic almost complex structures J, J' and hamiltonians H, H' intend to construct chain morphism:

$$\psi : \mathcal{C}(H, J) \to \mathcal{C}(H', J')$$

J_t generic homotopy; may assume $H \geq H'$; fix H_t monotone homotopy from H to H'.

For $x \in Per_H(L, L')$, $y \in Per_{H'}(L, L')$, moduli spaces:

$$\mathcal{N}_y^x = \{u \in C^\infty(\mathbf{R} \times [0,1], M) : u \text{ verifies } (8)\}$$

$$\frac{\partial u}{\partial s} + J(s,u)\frac{\partial u}{\partial t} + \nabla H(s,t,u) = 0 \tag{8}$$

with $u|_{\mathbf{R}\times\{0\}} \subset L$, $u|_{\mathbf{R}\times\{1\}} \subset L'$, $\lim_{s\to-\infty} u(s) = x$, $\lim_{s\to+\infty} u(s) = y$.

These moduli spaces admit compactifications $\overline{N}_y^x$ and maps to ΩL all compatible with:

$$\partial\overline{\mathcal{N}}_y^x = \bigcup_z \overline{\mathcal{M}}_z^x \times \overline{\mathcal{N}}_y^z \ \cup \ \bigcup_{z'} \overline{\mathcal{N}}_{z'}^x \times \overline{\mathcal{M}}_y^{z'}$$

May find a representing chain system for the $\mathcal{N}_y^x$ and corresponding cubical chains:

$$b_y^x \in C_*(\Omega L)$$

so that the matrix $\mathcal{B} = (b_y^x)$ verifies

$$d\mathcal{B} = \mathcal{A}'\mathcal{B} - \mathcal{B}\mathcal{A} \tag{9}$$

$\mathcal{A}$, $\mathcal{A}'$ correspond to (H, J), (H', J') respectively. Put

$$\psi(x) = \sum_y b_y^x y \ .$$

(9) $\Rightarrow \psi$ chain morphism; easy to see that ψ induces an iso. at E^2.

This shows i. and also that $\mathcal{E}^r(L, L; H) \approx E(f)$ by taking $H = f$ small in C^2.

If $\phi_1^G(L') = L$, then $\mathcal{E}^r(L, L'; 0, (\phi^G)^* J) \approx \mathcal{E}^r(L, L; G, J)$ because

$$\phi : \mathcal{M}_y^x(0, (\phi^G)^* J) \to \mathcal{M}_{y'}^{x'}(G, J)\ ,\ u(s,t) \to \phi_t^G(u(s,t))$$

is a homeomorphism compatible with the maps to ΩL.

This shows iii. Point ii. is immediate from similar arguments. $\surd$

Remark. We have shown that $\mathcal{E}^r(L, L')$ is isomorphic up to translation (for $r \geq 2$) to the Serre spectral sequence of the fibration $\Omega L \to PL \to L$. The differentials in $\mathcal{E}(L, L')$ measure algebraically moduli spaces of pseudoholomorphic strips. There are many such non-vaniching differentials because PL is contractible.

Localization.

A useful, immediate adaptation of the construction before.

Given: $l : L \to X$ continuous may change the coefficients in the construction above by:

$$\Omega l : \Omega L \to \Omega X \ , \ a_y^x \to l_{\#}(a_y^x) \in C_*(\Omega X) \ .$$

Resulting spectral sequence - denoted $\mathcal{E}_X^r(L, L')$ - has properties that shadow those in Theorem B.

Thus, it is also invariant (for $r \geq 2$) and, if L hamiltonian isotopic to L', then $\mathcal{E}_X^r(L, L') \approx E_X^r$ where E_X^r is the Serre spectral sequence of the fibration:

$$\Omega X \to E \to L$$

which is the pull-back of $\Omega X \to PX \to X$ over l.

A particular case of interest: l is a Thom-Pontryagin map.

Basic example. $X = S^n$, $l_{pt} : L \to S^n$ of degree 1 and sends the exterior of a disk D surrounding a point $P \in L$ to a single point in S^n.

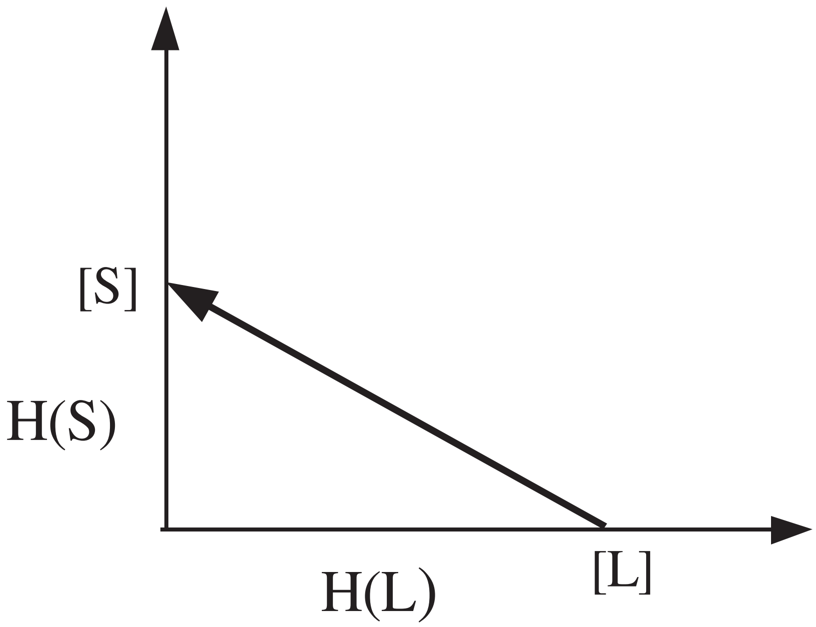


Figure 4. In E_{S^n} the fundamental classes of L and S^n are related by $d^n([L]) = \Omega[S] \in H_{n-1}(\Omega S^n)$.

Corollary B.1. *Given L and L' hamiltonian isotopic and a regular J, each point in $L \backslash L'$ belongs to a J-pseudoholomorphic strip of Maslov index at most n.*

Remark. Existence of strips through each point of L (without Maslov index restrictions) known before by work of Floer and Hofer.

Proof. Trivial: enough to show that the set of points belonging to J-strips of Maslov index n is dense.

Localize as above using any small disk $D \subset L$.

i. figure on last page

ii. isomorphism $\mathcal{E}^n_{S^n}(L, L') \approx E^n_{S^n}$

iii. the interpretation of the a^x_y

i.+ii.+iii. + (a bit of care needed to deal with the fact that there might be intersection pts. of arbitrary indexes) $\Rightarrow$ there have to be strips of Maslov index n passing through D. $\surd$

Stronger result:

Corollary B.2. *Assume L and L' are hamiltonian isotopic.*
For any J taming ω, each point $x \in L \backslash L'$ belongs to a J-strip u with

$$\int_{\mathbf{R}\times[0,1]} u^*\omega \leq \nabla(L, L') \ .$$

By definition:

$$\nabla(L, L') = \inf_{\psi \in \mathcal{H}, \psi(L) = L'} ||\psi||_H$$

Where $||\psi||_H$ is the Hofer norm of the Hamiltonian diffeomorphism ψ.

$$||\psi||_H = \inf_{\phi_1^G = \psi} (\sup G(x,t) - \inf G(x,t))$$

<u>Sketch of proof.</u> By Gromov compactness it is enough to work with regular J's.

Fix G, $\phi_1^G(L) = L'$ with

$$Var(G) = \nabla(L, L') + \delta$$

and $\inf(G) = 0$, $\sup(G) = m$.

By using the "naturality" map: $u(x,t) \to \phi_t^G(u(s,t))$ our problem is reduced to showing that for each point $x \in L \backslash L'$ there exists a semi-tube $u \in \mathcal{M}(L, L'; G, J')$ so that $x \in u(\mathbf{R}, 0)$ and

$$E_{L,L',G}(u) \leq Var(G) \ .$$

Consider $f : M \to \mathbf{R}$ Morse, small in C^2 norm with

$$\inf(f) = 0 \ , \ \sup(f) = \epsilon \ .$$

Fix $k > 0$ so that $f + k \geq G$.

Define monotone homotopies ϕ from $f + k$ to G and ϕ' from G to $f - \epsilon$.

They induce *action decreasing* morphisms of spectral sequences:

$$E_{S^n}(f + k) \xrightarrow{\hat{\phi}} \mathcal{E}_{S^n}(L, L'; G) \xrightarrow{\hat{\phi}'} E_{S^n}(f - \epsilon) \ .$$

This implies that, if the special differential d^n of $\mathcal{E}_{S^n}(L, L'; G)$ (considered before) decreases action by more that $\max(f + k) - \min(f - \epsilon)$ (or, equivalently, if the semi-tubes detected by it are *all* of higher energy than this value), then $\hat{\phi}' \circ \hat{\phi}$ can not be an isomorphism.

Therefore, there are semi-tubes passing through any apriori fixed disk D and of energy $\leq m + \epsilon - (-\epsilon) = \nabla(L, L') + \delta + 2\epsilon$. $\surd$.

A geometric consequence of Corollary B.2 (obtained by a reasonably standard application of the isoperimetric inequality):

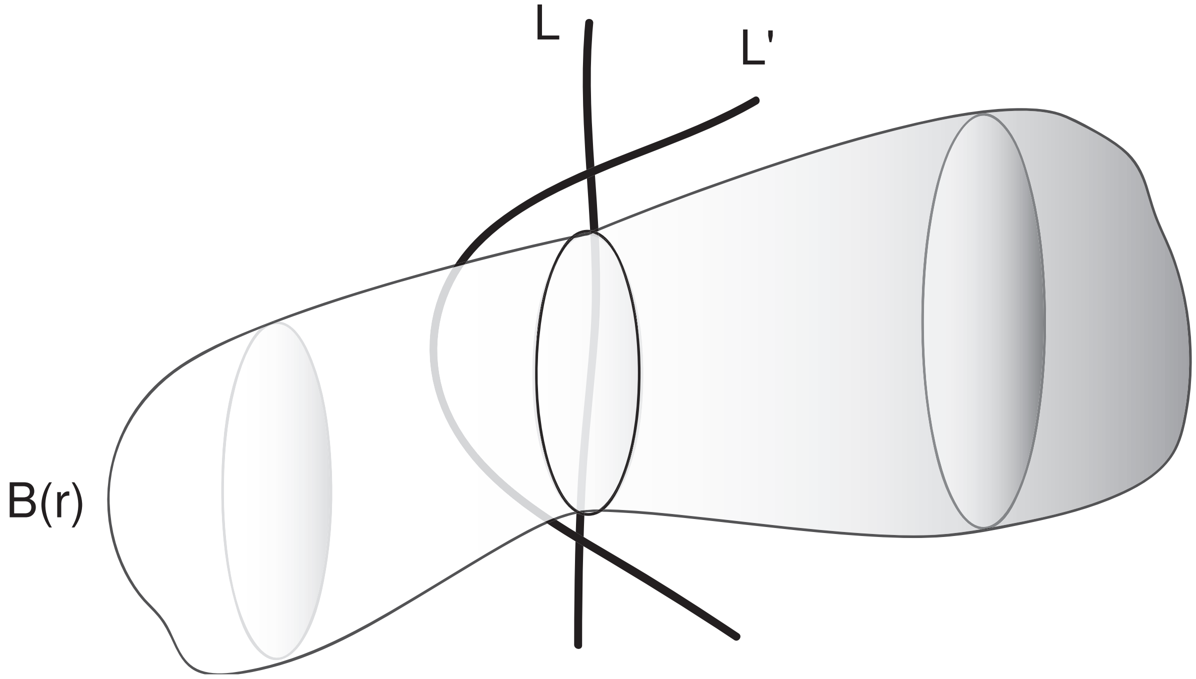


Figure 5: A standard ball $(B(r), \omega_0) \subset \mathbf{C}^n$ can not be embedded symplectically so that $e(B(r)) \cap L' = \emptyset$ and $e^{-1}(L) = B(r) \cap \mathbf{R}^n$ if

$$\frac{\pi}{2}r^2 \geq \nabla(L, L') \ .$$

Let $x \in L \backslash L'$ and let $\delta_x(L, L')$ be the supremum of the radii r of a standard symplectic ball $B(r)$ for which there exists a symplectic embedding

$e : (B(r), \omega_0) \to (M, \omega)$ such that

$$e(B(r)) \cap L' = \emptyset \ , \ e^{-1}(L) = B(r) \cap \mathbf{R}^n \ .$$

Easy to notice (Francois Lalonde) that $\delta_x(L, L')$ does not depend of x. Thus, put $\delta(L, L') = \delta_x(L, L')$.

Corollary B.3

$$\nabla(L, L') \geq \frac{\pi}{2} \delta(L, L')^2 \ . \tag{10}$$

Remark. i. This is a non-suqeezing type result which implies that the Hofer distance on Lagrangians is non-degenerate (under the connectivity conditions that we used).

ii. These connectivity conditions can be relaxed to $\omega|_{\pi_2(M,L)} = 0$.

iii. The inequality implies the usual displacement-energy estimate when $\omega|_{\pi_2}(M) = 0$.

Conjecture. For any two compact connected Lagangians which are hamiltonian isotopic (without connectivity assumptions) the inequality (10) remains true.

The full proofs of these results as well as other applications may be found in: "Lagrangian Intersections and the Serre Spectral Sequence" J. F. Barraud, O. Cornea, arXiv.math.DG/0401094.

III. The cluster complex. (joint work **in progress** with François Lalonde)

Why ?

No connectivity conditions from now on, then:

A. there might exist pseudoholomorphic spheres which bubble of strips - not too disruptive as it is a codimension two phenomenon.

B. there might exist pseudoholomorphic disks which bubble of strips - it *is* disruptive as it produces a codimension 1 subset.

C. The space of pseudoholomorphic disks is of interest in itself.

Will describe a reasonably simple method to manage algebraically the pseudoholomorphic disks with boundary on a fixed lagrangian submanifold $L \subset M$.

This will be achieved by defining a chain complex, called the cluster complex, whose homology is an invariant of L up to ambient symplectic diffeomorphism.

- our lagrangian will be connected, compact and relatively spin (which means that the second Stiefel-Whitney class of T_*L vanishes in $H^3(M,L;\mathbf{Z}_2)$), $dim(L)=n$; J almost complex structure which tames ω.

- fix $f: L \to \mathbf{R}$ a Morse function which is Morse-Smale with respect to a metric g.

- $\Lambda = \{\Sigma_{\lambda \in \pi_2(M,L)/\sim} q_\lambda e^\lambda \mid q_\lambda \in \mathbf{Q}$ and there are, for each $K>0$, only finitely many non vanishing q_λ with $\omega(\lambda) < K\}$ where

$$\lambda \sim \tau \text{ iff } \omega(\lambda)=\omega(\tau) \text{ and } \mu(\lambda)=\mu(\tau)$$

is the *Novikov ring*.

The *cluster complex* associated to (L,f,J) is:

$$\mathcal{C}\ell\, C_*(L,f,J) = (T(\mathbf{Q} < Crit(f) >) \otimes \Lambda, d)$$

where TV is the tensor algebra over the vector space V.

For $x \in Crit(f)$, dx is written:

$$dx = \sum \mathrm{m}^x_{x_1,\dots,x_k}(\lambda) x_1 \dots x_k e^\lambda$$

and satisfies the Leibniz rule.

Remark. i. In general, dx contains an "identity" term: $\Sigma_\lambda \mathrm{m}^x(\lambda)e^\lambda$.
ii. The variables x_i *do not* commute and may repeat.
iii. $m^x_x(\lambda)$ might be non-zero.

Grading given by:

$$|x| = ind_f(x) - 1 \text{ if } x \in Crit(f) \ ; \ |e^\lambda| = -\mu(\lambda)$$

and

$$|\sigma x_1 \dots x_k e^\lambda| = \sum |x_i| + |e^\lambda| \ .$$

The moduli spaces.

For $x, x_1, \dots, x_k \in Crit(f)$, $\lambda \in \Lambda$ we will define a moduli space:

$$\mathcal{M}^x_{x_1,\dots,x_k}(\lambda) \ .$$

It consists of all diagrams obtained from a connected tree T_0 with a finite number of vertices v_i and oriented edges e_j so that: the set of univalent vertices (called *boundary* vertices) contains exactly one entrance vertex (called the *root*); a subset of the remaining boundary vertices are called *ends* we assume they are in number of $k \in \mathbf{N}$ ($k = 0$ is allowed).

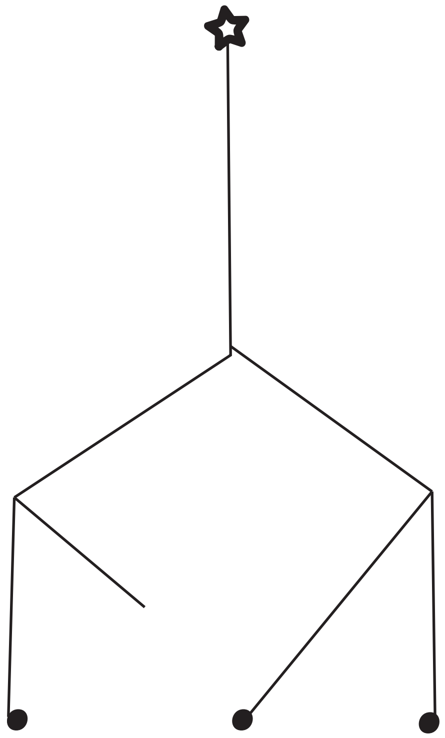


Figure 6: A simple T_0 - the root is on top; the sides are oriented towards the bottom; the ends are indicated by solid dots

Identify in an oriented fashion each edge with a copy of the interval $[0, \ell_j]$ of length $\ell_j \in [0, \infty)$ and replace the vertices v_i that are neither ends nor the root by closed 2-discs D_i so that after each change $v_i \mapsto D_i$, each oriented edge that had v_i as endpoint now has its endpoint on the boundary of D_i.

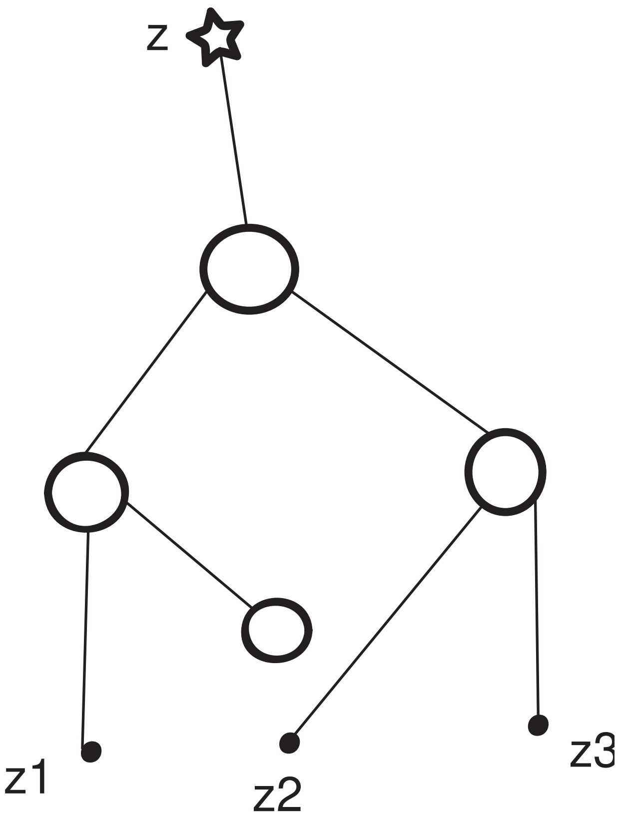


Figure 7: This is what is obtained out of T_0.

Let T be the resulting topological space. Denote the root by z and the endpoints by $z_1, \ldots, z_k$.

Consider the set of maps

$$w : T \to M$$

such that:

(1) $z, z_1, \ldots, z_k$ are mapped to $x, x_1, \ldots, x_k$ respectively.

(2) the restriction γ_j of w to each edge satisfies the negative gradient flow line equation
$$\frac{d\gamma_j}{dt} = -\nabla f(\gamma_j) \ .$$

(3) the restriction u_i of w to each disc D_i satisfies a generic perturbation of the $\bar{\partial}_J = 0$ equation:
$$\bar{\partial}_J u_i = \nu(u_i)$$
where the perturbations $\nu(u_i)$ will be chosen so that the resulting moduli spaces achieve transversality everywhere in a coherent way.

(4) each of the homotopy classes $[u_i] \in \pi_2(M, L)$ is *non-trivial* and their sum equals λ.

The resulting moduli space, is called the *clustered moduli space* corresponding to $x, x_1, \ldots, x_k, \lambda$. By Gromov's compactness theorem, there are only finitely many combinatorial types of such diagrams that can arise for given $\lambda, x, x_1, \ldots, x_k$. Under the transversality assumptions above this moduli space is a manifold of dimension
$$(ind_f(x) - 1) - (\sum_{i=1}^{k}(ind_f(x_i) - 1)) + \mu(\lambda) - 1 \ .$$

(Easy Exercise !).

Main property of the clustered moduli spaces *For generic J, f, g, there are virtual deformations $\nu(u_i)$ so that the moduli spaces $\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)$ admit natural compactifications $\bar{\mathcal{M}}^x_{x_1,\ldots,x_k}(\lambda)$ with a well-defined boundary which verifies:*

$$\partial\bar{\mathcal{M}}^x_{x_1,\ldots,x_k}(\lambda) = \bigcup_{s,y,\lambda'+\lambda''=\lambda} \bar{\mathcal{M}}^x_{x_1,\ldots,x_{i-1},y,x_{i+s},\ldots x_k}(\lambda') \times \bar{\mathcal{M}}^y_{x_i,\ldots,x_{i+s-1}}(\lambda'')$$

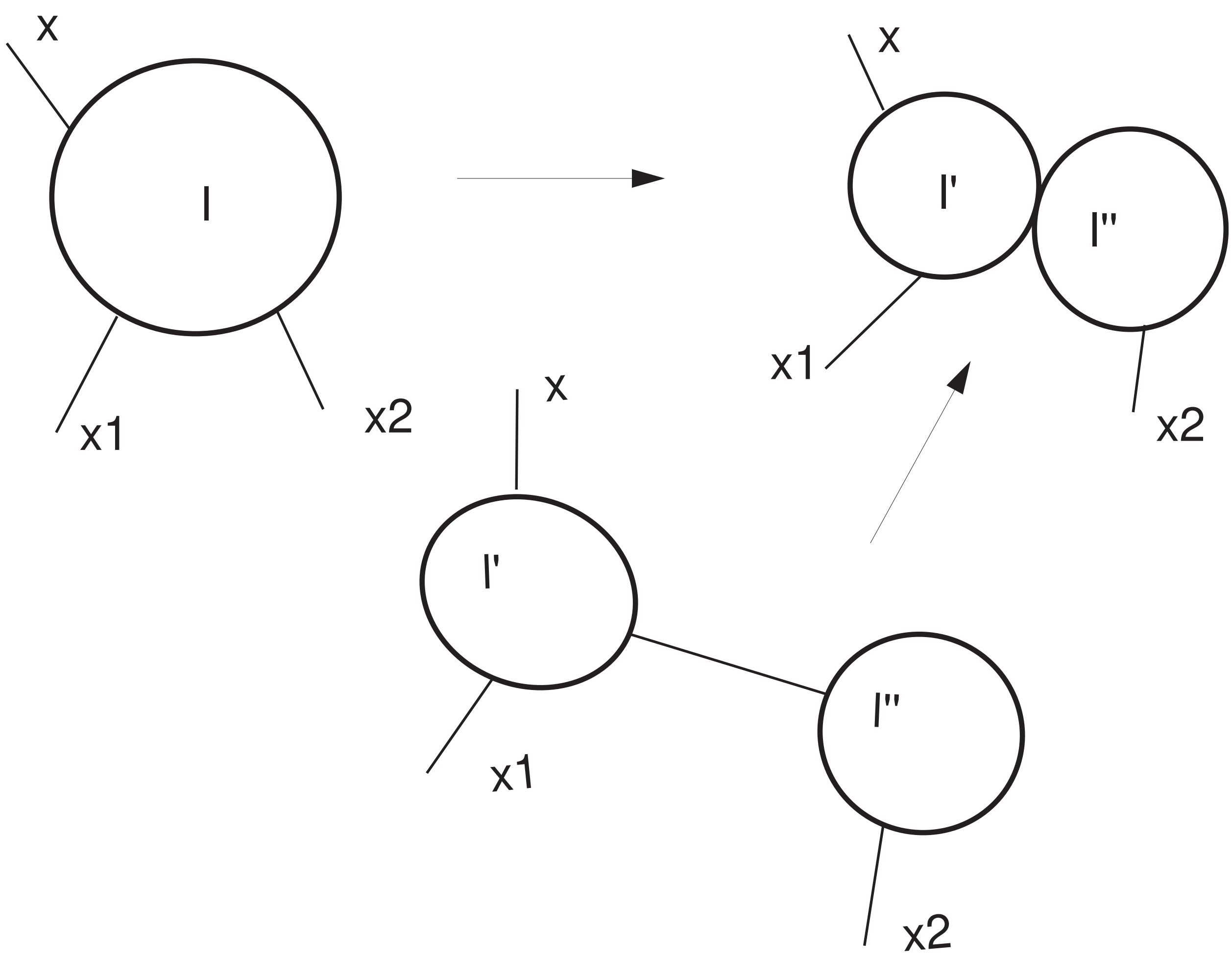
Key point to see this: consider the example below $\mathcal{M}^x_{x_1,x_2}(l)$.

One component, $\mathcal{M}^x_{x_1,x_2}(l;1)$, of this space appears when one disk D with $\mu(D) = l$ is connected to the root $x \in Crit(f)$ and to the two ends x_1 and x_2.

This is not a pseudo-cycle for two reasons:

- one of the connecting flow lines might break (as they are of variable length): this is OK - taken care of by the formula.

- a disk might bubble off: if this happens, the resulting configuration is also a "boundary" for the moduli space of configurations of two disks joined together by a flow line whose length tends to 0 (and with the respective connections to x, x_1, x_2).


x
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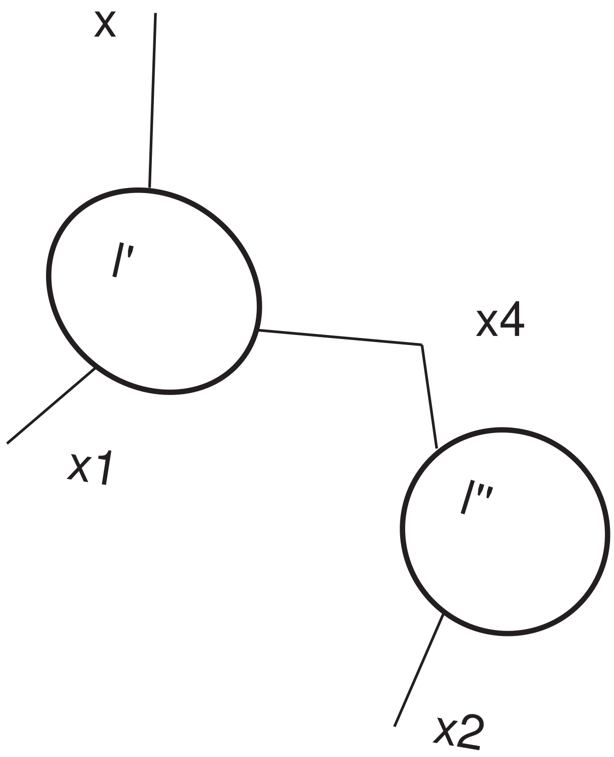


Denote the moduli space of configurations as in the second case by $\mathcal{M}^x_{x_1,x_2}(l',l'')$. It is top dimensional. This remark shows that the boundary of $\mathcal{M}^x_{x_1,x_2}(l)$ (the moduli space which assembles together the components $\mathcal{M}^x_{x_1,x_2}(l,1)$, $\mathcal{M}^x_{x_1,x_2}(l',l'' : l'+l''=l)$ etc) has a top dimensional stratum consisting of cluster trees which are only broken at some some critical point of f. This is exemplified above by an element which belongs to

$$\mathcal{M}^x_{x_1,x_4}(l') \times \mathcal{M}^{x_4}_{x_2}(l'') \ .$$

The cluster differential.

We put

$$\mathrm{m}^{x}_{x_1,\ldots,x_k}(\lambda) = \#(\mathcal{M}^{x}_{x_1,\ldots,x_k})$$

and

$$dx = \sum \mathrm{m}^{x}_{x_1,\ldots,x_k}(\lambda) x_1 \ldots x_k e^{\lambda} \ .$$

The boundary formula for the clustered moduli spaces implies

<u>**"Theorem" C.**</u> $d^2 = 0$.

We denote the resulting *cluster homology* by

$$\mathcal{C}\ell\, H_*(L, f, J) \ .$$

<u>Remark a.</u> If no bubbling is present, then $\mathcal{C}\ell\, H_*(L) = T(H_*(L))$

<u>Remark b.</u> M. Schwarz has used spheres with spikes in his work on the pair of pants/quantum product. Some use of disks connected by flow lines (without "bifurcations") appears in an older paper of Oh. b. While the analytical foundation needed for this construction is similar to that used by Fukaya, Ono, Oh, Ohta in their A^{∞} work, the moduli spaces counted here are *different* from theirs.

IV. An extension, invariance of cluster hlgy and an application.

An (obvious) extension.

It is possible to define a "richer" complex

$$\widetilde{\mathcal{C}\ell}\, C_*(L, f, J)$$

which encodes the higher dimensional moduli spaces.

As in the first two lectures identify all critical points of f to a single point in L.

Let S_k is the space of all the continuous maps from all cluster trees with k ends into L (no pseudoholomorphic condition !) sending the root and the ends to the base point.

The underlying vector space of the "rich" cluster complex is :

$$\oplus_{k\geq 0} C_*(S_k) \otimes \mathbf{Q} < Crit(f) >^{\otimes k}$$

completed with respect to the filtration $F^l = \oplus_{k \geq l} C_*(S_k) \otimes \mathbf{Q} < Crit(f) >^{\otimes k}$

There are certain products

$$S_k \times_i S_l \to S_{k+l-1}$$

which correspond to joining the i-th end of the first tree to the root of the second.

The differential in this case is:

$$\tilde{d}x = \sum \widetilde{\mathrm{m}}^{x}_{x_1,\ldots,x_k}(\lambda) x_1 \ldots x_k e^{\lambda}$$

where $\widetilde{\mathrm{m}}^{x}_{x_1,\ldots,x_k} \in C_*(S_k)$ form a representing chain system for the moduli spaces:

$$\mathcal{M}^{x}_{x_1,\ldots,x_k}(\lambda) \ .$$

Remark. a. This "rich" cluster complex has a strong (cacti ?) operadic flavor which remains to be explored.

b. Initially we intended to use instead of the S_k's a different parametrization with values in the the space $(\Omega L)^{k+1}$ but this turned out to not be appropriate.

Invariance.

<u>**"Theorem" D.**</u> *$\mathcal{C}\ell\, H_*(L, f, J)$ only depends of the (ambient) symplectic isotopy class of the embedding $L \subset M$.*

Notice that it suffices to show that $\mathcal{C}\ell\, H_*(L, f, J)$ does not depend on f or J.

Take f', J' so that $\mathcal{C}\ell\, C_*(L, f', J')$ is also defined. We intend to construct a morphism:

$$\phi : \mathcal{C}\ell\, C_*(L, f, J) \to \mathcal{C}\ell\, C_*(L, f', J')\ .$$

Once this construction is achieved it is not difficult to follow the standard Floer scheme to show that it induces an isomorphism in homology.

To construct ϕ we follow an obvious method:

We fix a Morse cobordism h between f and f'.

This is a homotopy

$$h : L \times [0,1] \to \mathbf{R}$$

so that

i. $h|_{L\times\{0\}} = f$, $h|_{L\times\{1\}} = f' + k$ for some constant k.

ii. $\frac{\partial h}{\partial t}(x,0) = 0 = \frac{\partial h}{\partial t}(x,1)$ for all $x \in L$, $\frac{\partial h}{\partial t}(x,t) > 0$ for all $x \in L$, $t \in (0,1)$.

iii. h is Morse so that

$$Crit_i(h) = Crit_i(f) \times \{0\} \cup Crit_{i-1}(f') \times \{1\} \ .$$

The relevant moduli spaces will be denoted by

$$\mathcal{N}^x_{y_1,\ldots,y_k}(\lambda)$$

where $x \in Crit f'$, $y_i \in Crit(f)$.

To define them, we fix a one parameter family of almost complex structures J_t on M so that $J_0 = J$ and $J_1 = J'$.

$\mathcal{N}^x_{y_1,\ldots,y_k}(\lambda)$ are analogous to $\mathcal{M}^x_{x_1,\ldots,x_k}(\lambda)$

The domains of the configurations assembled in $\mathcal{N}^x_{y_1,\ldots,y_k}$ are again clustered trees but the maps w satisfy a different set of conditions that we now list:

(0') $w : T \to M \times [0,1]$.

(1') $z, z_1, \ldots, z_k$ are mapped to $x \times \{1\}, y_1 \times \{0\}, \ldots, y_k \times \{0\}$ respectively.

(2') the restriction γ_j of w to each edge satisfies the negative gradient flow line equation

$$\frac{d\gamma_j}{dt} = -(\nabla h)(\gamma_j) \ .$$

(3') the restriction u_i of w to each disc D_i has its image contained in $M \times \{t_i\}$ for some *unique* t_i and satisfies a generic perturbation of the $\bar{\partial}_{J_{t_i}} = 0$ equation:

$$\bar{\partial}_{J_{t_i}} u_i = \nu(u_i)$$

where the perturbations $\nu(u_i)$ are chosen so that the resulting moduli spaces achieve transversality everywhere in a coherent way.

(4') each of the classes $[u_i]$ is *non-trivial* and their sum equals λ.

For suitable perturbations $\nu(u_i)$, $\mathcal{N}^x_{y_1,\ldots,y_k}(\lambda)$ has a top dimensional stratum of dimension

$$ind_{f'}(x) - (\sum_i (ind_f(y_i) - 1) + \mu(\lambda) - 1 \ .$$

By Gromov compactness it admits a compactification $\bar{\mathcal{N}}^x_{y_1,\ldots,y_k}(\lambda)$ whose boundary verifies:

$$\partial\bar{\mathcal{N}}^x_{y_1,\ldots,y_k}(\lambda) = \cup_{s,y,\lambda'+\lambda''=\lambda} \bar{\mathcal{N}}^x_{y_1,\ldots,y_{i-1},y,y_{i+s},\ldots y_k}(\lambda') \times \bar{\mathcal{M}}^y_{y_i,\ldots,y_{i+s-1}}(\lambda'', f) \ \cup$$
$$\cup_{\lambda'+\lambda''=\lambda, x_1,\ldots,x_s} [\bar{\mathcal{M}}^x_{x_1,\ldots x_s}(\lambda', f') \times_{\cup y_{i_j}=\{y_1,\ldots,y_k\}, \sum \lambda_i = \lambda''} \ (\bar{\mathcal{N}}^{x_i}_{y_{i_1},\ldots,y_{i_r}}(\lambda_i))] \ . \tag{11}$$

The proof of this formula is similar to the one for the $\mathcal{M}$'s + two key remarks:

A. each pseudoholomorphic disk is "vertical" & properties of $h \Rightarrow u \in \bar{\mathcal{N}}^x_{y_1,\ldots,y_k}(\lambda)$ can only leave $L \times \{1\}$ via a critical point of f'.

Similarly, the only way such a u can reach $L \times \{0\}$ is through a critical point of f.

This also means that such a u can break only on precisely one of $L \times \{0\}$ or $L \times \{1\}$ (bubbling is, of course, taken care of as before).

B. The second remark is that a u as above whose break points are on

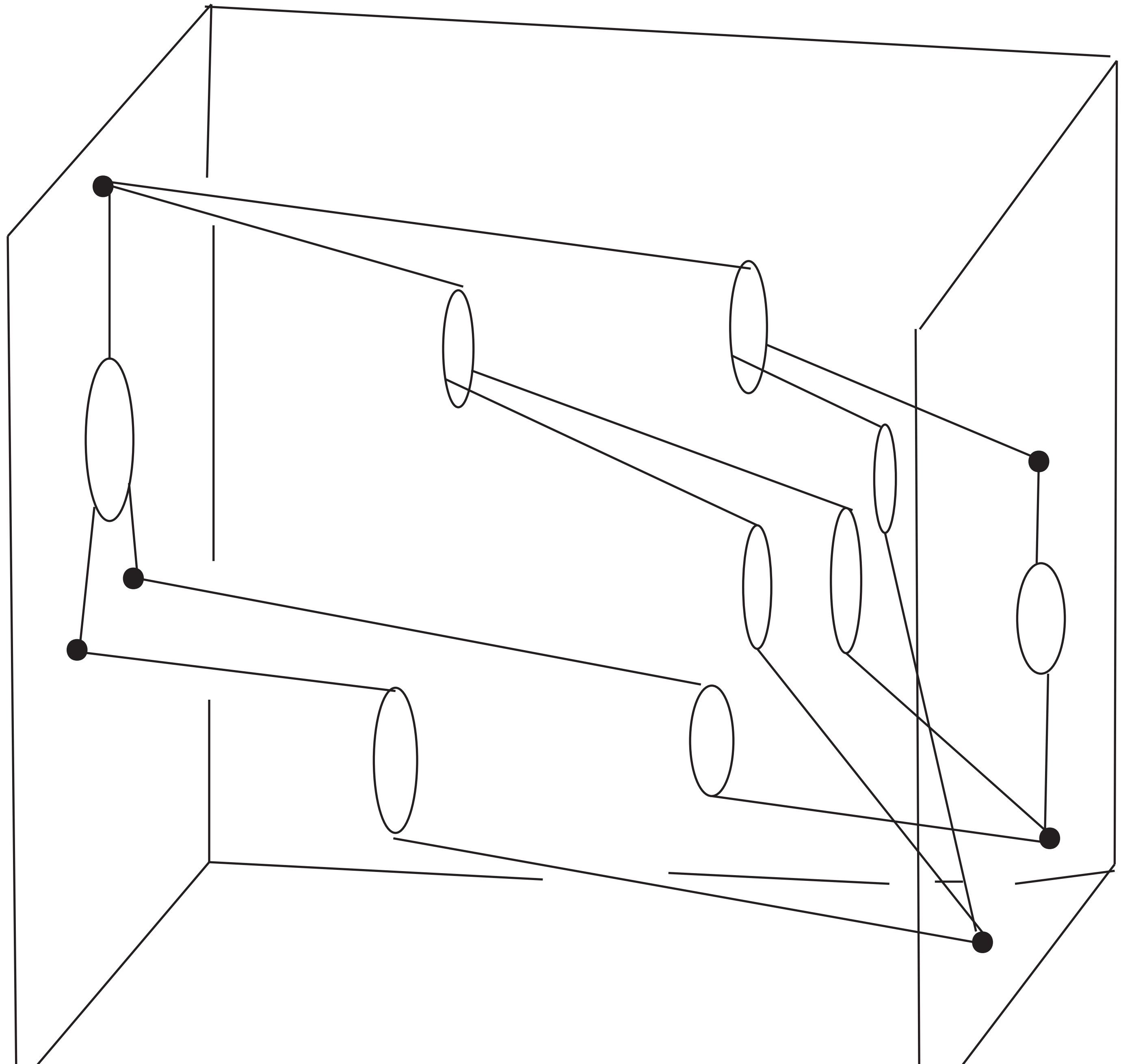
$$L \times \{1\}$$

might present more than one break and still belong to the top dimensional stratum of $\partial \bar{\mathcal{N}}^{x}_{y_1,\ldots,y_k}(\lambda)$.

A simple dimension calculation indicates that this is so:

$$\begin{aligned} & dim(\bar{\mathcal{M}}^{x}_{x_1,\ldots x_s}(\lambda', f') \times_{\cup y_{i_j} = \{y_1,\ldots,y_k\}, \sum \lambda_i = \lambda''} (\bar{\mathcal{N}}^{x_i}_{y_{i_1},\ldots,y_{i_r}}(\lambda_i)) = \\ & = (ind_{f'}(x) - 1 - \Sigma_i(ind_{f'}(x_i) - 1) + \mu(\lambda') - 1) + \\ & \qquad + \Sigma_i[ind_{f'}(x_i) - \Sigma_j(ind_f(y_{i_j}) - 1) + \mu(\lambda_i) - 1] = \\ & = ind_{f'}(x) - \Sigma_j(ind_f(y_j) - 1) + \mu(\lambda' + \lambda'') - 2 \; . \end{aligned}$$

See the next picture which represents $L \times [0, 1]$ and where $L \times \{1\}$ is at the left and $L \times \{0\}$ is at the right. The dots are critical points of h.



An application.

Assume that L can be disjoined from itself by a Hamiltonian isotopy.

Expected value of cluster homology in this case

$$\mathcal{C}\ell\, H_*(L) = \mathbf{Q}\ .$$

Corollary. If L can be disjoined from itself by a Hamiltonian isotopy, then through each point of L passes a pseudoholomorphic disk.

Proof. Fix $P \in L$. Take f Morse with a single minimum $m = P$.

we have a continuous, forgetful map:

$$\psi_m : \mathcal{M}^x_{m,x_1,x_2,\ldots,x_k}(\lambda) \to \mathcal{M}^x_{x_1,x_2,\ldots,x_k}(\lambda)$$

which forgets the edge arriving in m.

Notice that:

$$dim(\mathcal{M}^x_{x_1,x_2,\ldots,x_k}(\lambda)) = dim(\mathcal{M}^x_{m,x_1,x_2,\ldots,x_k}(\lambda)) - 1\ .$$

This means that $\mathcal{M}^x_{m,x_1,x_2,\ldots,x_k}(\lambda) \neq \emptyset \Rightarrow dim(\mathcal{M}^x_{m,x_1,x_2,\ldots,x_k}(\lambda)) > 0$.

Recall $|m| = -1$. Suppose that $dm = 0$.

The remark above shows that m can not be a boundary in the cluster complex $\Rightarrow [m] \neq 0$. But this contradicts $\mathcal{C}\ell\, H_*(L) = \mathbf{Q}$.

Therefore, $dm \neq 0$. In other words $\exists\, \mathcal{M}^m_{x_1,x_2,\ldots,x_k}(\lambda) \neq \emptyset$.
But m is a minimum $\Rightarrow$ there is a pseudoholomorphic disk passing through m.

Putting this together with the results of the first two lectures:

Conjecture.
Any two lagrangian submanifolds L, L' which are Hamiltonian isotopic have the property that for any $P \in L \backslash L'$ there exists either a pseudoholomorphic strip or a pseudoholomorphic disk passing through P.

Cautionary final note: The material included in the last two lectures still requires at this time a sound analytical foundation (maybe, in part at least, this will be provided by the framework described by Helmut Hofer at this SMS). Hence the quotation marks in Theorems C and D.