MOTIVATING THE MULTIPLICATIVE SPECTRUM

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Dedicated to the memory of S. D. Chowla

Abstract. In this article, we describe and motivate some of the results and notions from our ongoing project [2]. The results stated here are substantially new (unless otherwise attributed) and detailed proofs will appear in [2].

1. Definitions and Properties of the Spectrum

Let S be a subset of the unit disc U. By $\mathcal{F}(S)$ we denote the class of completely multiplicative functions f such that $f(p) \in S$ for all primes p. Our main concern is: What numbers arise as mean-values of functions in $\mathcal{F}(S)$?

Precisely, we define

$$\Gamma_N(S) = \left\{ \frac{1}{N} \sum_{n \le N} f(n) : f \in \mathcal{F}(S) \right\} \text{ and } \Gamma(S) = \lim_{N \to \infty} \Gamma_N(S).$$

Here and henceforth, if we have a sequence of subsets J_N of the unit disc $U := \{|z| \le 1\}$, then by writing $\lim_{N\to\infty} J_N = J$ we mean that $z \in J$ if and only if there is a sequence of points $z_N \in$

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J_N with $z_N \to z$ as $N \to \infty$. We call $\Gamma(S)$ the *spectrum* of the set S and our main object is to understand the spectrum. Although we can determine the spectrum explicitly only in a few cases (see Theorem 1 below for the most interesting of these cases), we are able to qualitatively describe it, and obtain a lot of its geometric structure. For example, we can always determine the boundary points of the spectrum (that is the elements of $\Gamma(S) \cap \mathbb{T}$ where $\mathbb{T}$ is the unit circle). Another property is that the spectrum is always connected. Qualitatively, the spectrum may be described in terms of Euler products and solutions to certain integral equations.

We begin with a few immediate consequences of our definition:

- $\Gamma(S)$ is a closed subset of the unit disc U.
- $\Gamma(S) = \Gamma(\overline{S})$ (where $\overline{S}$ denotes the closure of S). Henceforth, we shall assume that S is always closed.
- If $S_1 \subset S_2$ then $\Gamma(S_1) \subset \Gamma(S_2)$.
- $\Gamma(\{1\}) = \{1\}$.

One of the main results of [2], which formed the initial motivation to study the questions discussed herein, is a precise description of the spectrum of $[-1, 1]$.

Theorem 1. *The spectrum of the interval* $[-1, 1]$ *is the interval* $\Gamma([-1, 1]) = [\delta_1, 1]$ *where* $\delta_1 = 2\delta_0 - 1 = -0.656999\ldots$, *and*

$$\begin{aligned}\delta_0 &= 1 - \frac{\pi^2}{6} - \log(1 + \sqrt{e}) \log \frac{e}{1 + \sqrt{e}} + 2 \sum_{n=1}^{\infty} \frac{1}{n^2} \frac{1}{(1 + \sqrt{e})^n} \\ &= 0.17150049\ldots.\end{aligned}$$

Theorem 1 tells us that for any real-valued completely multiplicative function f with $|f(n)| \leq 1$,

$$\sum_{n \leq x} f(n) \geq (\delta_1 + o(1))x. \tag{1}$$

In 1994, Roger Heath-Brown conjectured that there is some constant $c > -1$ such that $\sum_{n \leq x} f(n) \geq (c + o(1))x$. Richard Hall [5] proved this conjecture, and, in turn, conjectured (as did Hugh L.

Montgomery independently) the stronger estimate (1). Both Hall and Montgomery noticed that the estimate (1) is best possible by taking

$$f(q) = \begin{cases} 1 & \text{for primes } q \le x^{1/(1+\sqrt{e})} \\ -1 & \text{for primes } x^{1/(1+\sqrt{e})} \le q \le x. \end{cases} \tag{2}$$

In this example, the reader can verify (or see [5]) that equality holds in (1). Our proof shows that this is essentially the only case when equality holds in (1). The proof of Theorem 1 is too complicated to be given here; however, in §3, we shall sketch its salient features and prove a weaker estimate with $-(2 - 2/\sqrt{e}) = -0.78693\ldots$ replacing δ_1 in (1).

By applying our Theorem to the completely multiplicative function $f(n) = \left(\frac{n}{p}\right)$, for some prime p, we deduce that $\ge (\delta_0 + o(1))x$ integers below x are quadratic residues $\pmod p$. In fact, the constant δ_0 here is best possible. To see this, we choose p such that $\left(\frac{q}{p}\right)$ is given as in the Hall-Montgomery example (2); that infinitely many such primes exist follows from quadratic reciprocity and Dirichlet's theorem on primes in arithmetic progressions.

For a given $f \in \mathcal{F}(S)$, the mean-value of f (that is, $\lim_{x\to\infty} x^{-1} \sum_{n \le x} f(n)$), if it exists, is obviously an element of the spectrum $\Gamma(S)$. We begin by trying to understand the subset of the spectrum consisting of such mean-values.

Suppose that $1 \in S$ and consider $f \in \mathcal{F}(S)$ satisfying $f(p) = 1$ for all large primes p. Writing $f(n) = \sum_{d|n} g(d)$ for some multiplicative function $g(d)$, an easy argument shows that

$$\begin{aligned}\lim_{x\to\infty} \frac{1}{x} \sum_{n\le x} f(n) &= \sum_{d=1}^{\infty} \frac{g(d)}{d} \\ &= \prod_p \left(1 + \frac{f(p)}{p} + \frac{f(p^2)}{p^2} + \dots\right)\left(1 - \frac{1}{p}\right) \\ &=: \Theta(f, \infty). \end{aligned} \tag{3}$$

Thus the element $\Theta(f, \infty) \in \Gamma(S)$, and if we define $\Gamma_\Theta(S)$ to be the closure of the set of values $\Theta(f, \infty)$ we have $\Gamma_\Theta(S) \subset \Gamma(S)$.

Here, we assumed that $1 \in S$ and if $1 \notin S$ we define $\Gamma_\Theta(S) = \{0\}$. We call $\Gamma_\Theta(S)$ the Euler product spectrum of S.

Proving an old conjecture of Erdős and Wintner, Wirsing [10] showed that every real multiplicative function with $|f(n)| \leq 1$ has a mean-value. In fact, he proved that (3) always holds for such functions. Thus, when $S \subset [-1, 1]$ Wirsing's Theorem gives that $\Gamma_\Theta(S)$ is precisely the set of mean-values of elements in $\mathcal{F}(S)$. The critical point in Wirsing's Theorem is to show that if f is real valued and $\sum_p (1 - f(p))/p$ diverges, then $x^{-1} \sum_{n \leq x} f(n) \to 0$.

The situation is more delicate for complex valued multiplicative functions. For example, the function $f(n) = n^{i\alpha}$ (α a non-zero real) does not have a mean-value; indeed $\sum_{n \leq x} f(n) \sim x^{1+i\alpha}/(1+i\alpha)$. Again, note that here $\sum_p (1 - \text{Re } p^{i\alpha})/p$ diverges but $x^{-1} \sum_{n \leq x} n^{i\alpha}$ does not tend to 0. Halász [3] realised that the problem with this example is that the set $\{f(p)\}$ is everywhere dense on $\mathbb{T}$. He proved that if $\sum_p (1 - \text{Re } f(p)p^{-i\alpha})/p$ diverges (which obviously does not hold for the troublesome example $n^{i\alpha}$) for all real α then $x^{-1} \sum_{n \leq x} f(n) \to 0$; and he quantified how fast this tends to 0.

Over the years Halász' Theorem has been considerably refined, and recently Hall [4] found the following useful formulation: Let D be a convex subset of U containing 0. If $f \in \mathcal{F}(D)$ then

$$\sum_{n \leq x} f(n) \ll x \exp\left(-\eta(D) \sum_{p \leq x} \frac{1 - \text{Re } f(p)}{p} \right), \tag{4}$$

where $\eta(D)$ is a constant defined in terms of the geometry of D, and $\eta(D) > 0$ when the perimeter length of the closure of D is $< 2\pi$. Using (4), it is easy to check that if $1 \notin S$ then $\Gamma(S) = \{0\}$ (recall that S is assumed to be closed). Thus we may assume henceforth that $1 \in S$.

But all this doesn't answer when (3) holds. We now give a criterion for the set S, such that for all $f \in \mathcal{F}(S)$, (3) is true, thus generalizing Wirsing's Theorem. To formulate this fluidly, and for subsequent results, we introduce the notion of the *angle* of a set.

For any $V \subseteq U$, define

$$\text{Ang}(V) := \sup_{\substack{v \in V \\ v \neq 1}} |\arg(1-v)|. \tag{5}$$

Note that each such $1-v$ has positive real part, so $0 \leq \text{Ang}(V) \leq \pi/2$. We adopt the convention that $\text{Ang}(\{1\}) = \text{Ang}(\emptyset) = 0$. Sometimes we will speak of the angle of a point $z \in U$ ($z \neq 1$); by this we mean $\text{Ang}(z) = |\arg(1-z)|$.

Theorem 2. *Suppose $S \subset U$ and $Ang(S) < \pi/2$. Then (3) holds for every $f \in \mathcal{F}(S)$; that is, every $f \in \mathcal{F}(S)$ has a mean-value. Thus,*

$$\begin{aligned}\Gamma_\Theta(S) &= \left\{ \lim_{N\to\infty} \frac{1}{N} \sum_{n \leq N} f(n) : \; f \in \mathcal{F}(S) \right\} \\ &= \left\{ \prod_p \left(\frac{p-1}{p-f(p)} \right) : \; f \in \mathcal{F}(S) \right\}.\end{aligned}$$

If $S \subset [-1,1]$ then $\text{Ang}(S) = 0$, and so Theorem 2 generalizes Wirsing's result. If $\alpha \neq 0$ is real then $\text{Ang}(\{p^{i\alpha}\}) = \pi/2$, and thus Theorem 2 avoids the example $f(n) = n^{i\alpha}$. What happens when $\text{Ang}(S) = \pi/2$? It will follow from results stated below that here $\Gamma(S) = \Gamma_\Theta(S) = U$. Thus the spectrum is quite easy to understand here.

In general, the spectrum contains more elements than simply the Euler products. For example, the spectrum of Euler products for $S = [-1,1]$ is simply the interval $[0,1]$. However, as Theorem 1 shows, the spectrum of S is more exotic. We now describe a family of integral equations whose solutions belong to the spectrum. In Theorem 3, we shall show that all points of the spectrum may be obtained by suitably combining an Euler product and a solution to one of these integral equations.

Recall that we assume S is closed and $1 \in S$. We define $\Lambda(S)$ to be the set of values $\sigma(u)$ obtained as follows. Let $\chi(t)$ be any measurable function with $\chi(t) = 1$ for $0 \leq t \leq 1$ and with $\chi(t)$

belonging to the convex hull of S, for all t. Next define $\sigma(u) = 1$ for $0 \le u \le 1$ and for $u > 1$ by the integral equation

$$u\sigma(u) = \sigma * \chi = \int_0^u \sigma(u-t)\chi(t)dt. \tag{6}$$

Here, and throughout, $f * g$ denotes the convolution of the two functions f and g: that is, $f * g(x) = \int_0^x f(t)g(x-t)dt$. As we prove in [2] (see Theorem 7 below), there is a unique solution $\sigma(u)$ to (6). This solution is continuous and satisfies $|\sigma(u)| \le 1$ for all u.

That the integral equation (6) is relevant to the study of multiplicative functions was already observed by Wirsing [10]. This connection may be seen from the following Proposition.

Proposition 1. *Let f be a multiplicative function with $|f(n)| \le 1$ for all n and $f(n) = 1$ for $n \le y$. Let $\vartheta(x) = \sum_{p \le x} \log p$ and define*

$$\chi(u) = \chi_f(u) = \frac{1}{\vartheta(y^u)} \sum_{p \le y^u} f(p) \log p.$$

Then $\chi(t)$ is a measurable function taking values in the unit disc and with $\chi(t) = 1$ for $t \le 1$. Let $\sigma(u)$ be the corresponding unique solution to (6). Then

$$\frac{1}{y^u} \sum_{n \le y^u} f(n) = \sigma(u) + O\left(\frac{u}{\log y}\right).$$

The converse to Proposition 1 is also true:

Proposition 1 (Converse). *Let $S \subset U$ and suppose χ takes on values in the convex hull of S with $\chi(t) = 1$ for $t \le 1$. Given $\epsilon > 0$ and $u \ge 1$ there exist arbitrarily large y and $f \in \mathcal{F}(S)$ with $f(n) = 1$ for $n \le y$ and*

$$\left| \chi(t) - \frac{1}{\vartheta(y^u)} \sum_{p \le y^u} f(p) \log p \right| \le \epsilon \quad \textit{for almost all } 0 \le t \le u.$$

Consequently, if $\sigma(u)$ is the solution to (6) for this χ then

$$\sigma(t) = \frac{1}{y^t} \sum_{n \le y^t} f(n) + O(\epsilon \log(u+2)) + O\left(\frac{u}{\log y}\right) \quad \textit{for all } t \le u.$$

If J and K are two subsets of the unit disc, we define $J \times K$ to be the set of elements $z = jk$ where $j \in J$ and $k \in K$.

Theorem 3. *For any $S \subset U$, $\Gamma(S) = \Gamma_\Theta(S) \times \Lambda(S)$.*

Researchers in the field have long "known" that results like Proposition 1 and Theorem 3 can used when needed (see [10] and [6], for instance). But this appears to be the first attempt to provide a complete proof of such a result in this generality. The idea of the proof of Theorem 3 is to decompose $f \in \mathcal{F}(S)$ into two parts: $f_s(p) = f(p)$ for $p \le y$ and $f_s(p) = 1$ for $p > y$, and $f_l(p) = 1$ for $p \le y$ and $f_l(p) = f(p)$ for $p > y$. For appropriately chosen y, the average of f until x is approximated by the product of the averages of f_s and f_l. If y is small enough compared with x, then the average of f_s is approximated by $\Theta(f_s, \infty) \in \Gamma_\Theta(S)$. Proposition 1 shows that if y is not too small, the average of f_l is approximated by the solution to an integral equation. Combining these, one gets that $\Gamma(S) \subset \Gamma_\Theta(S) \times \Lambda(S)$. The proof that $\Gamma_\Theta(S) \times \Lambda(S) \subset \Gamma(S)$ is similar, invoking the converse of Proposition 1.

As the case $S = [-1, 1]$ illustrates, $\Gamma_\Theta(S)$ represents the easy part of the spectrum while $\Lambda(S)$ is more mysterious. Here Theorems 1 and 3 tell us that $\Lambda(S) \subset [\delta_1, 1]$. That is, given any measurable function χ with $\chi(t) = 1$ for $t \le 1$, and $-1 \le \chi(t) \le 1$ always, then $\sigma(u) \ge \delta_1$ for all u (where σ is the corresponding solution to (6)). An important example is the function $\chi(t) = 1$ for $t \le 1$ and $\chi(t) = -1$ for $t > 1$. Denote by $\rho_1(u)$ the corresponding solution to (6). Then $\rho_1(u)$ satisfies a differential-difference equation very similar to that satisfied by the Dickman-de Bruijn function. Namely, $\rho_1(u) = 1$ for $u \le 1$ and for $u > 1$,

$$u\rho_1'(u) = -2\rho_1(u-1).$$

It is not hard to verify that $\rho_1(u)$ decreases for u in $[1, 1+\sqrt{e}]$ and increases for $u > 1+\sqrt{e}$. The absolute minimum $\rho_1(1+\sqrt{e})$ is guaranteed by Theorem 1 to be $\geq \delta_1$ and in fact $\rho_1(1+\sqrt{e}) = \delta_1$. By continuity, $\rho_1(u)$ takes on all values in the interval $[\delta_1, 1]$ showing that $\Lambda(S) = [\delta_1, 1]$.

We now describe a lot of properties of $\Gamma_\Theta(S)$ which are also inherited by $\Gamma(S)$; in many cases, we get an explicit description of $\Gamma_\Theta(S)$. To state our results we introduce the set $\mathcal{E}(S)$ defined as follows. Let $\mathcal{E}(S) = \{0\}$ if $1 \notin \overline{S}$ and if $S \subset U$ is closed with $1 \in S$, we define

$$\mathcal{E}(S) = \{e^{-k(1-\alpha)} : \quad k \geq 0, \quad \alpha \text{ is in the convex hull of } S\}.$$

Thus $\mathcal{E}(S)$ consists of various 'spirals' connecting 1 to 0.

Theorem 4. *For all subsets S of U,*

$$\mathcal{E}(S) \times [0,1] \supset \Gamma_\Theta(S) = \Gamma_\Theta(S) \times \mathcal{E}(S) \supset \mathcal{E}(S).$$

If the convex hull of S contains a real point other than 1, (in other words, if $-1 \in S$ or if $\alpha, \beta \in S$ with $\mathrm{Im}\ (\alpha) > 0 > \mathrm{Im}\ (\beta)$) then

$$\Gamma_\Theta(S) = \mathcal{E}(S) = \mathcal{E}(S) \times [0,1],$$

and $\Gamma_\Theta(S)$ is starlike *(that is, if $z \in \Gamma_\Theta(S)$ then $\Gamma_\Theta(S)$ contains the line joining 0 and z).*

If $\mathrm{Ang}(S) = \pi/2$, then it is easy to see that $\mathcal{E}(S) = U$. Hence in this case $\Gamma(S) = \Gamma_\Theta(S) = U$, as we claimed earlier. Theorems 3 and 4 enable us to deduce some basic properties of the spectrum.

Corollary 1. *For all subsets S of U, $\Gamma(S) = \Gamma(S) \times \mathcal{E}(S)$. Consequently, the spectrum of S is connected. If the convex hull of S contains a real point other than 1, then the spectrum is starlike. If 1, $e^{i\alpha}$ and $e^{i\beta}$ are distinct elements of S, then the spectrum contains the disc centered at the origin with radius $\exp(-2\pi/(|\cot(\alpha/2) - \cot(\beta/2)|))$. In fact, $\mathcal{E}(S)$ contains this disc.*

The sets $\Gamma_\Theta(S)$ and $\Gamma(S)$ have the property that multiplying by $\mathcal{E}(S)$ leaves them unchanged. It turns out that $\Lambda(S)$ also has this property: $\Lambda(S) = \Lambda(S) \times \mathcal{E}(S)$. This gives us the following variant of Theorem 3 which reveals that $\Lambda(S)$ contains all the information about the spectrum.

Theorem 3′. *For all subsets S of U,*

$$\Lambda(S) \subset \Gamma(S) \subset \Lambda(S) \times [0,1].$$

If the convex hull of S contains a real point different from 1 then $\Gamma(S) = \Lambda(S)$.

Next we give a bound on the spectrum and determine $\Gamma(S) \cap \mathbb{T}$.

Theorem 5. *Suppose S is a subset of U with $1 \in \overline{S}$. The spectrum of S is U if and only if $Ang(S) = \pi/2$. If $Ang(S) = \theta < \pi/2$, then there exists a positive constant $A(\theta)$, depending only on θ, such that $\Gamma(S)$ is contained in a disc centered at $A(\theta)$ with radius $1 - A(\theta)$. In fact, $A(\theta) = (28/411)\cos^2\theta$ is permissible. Thus*

$$\Gamma(S) \cap \mathbb{T} = \begin{cases} \{1\} & \text{if } Ang(S) < \pi/2 \\ \mathbb{T} & \text{if } Ang(S) = \pi/2. \end{cases}$$

Applied to the set $S = [-1,1]$, Theorem 5 shows that there exists $c > -1$ such that $\Gamma(S) \subset [c,1]$. Thus Theorem 5 generalises Hall's result on Heath-Brown's conjecture.

By a simple calculation, we can show that $\mathcal{E}(S)$, $\Gamma_\Theta(S)$ and S all have the same angle: $\mathrm{Ang}(\Gamma_\Theta(S)) = \mathrm{Ang}(\mathcal{E}(S)) = \mathrm{Ang}(S)$. From Theorems 3 and 3′ we see that $\mathrm{Ang}(\Gamma(S)) = \mathrm{Ang}(\Lambda(S)) \geq \mathrm{Ang}(S)$.

Conjecture 1. *The angle of the set equals the angle of the spectrum. Thus*

$$Ang(\Gamma_\Theta(S)) = Ang(\mathcal{E}(S)) = Ang(\Lambda(S)) = Ang(\Gamma(S)) = Ang(S).$$

We support this conjecture by showing that $\mathrm{Ang}(S)$ and $\mathrm{Ang}(\Gamma(S))$ are comparable in the situations $\mathrm{Ang}(S) \to 0$ and $\mathrm{Ang}(S) \to \pi/2$.

Theorem 6. *Suppose $S \subset U$ and $Ang(S) = \theta = \pi/2 - \delta$. Then $Ang(S) \ll Ang(\Gamma(S))$ and*

$$\frac{\pi}{2} - \delta = Ang(S) \leq Ang(\Gamma(S)) \leq \frac{\pi}{2} - \frac{\sin\delta}{2}.$$

The first part of the Theorem says that $\mathrm{Ang}(S)$ and $\mathrm{Ang}(\Gamma(S))$ are comparable when $\mathrm{Ang}(S)$ is small. The second part of the Theorem is mainly interesting in the complementary case when $\mathrm{Ang}(S)$ is close to $\pi/2$. In fact, when δ is small we see that we are away from the truth only by a factor of 2 (as $\sin\delta \sim \delta$).

Example. Let $k \geq 3$ and S_k denote the set of k-th roots of unity. If $f \in \mathcal{F}(S_k)$ then $f(n) \in S_k$ for all n. Hence $\Gamma(S_k)$ is contained in the convex hull of S_k: that is, in the regular k-gon with vertices the k-th roots of unity. Notice that this implies $\mathrm{Ang}(\Gamma(S_k)) \leq \mathrm{Ang}(S_k)$, so that $\mathrm{Ang}(S_k) = \mathrm{Ang}(\Gamma(S_k))$ supporting Conjecture 1. Applying Corollary 1 with the two points $e^{\pm 2\pi i/k}$, we conclude that $\Gamma(S_k)$ is starlike and contains the disc centered at 0 with radius $\exp(-\pi\tan(\pi/k))$.

We define *the projection of* (a complex number) z *in the direction* $e^{i\alpha}$ to be $\mathrm{Re}\,(e^{-i\alpha}z)$. Theorem 1 may be re-interpreted as stating that if $z \in \Gamma(\{\pm 1\})$ then the projection of z along -1 is $\leq -\delta_1$. Evidently if $1 \in S$ then $1 \in \Gamma(S)$ so there is always a $z \in \Gamma(S)$ whose projection in the direction 1, is 1, and thus uninteresting to us. This motivates us to define the *maximal projection of the spectrum of a set* $S \subset \mathbb{T}$ as

$$\max_{1\neq\zeta\in S}\ \max_{z\in\Gamma(S)} \mathrm{Re}\,(\zeta^{-1}z).$$

Conjecture 2. *If* $S \subset \mathbb{T}$ *with* $1 \in S$ *and Ang* $(S) = \theta$ *then the maximal projection of* $\Gamma(S)$ *is*

$$\max_{1\neq\zeta\in S}\ \max_{z\in\Gamma(S)} \mathrm{Re}\,(\zeta^{-1}z) = 1 - (1+\delta_1)\cos^2\theta.$$

One half of this conjecture is easy to establish: namely, the maximal projection is $\geq 1-(1+\delta_1)\cos^2\theta$. To see this, let $z = x^{-1}\sum_{n\leq x} f(n)$ where f is the completely multiplicative function defined by $f(p) = 1$ for all $p \leq x^{1/(1+\sqrt{e})}$, and $f(p) = \zeta$ for $x^{1/(1+\sqrt{e})} \leq p \leq x$, where $\zeta \in \mathbb{T}$ and $\mathrm{Ang}(\zeta) = \theta$. Then, a simple calculation (analogous to the calculation in the Hall-Montgomery example (2)) gives that the projection of z along ζ is $1-(1+\delta_1)\cos^2\theta + o(1)$.

Theorem 7. *Conjecture 2 is true for the sets* $S = \{1, -1\}$ *and* $S = \{1, -1, i, -i\}$. *If* $S \subset \mathbb{T}$ *with* $1 \in S$ *and* $Ang\ (S) = \theta$ *then the maximal projection of* $\Gamma(S)$ *is* $\leq 1 - (56/411)\cos^2\theta$.

To facilitate comparison between Theorem 7 and Conjecture 2, we observe that $1 + \delta_1 = 0.3430\ldots$ whereas $56/411 = 0.1362\ldots$. Thus Theorem 6 is not too far away from the (conjectured) truth.

2. Solutions to Integral Equations

In §1, we described the Euler product spectrum and deduced information about the geometric part of the spectrum. We now outline some of our results on the integral equation (6). Our tool in analysing (6) is the Laplace transform, which, for a measurable function $f:\ [0,\infty) \to \mathbb{C}$ is given by

$$\mathcal{L}(f,s) = \int_0^\infty f(t)e^{-ts}dt$$

where s is some complex number. If f is integrable and grows sub-exponentially (that is, for every $\epsilon > 0$, $|f(t)| \ll_\epsilon e^{\epsilon t}$ almost everywhere) then the Laplace transform is well defined for all complex numbers s with Re $(s) > 0$. Laplace transforms occupy a role in the study of differential equations analogous to Dirichlet series in multiplicative number theory.

Theorem 8. *Suppose that* $\chi(t) = 1$ *for* $t \leq 1$ *and* $|\chi(t)| \leq 1$ *for all* t. *There exists a unique solution* $\sigma(u)$ *to (6), which satisfies* $\sigma(u) = 1$ *for* $0 \leq u \leq 1$ *and* $|\sigma(u)| \leq 1$ *for all* u. *In terms of* χ *the solution* σ *is given explicitly by*

$$\sigma(u) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{k!} I_k(u;\chi) \tag{7a}$$

where

$$I_k(u;\chi) := \int_{\substack{t_1,\ldots,t_k \geq 1 \\ t_1+\cdots+t_k \leq u}} \prod_{1 \leq i \leq k} \left(\frac{1-\chi(t_i)}{t_i} \right) dt_1 \ldots dt_k. \tag{7b}$$

In terms of Laplace transforms we have

$$\mathcal{L}(\sigma, s) = \frac{1}{s} \exp\Big(-\mathcal{L}\Big(\frac{1-\chi(t)}{t}, s\Big)\Big).$$

The formula (7ab) has the appearance of an inclusion-exclusion identity, and indeed if we translate this result back in terms of multiplicative functions (via Proposition 1) then we find it can be so interpreted. Therefore if χ is real valued one can obtain upper and lower bounds by truncating the sum in (7a) at odd or even values of k, as in the combinatorial sieve. This is one of the main tools in our proof of Theorem 1.

Proposition 2. *Suppose χ is real valued so that $\chi(t) = 1$ for $t \leq 1$ and $-1 \leq \chi(t) \leq 1$ for all t. Then the corresponding solution $\sigma(u)$ to (6) satisfies, for all $m \geq 0$,*

$$1 + \sum_{k=1}^{2m+1} \frac{(-1)^k}{k!} I_k(u;\chi) \leq \sigma(u) \leq 1 + \sum_{k=1}^{2m} \frac{(-1)^k}{k!} I_k(u;\chi). \tag{8}$$

Our other main tools in the proof of Theorem 1 (and many other results stated in §1) are the non-increasing property of the average $v^{-1}\int_0^v |\sigma(t)|dt$, and a convolution identity connecting $I_1(t,\chi)$ and $I_2(t,\chi)$, which we now describe.

Lemma 1. *Suppose χ is a complex valued measurable function with $\chi(t) = 1$ for $t \leq 1$ and $|\chi(t)| \leq 1$ for all t and let σ be the corresponding solution to (6). Then*

$$A(v) := \frac{1}{v}\int_0^v |\sigma(t)|dt$$

is a non-increasing function of v. Hence, for all $u \geq v$,

$$|\sigma(u)| \leq A(v) = \frac{1}{v}\int_0^v |\sigma(t)|dt. \tag{9}$$

Proof. From (6), we have $|\sigma(u)| \leq A(u)$ for all u. Hence (9) follows once we have shown that $A(v)$ is non-increasing. Differentiating the definition of $A(v)$, we have $A'(v) = |\sigma(v)|/v - A(v)/v \leq 0$, so $A(v)$ is non-increasing and the result follows.

Lemma 2. *Let χ be as in Lemma 1. Then*

$$\int_0^u I_2(t,\chi)dt = \int_0^u I_1(t,\chi)I_1(u-t,\chi)dt.$$

Proof. For brevity, let $\chi_1(t) = (1-\chi(t))/t$. Then $I_1(t,\chi) = (1 * \chi_1)(t)$ and $I_2(t,\chi) = (1 * \chi_1 * \chi_1)(t)$. The left side of our claimed identity is

$$\begin{aligned}(1 * I_2)(u) &= (1 * 1 * \chi_1 * \chi_1)(u) = ((1 * \chi_1) * (1 * \chi_1))(u)\\ &= (I_1 * I_1)(u),\end{aligned}$$

which is the right side of the claimed identity.

A fundamental question is: when does $\lim_{u\to\infty} \sigma(u)$ exist? If it exists, what does it equal? In view of Proposition 1 and its converse, this is equivalent to the question of the existence of the mean-value of a multiplicative function. Thus Theorem 2 has the following implication for integral equations.

Theorem 2′. *Suppose $S \subset U$ and $Ang(S) < \pi/2$. Let $\chi(t) = 1$ for $t \le 1$ and suppose $\chi(t)$ lies in the convex hull of S for all t. Then*

$$\lim_{u\to\infty} \sigma(u) = \exp\left(-\int_1^\infty \frac{1-\chi(t)}{t}dt\right) = \exp(-I_1(\infty,\chi)).$$

Translated in terms of integral equations, Hall's result (4) can be re-interpreted as follows: Suppose that D is a closed, convex subset of the unit disc containing 0 and suppose that χ is measurable with $\chi(t) = 1$ for $t \le 1$ and $\chi(t) \in D$ for $t > 1$. Then

$$\sigma(u) \ll \exp\left(-\eta(D)\int_1^u \frac{1-\text{Re }\chi(t)}{t}dt\right). \tag{10}$$

In [2] we give a direct proof of this result using Laplace transforms. The advantages of our treatment is that the details are considerably simpler, and so we prove (10) giving explicit values to all of the implicit constants. Translating back to multiplicative functions, this allows us to give a completely explicit version of (4) (we are still refining the constants obtained, so do not state this result here). To whet the reader's appetite we now sketch a proof of the integral equations analogue of Montgomery's lemma (see [8] and [9]), which is the key to proving results such as (4) and (10).

For $t > 0$ we define $H(t) = \max_{y\in\mathbb{R}} |\mathcal{L}(\sigma, t+iy)|$ and

$$M(t) = \min_{y\in\mathbb{R}} \text{Re } \mathcal{L}\left(\frac{1-\chi(v)e^{-ivy}}{v}, t\right).$$

Note that

$$\begin{aligned}\mathcal{L}(\sigma, t+iy) &= \frac{1}{t+iy}\exp\left(-\mathcal{L}\left(\frac{1-\chi(v)}{v}, t+iy\right)\right)\\ &= \frac{1}{t+iy}\exp\left(-\mathcal{L}\left(\frac{1-\chi(v)e^{-ivy}}{v}, t\right)\right.\\ &\qquad\qquad \left.+\mathcal{L}\left(\frac{1-e^{-ivy}}{v}, t\right)\right).\end{aligned}$$

The identity

$$\text{Re } \mathcal{L}\left(\frac{1-e^{-ivy}}{v}, t\right) = \log|1+iy/t|$$

is easily proved by differentiating both sides with respect to y, so that

$$t|\mathcal{L}(\sigma, t+iy)| = \exp\left(-\text{Re } \mathcal{L}\left(\frac{1-\chi(v)e^{-ivy}}{v}, t\right)\right)$$

combining the last two estimates. Therefore $tH(t) = \exp(-M(t))$.

Proposition 3. *For any $\alpha > 0$ and any $u \ge 1$, we have*

$$|\sigma(u)| \le \frac{e^{2\alpha u}}{u}\int_\alpha^\infty \frac{H(t)}{t}dt = \frac{e^{2\alpha u}}{u}\int_\alpha^\infty \frac{\exp(-M(t))}{t^2}dt.$$

Note that $H(t) \le \int_0^\infty |\sigma(v)|e^{-tv}dv \le 1/t$ and so the integral in the Proposition converges.

Proof. By Lemma 1 we have

$$\begin{aligned} |\sigma(u)| &\le \frac{1}{u}\int_0^u |\sigma(v)|dv \le \frac{e^{2\alpha u}}{u}\int_0^\infty |\sigma(v)|e^{-2\alpha v}dv \\ &= \frac{e^{2\alpha u}}{u}\int_{v=0}^\infty 2v|\sigma(v)|\int_{t=\alpha}^\infty e^{-2tv}dtdv \\ &= \frac{e^{2\alpha u}}{u}\int_{t=\alpha}^\infty \left(\int_{v=0}^\infty 2v|\sigma(v)|e^{-2tv}dv\right)dt. \end{aligned} \tag{11}$$

We shall prove that for all $t > 0$

$$2\int_0^\infty v|\sigma(v)|e^{-2tv}dv \le \frac{H(t)}{t}, \tag{12}$$

which when inserted in (11) furnishes the Proposition.

By Cauchy-Schwarz

$$\left(2\int_0^\infty v|\sigma(v)|e^{-2tv}dv\right)^2 \le \frac{2}{t}\int_0^\infty |v\sigma(v)|^2e^{-2tv}dv.$$

By Plancharel's formula (since the Fourier transform is an isometry on L^2)

$$\int_0^\infty |v\sigma(v)|^2e^{-2tv}dv = \frac{1}{2\pi}\int_{-\infty}^\infty |\mathcal{L}(v\sigma(v), t+iy)|^2dy.$$

Now $v\sigma(v) = (\sigma * \chi)(v)$ and, since $\mathcal{L}(f*g, x) = \mathcal{L}(f,x)\mathcal{L}(g,x)$, this is

$$\begin{aligned} &= \frac{1}{2\pi}\int_{-\infty}^\infty |\mathcal{L}(\sigma, t+iy)|^2|\mathcal{L}(\chi, t+iy)|^2dy \\ &\le H(t)^2\frac{1}{2\pi}\int_{-\infty}^\infty |\mathcal{L}(\chi, t+iy)|^2dy. \end{aligned}$$

Applying Plancharel again we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |\mathcal{L}(\chi, t+iy)|^2 dy = \int_0^{\infty} |\chi(v)|^2 e^{-2tv} dv$$
$$\le \int_0^{\infty} e^{-2tv} dv = \frac{1}{2t}.$$

Assembling the above estimates we obtain (12) and hence the Proposition.

Given χ, how does the solution $\sigma(u)$ vary with u? In terms of multiplicative functions we want to know how the average $x^{-1}\sum_{n\le x} f(n)$ varies with x. By considering the problem example $f(n) = n^{i\alpha}$ we see that these averages can fluctuate a lot:

$$\left| \frac{1}{x}\sum_{n\le x} n^{i\alpha} - \frac{w}{x}\sum_{n\le x/w} n^{i\alpha} \right| = \frac{1}{|1+i\alpha|}|1-w^{-i\alpha}|.$$

However, Elliott [1] observed that the absolute values of these averages is slowly varying. He showed that for any multiplicative function f with $|f(n)| \le 1$ and all $2 \le w \le \sqrt{x}$

$$\frac{1}{x}\left|\sum_{n\le x} f(n)\right| - \frac{w}{x}\left|\sum_{n\le x/w} f(n)\right| \ll \left(\frac{\log w}{\log x}\right)^{1/19}.$$

Translated in terms of integral equations, this means that $|\sigma(u)|$ satisfies the strong Lipschitz estimate:

$$|\sigma(u)| - |\sigma(v)| \ll \left|\frac{u-v}{u+v}\right|^{1/19}.$$

Estimates like this are of interest because, as Hildebrand [7] observed, they allow one to (slightly) extend the range of validity of Burgess' character sum estimate. Our next Theorem improves the Lipschitz exponent 1/19 considerably.

Theorem 9. *Suppose that χ is a measurable function with $\chi(t) = 1$ for $t \le 1$ and $|\chi(t)| \le 1$ for all $t > 1$. Let $\sigma(u)$ denote the corresponding solution to the integral equation (6). Then for all u, v we have*

$$|\sigma(u)| - |\sigma(v)| \ll \left|\frac{u-v}{u+v}\right|^{\kappa}$$

where κ is any real number less than $2(1-2/\pi)/3 = 0.24225\ldots$ and the implied constant depends only on κ.

The constant $2(1-2/\pi)/3$ can be improved slightly by our methods. But we have been unable to attain the optimal exponent 1.

3. $\Gamma([-1,1]) \subset [-(2-2/\sqrt{e}), 1]$ and a sketch of the proof of Theorem 1

Notice that $\Gamma([-1,1]) = \Lambda([-1,1])$ by Theorem 3′. Let χ be any real valued measurable function with $\chi(t) = 1$ for $t \le 1$ and $-1 \le \chi(t) \le 1$ for $t > 1$ and let $\sigma(u)$ be the corresponding solution to (6). Then we need to show that $\sigma(u) \ge -(2-2/\sqrt{e})$ for all u.

Let u_0 be such that $I_1(u_0,\chi) = 1$; if no such u_0 exists then $I_1(u,\chi) < 1$ for all u, and define $u_0 = \infty$. If $u \le u_0$ then, using the lower bound of (8) with $m = 0$, $\sigma(u) \ge 1 - I_1(u,\chi) \ge 1 - I_1(u_0,\chi) = 0$ which is stronger than our claimed bound.

Now suppose $u > u_0$ so that by (9), and since $\sigma(t) \ge 0$ for $t \le u_0$

$$|\sigma(u)| \le A(u_0) = \frac{1}{u_0}\int_0^{u_0} |\sigma(t)|dt = \frac{1}{u_0}\int_0^{u_0} \sigma(t)dt.$$

Using the upper bound of (8) with $m = 1$ we get

$$A(u_0) \le 1 - \frac{1}{u_0}\int_0^{u_0} I_1(t,\chi)dt + \frac{1}{2u_0}\int_0^{u_0} I_2(t,\chi)dt.$$

Using Lemma 2 here, we see with a little manipulation

$$A(u_0) \le \frac{1}{2} + \frac{1}{2u_0}\left((1-I_1) * (1-I_1)\right)(u_0). \tag{13}$$

Notice that for $0 \le t \le u_0$,

$$
\begin{aligned}
1 - I_1(t,\chi) &= \int_t^{u_0} \frac{1-\chi(v)}{v} dv \le \min(1, 2\log(u_0/t)) \\
&= \begin{cases} 1 & \text{if } t \le u_0/\sqrt{e} \\ 2\log(u_0/t) & \text{if } u_0/\sqrt{e} < t \le u_0. \end{cases}
\end{aligned}
$$

Substituting this into (13) we deduce that

$$
\begin{aligned}
A(u_0) &\le \frac{1}{2} + \frac{1}{2u_0}\Big(\int_0^{u_0(1-1/\sqrt{e})} 2\log\frac{u_0}{u_0-t} dt \\
&\qquad\qquad + \int_{u_0(1-1/\sqrt{e})}^{u_0/\sqrt{e}} dt + \int_{u_0/\sqrt{e}}^{u_0} 2\log\frac{u_0}{t} dt \Big) \\
&= 2 - \frac{2}{\sqrt{e}},
\end{aligned}
$$

which proves our desired bound.

We have shown more than just $\sigma(u) \ge -(2-2/\sqrt{e})$ above; in fact we have shown that either $\sigma(t)$ is non-negative for all $t \le u$ or $|\sigma(u)| \le (2-2/\sqrt{e})$. Analogously, in [2] we prove that either $\sigma(t)$ is non-negative for all $t \le u$ or $|\sigma(u)| \le |\delta_1|$.

We now give the barest outline of our proof that $\sigma(u) \ge \delta_1$. As above we may assume that $u > u_0$ and for simplicity we assume that $u > u_0(1+1/\sqrt{e})$ (the case $u_0 < u \le u_0(1+1/\sqrt{e})$ succumbs to similar arguments). Here we know that

$$
\begin{aligned}
|\sigma(u)| &\le A(u_0(1+1/\sqrt{e})) \\
&= \frac{1}{u_0(1+1/\sqrt{e})}\Big(\int_0^{u_0} \sigma(t) dt + \int_{u_0}^{u_0(1+1/\sqrt{e})} |\sigma(t)| dt \Big). \qquad (14)
\end{aligned}
$$

We shall bound the above quantities in terms of the parameters

$$
\lambda = I_1(u_0(1-1/\sqrt{e}), \chi), \qquad \text{and} \qquad \tau = I_1(u_0/\sqrt{e}, \chi),
$$

which satisfy $0 \le \lambda \le \tau \le 1$.

With these conditions, one can easily show (by techniques similar to those used above) that $I_1(tu_0, \chi)$ exceeds

$$\begin{cases} \max\left(0, \lambda + 2\log(t(1-1/\sqrt{e})^{-1})\right) & \text{if } 0 \le t \le 1 - 1/\sqrt{e} \\ \max(\lambda, \tau + 2\log(\sqrt{e}t)) & \text{if } 1 - 1/\sqrt{e} \le t \le 1/\sqrt{e} \\ \max(\tau, 1 + 2\log t) & \text{if } 1/\sqrt{e} \le t \le 1. \end{cases}$$

By inserting these bounds into (13), we deduce that
(15)

$$\frac{1}{u_0}\int_0^{u_0} \sigma(t)dt \le 2 - \frac{2}{\sqrt{e}} - E_1(\lambda,\tau) = \frac{1}{\sqrt{e}}\int_0^{\sqrt{e}} \rho_1(t)dt - E_1(\lambda,\tau),$$

where $E_1(\lambda, \tau)$ is an explicit (but complicated) non-negative function of τ and λ.

For $u_0 \le t \le u_0(1 + 1/\sqrt{e})$, we use the inclusion exclusion inequalities (8) (with $m = 0$ or 1) to obtain inequalities of the form

$$|\sigma(t)| \le |\rho_1(\sqrt{e}t/u_0)| + E_2(\lambda, \tau, t/u_0),$$

for some non-negative function $E_2(\lambda, \tau, t/u_0)$. The key is to obtain very precise estimates for $E_2(\lambda, \tau, t/u_0)$ such that

$$\frac{1}{u_0(1+1/\sqrt{e})}\int_{u_0}^{u_0(1+1/\sqrt{e})} E_2(\lambda,\tau,t/u_0)dt \le \frac{E_1(\lambda,\tau)}{1+1/\sqrt{e}}.$$

In fact, equality above holds only when $\lambda = \tau = 0$. Combining this with (14) and (15), we obtain (after a suitable change of variables)

$$|\sigma(u)| \le \frac{1}{1+\sqrt{e}}\int_0^{1+\sqrt{e}} |\rho_1(t)|dt.$$

Miraculously, the right side above equals $|\rho_1(1 + \sqrt{e})| = |\delta_1|$ and everything is proved!

4. Open Problems

Our main objective is to explicitly determine the spectra of interesting sets S (as in Theorem 1). For example, what is the spectrum of S_k, the k-th roots of unity for $k \geq 3$? Failing this, one can ask for information on various geometric aspects of the spectrum, as in Conjectures 1 and 2.

What is the largest disc centered at the origin that can be contained in the spectrum? Corollary 1 gives a lower bound on this. In the other direction, one can ask for the largest positive $A(\theta)$ such that, for any S with $\text{Ang}(S) = \theta$, $\Gamma(S)$ is contained in the disc centered at $A(\theta)$ with radius $1 - A(\theta)$. Theorem 5 gives that $A(\theta) \geq (28/411) \cos^2 \theta$, and using Corollary 1, we can check that $A(\theta) \leq (1 - \exp(-\pi \cot \theta))/2 \leq (38/75) \cos \theta$. Is it true that $A(\theta) \asymp \cos \theta$ as $\theta \to \pi/2$?

What is $\mu(S) := \min_{z \in \Gamma(S)} \text{Re } (z)$? When $S = [-1, 1]$, Theorem 1 shows that $\mu(S) = \delta_1$, and, as we saw in §1, this has the application that $\geq (1 + \delta_1 + o(1))x/2$ integers below x are quadratic residues $\pmod p$. It is especially interesting to determine $\mu(S_3)$ where S_3 denotes the cube roots of unity. It is easy to see that $\mu(S_3) \geq -1/2$ and we have shown that $\mu(S_3) > -1/2$. This demonstrates that a positive proportion of the integers below x are cubic residues $\bmod p$: suppose $p \equiv 1 \pmod 3$ and that χ is a cubic character $\pmod p$,

$$\begin{aligned} \#\{n \leq x : \ n \equiv m^3 \pmod p\} &= \frac{1}{3} \sum_{n \leq x} (1 + \chi(n) + \chi(n)^2) \\ &\geq \frac{1}{3}(1 + 2\mu(S_3) + o(1))x \gg x. \end{aligned}$$

We can determine explicit bounds here, but these are quite weak. In general, we know that $\mu(S) \geq -1 + 2A(\theta)$.

Let γ_m denote the largest real number such that if p is a prime then $\geq (\gamma_m + o(1))x$ integers below x are m-th power residues $\pmod p$. Theorem 1 gives $\gamma_2 = \delta_0$ and from the above paragraph we know $\gamma_3 > 0$. In fact, we have shown that $\gamma_m > 0$ for all m: that is, a positive proportion of the integers below

x are m-th power residues $\pmod p$. It is not hard to show that $\gamma_m \le \rho(m) = 1/m^{m+o(1)}$, where ρ denotes the Dickman-de Bruijn function. However, we have not been able to obtain good lower bounds for γ_m. So determining γ_m for $m \ge 3$ is a key open problem.

We may generalize the notion of spectrum by considering weighted averages of elements of $\mathcal{F}(S)$. Of particular interest is the *logarithmic spectrum*

$$\Gamma_0(S) = \lim_{N\to\infty} \left\{ \frac{1}{\log N} \sum_{n \le N} \frac{f(n)}{n} : \qquad f \in \mathcal{F}(S) \right\}.$$

The logarithmic spectrum is contained in the convex hull of the spectrum, and its geometric properties are a lot easier to characterize. For example, while it is hard to determine the spectrum of $[-1, 1]$, it is an easy exercise that its logarithmic spectrum is $[0, 1]$. Further, the angle of the logarithmic spectrum equals the angle of the set; that is, the analogue of Conjecture 1 is true. In fact, if $\text{Ang}(S) = \theta = \pi/2 - \delta$ and $z \in \Gamma_0(S)$ with $|\arg(z)| = \varphi$ where $0 \le \varphi \le \pi$ then $|z| \le (\cos\delta)^{\varphi/\delta}$. This allows us to bound $\Gamma_0(S)$ quite precisely. As for the spectrum, one can ask for the largest real number $A_0(\theta)$ such that, for all S with $\text{Ang}(S) = \theta = \pi/2 - \delta$, $\Gamma_0(S)$ is contained in the disc with center $A_0(\theta)$ and radius $1 - A_0(\theta)$. We can show that

$$\frac{\pi}{4}\delta + O(\delta^2) = \frac{1 - (\cos\delta)^{\pi/\delta}}{2} \le A_0(\theta) \le \frac{1 - \exp(-\pi\cot\theta)}{2} = \frac{\pi}{2}\delta + O(\delta^2)$$

so that we have understood this up to a factor of 2.

Motivated by the logarithmic spectrum, we ask for the logarithmic density of m-th power residues $\pmod p$. That is, what is the largest real number γ'_m such for all primes p

$$\sum_{\substack{n \le x \\ n \equiv a^m \pmod p}} \frac{1}{n} \ge (\gamma'_m + o(1)) \log x \ ?$$

By partial summation, it is clear that $\gamma'_m \geq \gamma_m$. Since $\Gamma_0([-1,1]) = [0,1]$, it is easy to see that $\gamma'_2 = 1/2$. We do not know the precise value of γ'_m for any other value of m. However, by a combinatorial argument, we have shown that $\gamma'_m \geq 1/2^{m-1}$; and by an easy construction $\gamma'_m \leq \exp(-m/e + o(m))$.

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