A DECOMPOSITION OF RIEMANN'S ZETA FUNCTION

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It is currently very much in vogue to study sums of the form

$$\zeta(p_1, p_2, \dots, p_g) := \sum_{a_1 > a_2 > \cdots > a_g \geq 1} \frac{1}{a_1^{p_1}} \frac{1}{a_2^{p_2}} \cdots \frac{1}{a_g^{p_g}}$$

where all the a_is and p_is are positive integers, with $p_1 \geq 2$. Note that it is necessary that $p_1 \geq 2$ else $\zeta(\mathbf{p})$ diverges. These sums are related to polylogarithm functions (see [1,2,3]) as well as to zeta functions (the Riemann zeta function is of course the case $g = 1$). In this note we prove an identity that was conjectured by Moen [5] and Markett [7]:

Proposition. *If g and N are positive integers with $N \geq g + 1$ then*

$$\zeta(N) = \sum_{\substack{p_1 + p_2 + \cdots + p_g = N \\ \text{Each } p_j \geq 1, \text{ and } p_1 \geq 2}} \zeta(p_1, p_2, \dots, p_g). \tag{1}$$

This identity was proved for $g = 2$ by Euler, and for $g = 3$ by Hoffman and Moen [6]. The above proposition has been proved independently by Zagier [9], who writes of his proof, '*Although this proof is not very long, it seems too complicated compared with the elegance of the statement. It would be nice to find a more natural proof*': Unfortunately much the same can be said of the proof that I have presented here.

Markett [7] and J. Borwein and Girgensohn [3] were able to evaluate $\zeta(p_1, p_2, p_3)$ in terms of values of $\zeta(p)$ whenever $p_1 + p_2 + p_3 \leq 6$, and in terms of $\zeta(p)$ and $\zeta(a, b)$ whenever $p_1 + p_2 + p_3 \leq 10$ — it would be interesting to know whether such 'descents' are always possible or, as most researchers seem to believe, that there is only a small class of such sums that can be so evaluated.

Proof of (1). We may re-write the sum on the right side of (1) as

$$\sum_{a_1 > a_2 > \cdots > a_g \geq 1} \sum_{\substack{p_1 + p_2 + \cdots + p_g = N \\ \text{Each } p_j \geq 1, \text{ and } p_1 \geq 2}} \frac{1}{a_1^{p_1}} \frac{1}{a_2^{p_2}} \cdots \frac{1}{a_g^{p_g}}.$$

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The second sum here is the coefficient of x^n in the power series

$$\left(\sum_{p_1\geq 2}\left(\frac{x}{a_1}\right)^{p_1}\right)\prod_{j=2}^{g}\left(\sum_{p_j\geq 1}\left(\frac{x}{a_j}\right)^{p_j}\right)=\frac{x^2/a_1^2}{(1-x/a_1)}\prod_{j=2}^{g}\frac{x/a_j}{(1-x/a_j)}$$
$$=\frac{x^{g+1}}{a_1}\prod_{j=1}^{g}\frac{1}{(a_j-x)}=\frac{x^{g+1}}{a_1}\sum_{j=1}^{g}\frac{1}{(a_j-x)}\prod_{\substack{i=1\\ i\neq j}}^{g}\frac{1}{(a_i-a_j)}.$$

Therefore the sum above is
(2)
$$\sum_{a_1>a_2>\cdots>a_g\geq 1}\frac{1}{a_1}\sum_{j=1}^{g}\frac{1}{a_j^{N-g}}\prod_{\substack{i=1\\ i\neq j}}^{g}\frac{1}{(a_i-a_j)}=\sum_{m\geq 1}\frac{1}{m^{N-g}}\sum_{j=1}^{g}A(m,j-1)(-1)^{g-j}B(m,g-j)$$

where we take each $a_j=m$ in turn, with

$$A(m,j-1):=\sum_{a_1>a_2>\cdots>a_{j-1}>m}\frac{1}{a_1}\prod_{i=1}^{j-1}\frac{1}{(a_i-m)}=\sum_{b_1>b_2>\cdots>b_{j-1}\geq 1}\frac{1}{(b_1+m)b_1b_2\dots b_{j-1}}$$

taking each $b_i=a_i-m$, and

$$B(m,g-j):=\sum_{m>a_{j+1}>a_{j+2}>\cdots>a_g\geq 1}\prod_{i=j+1}^{g}\frac{1}{(m-a_i)}=\sum_{0<b_{j+1}<b_{j+2}<\cdots<b_g<m}\frac{1}{b_{j+1}b_{j+2}\dots b_g},$$

now taking each $b_i=m-a_i$. Note that the generating function for B is

(3)
$$\sum_{i\geq 0}B(m,i)x^i=\prod_{b=1}^{m-1}\left(1+\frac{x}{b}\right).$$

Dealing with A is somewhat more difficult. We start by noting that

$$\sum_{b_1>b_2}\frac{1}{(b_1+m)b_1}=\frac{1}{m}\sum_{b_1>b_2}\left(\frac{1}{b_1}-\frac{1}{b_1+m}\right)=\frac{1}{m}\sum_{c_2=1}^{m}\frac{1}{b_2+c_2},$$

as this is a telescoping sum. Substituting this back into the definition for A we next have to deal with

$$\sum_{b_2>b_3}\sum_{c_2=1}^{m}\frac{1}{(b_2+c_2)b_2}=\sum_{c_2=1}^{m}\frac{1}{c_2}\sum_{b_2>b_3}\left(\frac{1}{b_2}-\frac{1}{(b_2+c_2)}\right)=\sum_{c_2=1}^{m}\frac{1}{c_2}\sum_{c_3=1}^{c_2}\frac{1}{b_3+c_3},$$

for the same reasons. Putting this back into the definition we have to do the same calculation again, now with the indices moved up one. Iterating this procedure we end up with

$$A(m,j-1):=\frac{1}{m}\sum_{m\geq c_2\geq c_3\geq\cdots\geq c_j\geq 1}\frac{1}{c_2c_3\dots c_j},$$

which has generating function

$$\sum_{i\geq 0} A(m,i)x^i = \frac{1}{m}\prod_{c=1}^{m}\left(1+\frac{x}{c}+\left(\frac{x}{c}\right)^2+\left(\frac{x}{c}\right)^3+\dots\right) = \frac{1}{m}\prod_{c=1}^{m}\left(1-\frac{x}{c}\right)^{-1}. \tag{4}$$

Therefore, by (2), (3) and (4), the sum on the right side of (1) is $\sum_{m\geq 1} 1/m^{N-g+1}$ times the coefficient of x^{g-1} in the power series

$$\prod_{b=1}^{m-1}\left(1-\frac{x}{b}\right)\prod_{c=1}^{m}\left(1-\frac{x}{c}\right)^{-1} = \left(1-\frac{x}{m}\right)^{-1} = \sum_{i\geq 0}\left(\frac{x}{m}\right)^i.$$

We thus get $\sum_{m\geq 1} 1/m^N = \zeta(N)$, giving (1).

2. Evaluations of $\zeta(r,s)$.

Euler demonstrated that if $N = r+s$ is odd wih s even then

$$\zeta(r,s) = -\frac{1}{2}\left\{\binom{N}{r}+1\right\}\zeta(N) + \sum_{\substack{a+b=N\\ a,b\geq 2\\ a \text{ odd}}}\left\{\binom{a-1}{s-1}+\binom{a-1}{t-1}\right\}\zeta(a)\zeta(b). \tag{5}$$

One can then obtain the value of $\zeta(s,r)$, provided $r>1$, from the trivial identity

$$\zeta(r,s)+\zeta(s,r) = \zeta(r)\zeta(s) - \zeta(r+s). \tag{6}$$

(To prove this just write out the zeta-functions on both sides and compare terms). In the special case $r=1$ he proved, for any $N\geq 3$ that

$$\zeta(N-1,1) = \frac{N-1}{2}\zeta(N) - \frac{1}{2}\sum_{\substack{a+b=N\\ a,b\geq 2}}\zeta(a)\zeta(b). \tag{7}$$

Pages 47-49 of [8], Equation (2) of [2] and Theorem 4.1 of [7] are all equivalent to, for $N\geq 4$,

$$\begin{aligned}\zeta(N-2,1,1) = \frac{(N-1)(N-2)}{6}\zeta(N) + \frac{1}{2}\zeta(2)\zeta(N-2) - \frac{N-2}{4}\sum_{\substack{a+b=N\\ a,b\geq 2}}\zeta(a)\zeta(b)\\ + \frac{1}{6}\sum_{\substack{a+b+c=N\\ a,b,c\geq 2}}\zeta(a)\zeta(b)\zeta(c).\end{aligned} \tag{8}$$

Proof of (7). We evaluate the last sum in (7), using (6):

$$\begin{aligned}\sum_{\substack{a+b=N\\ a,b\geq 2}}\zeta(a)\zeta(b) &= \sum_{\substack{a+b=N\\ a,b\geq 2}}(\zeta(a,b)+\zeta(b,a)+\zeta(N))\\ &= 2\sum_{\substack{a+b=N\\ a,b\geq 2}}\zeta(a,b) + (N-3)\zeta(N)\\ &= 2(\zeta(N)-\zeta(N-1,1)) + (N-3)\zeta(N)\end{aligned}$$

using (1) with $g=2$, and the result follows after some re-arrangement.

Proof of (8). We begin by proving

$$\sum_{\substack{p+q=N-1\\ p\geq 2,\ q\geq 1}} \zeta(p,1,q) = \zeta(2,N-2)+\zeta(N-1,1). \tag{9}$$

Now the sum here equals

$$\sum_{a>b>c\geq 1} \frac{1}{b} \sum_{\substack{p+q=N-1\\ p\geq 2,\ q\geq 1}} \frac{1}{a^p c^q} = \sum_{a>b>c\geq 1} \frac{1}{ab(a-c)}\left(\frac{1}{c^{N-3}}-\frac{1}{a^{N-3}}\right)$$

The first term is

$$\begin{aligned}\sum_{c\geq 1}\frac{1}{c^{N-2}}\sum_{b>c}\frac{1}{b}\sum_{a>b}\left(\frac{1}{a-c}-\frac{1}{a}\right) &= \sum_{c\geq 1}\frac{1}{c^{N-2}}\sum_{b>c}\frac{1}{b}\sum_{i=0}^{c-1}\frac{1}{b-i}\\ &= \zeta(2,N-2)+\sum_{c>i\geq 1}\frac{1}{c^{N-2}}\frac{1}{i}\sum_{b>c}\left(\frac{1}{b-i}-\frac{1}{b}\right)\\ &= \zeta(2,N-2)+\zeta(N-1,1)+\sum_{c>i\geq 1}\frac{1}{c^{N-2}}\frac{1}{i}\sum_{j=1}^{i-1}\frac{1}{c-j}\end{aligned}$$

But the final sum in both of the last two displays are identical (after the change of variables $(a,b,c)\to(c,i,j)$), and so we have proved (9).

We next prove that

$$\sum_{\substack{p+q=N-1\\ p\geq 2,\ q\geq 1}} \zeta(p,q,1) = \zeta(2)\zeta(N-2)-\zeta(2,N-2)+\zeta(N-1,1)-\zeta(N). \tag{10}$$

Using (7) and then (1) we have

$$\begin{aligned}(N-1)\zeta(N)-2\zeta(N-1,1)-\zeta(2)\zeta(N-2) &= \sum_{N-3\geq a\geq 2}\zeta(a)\zeta(N-a)\\ &= \sum_{\substack{a+b+c=N\\ a,b\geq 2,\ c\geq 1}}\zeta(a)\zeta(b,c)\end{aligned}$$

Just as in the proof of (6) we may determine such a product in terms of zeta-functions by considering each term. We thus get $\zeta(a)\zeta(b,c)=\zeta(a,b,c)+\zeta(b,a,c)+\zeta(b,c,a)+\zeta(a+b,c)+\zeta(b,a+c)$. Summing up over all possibilities with $a+b+c=N$ and $a,b\geq 2$ we get three times the sum over all $\zeta(A,B,C)$ in the sum (1) other than a few terms corresponding to when $a=1$ or $b=1$, and some multiples of

$\zeta(A,B)$. Precisely we get:

$$\begin{aligned}
&3\sum_{\substack{a+b+c=N\\ a,b\geq 2,\ c\geq 1}}\zeta(a,b,c)-\sum_{\substack{a+c=N-1\\ a\geq 2,\ c\geq 1}}(2\zeta(a,1,c)+\zeta(a,c,1))\\
&\qquad\qquad+\sum_{\substack{d+c=N\\ d\geq 3,\ c\geq 1}}(d-3)\zeta(d,c)+\sum_{\substack{f+b=N\\ b\geq 2,\ f\geq 2}}(f-2)\zeta(b,f)\\
&\quad=3\zeta(N)-\sum_{\substack{a+c=N-1\\ a\geq 2,\ c\geq 1}}\zeta(a,c,1)+\sum_{\substack{a+b=N\\ a\geq 2,\ b\geq 1}}(a+b-5)\zeta(a,b)-\zeta(2,N-2)-\zeta(N-1,1)\\
&\quad=(N-2)\zeta(N)-\sum_{\substack{a+c=N-1\\ a\geq 2,\ c\geq 1}}\zeta(a,c,1)-\zeta(2,N-2)-\zeta(N-1,1)
\end{aligned}$$

using (1) and (9). Combining the last two displays gives (10).

We now try to evaluate the last sum in (8):

$$\sum_{\substack{a+b+c=N\\ a,b,c\geq 2}}\zeta(a)\zeta(b)\zeta(c)=\sum_{\substack{a+b+c=N\\ a,b,c\geq 2}}\sum_{x,y,z\geq 1}\frac{1}{x^a y^b z^c}.$$

By analogy with the proof of (6), with a,b,c fixed we break up this sum according to how x,y,z are ordered by size. For example, when $x>y>z$ we get precisely $\zeta(a,b,c)$. We thus get the sum of $\zeta(u,v,w)$ as u,v,w ranges over all six orderings of a,b,c; plus the sum of $\zeta(u,N-u)+\zeta(N-u,u)$ for each $u\in\{a,b,c\}$; plus $\zeta(N)$. Thus, using (6), we have

$$6\sum_{\substack{a+b+c=N\\ a,b,c\geq 2}}\zeta(a,b,c)+3\sum_{\substack{a+b+c=N\\ a,b,c\geq 2}}(\zeta(c)\zeta(a+b)-\zeta(N))+\sum_{\substack{a+b+c=N\\ a,b,c\geq 2}}\zeta(N).$$

Using (1) with $g=3$, we thus have

$$\begin{aligned}
&6\left(\zeta(N)+\zeta(N-2,1,1)-\sum_{\substack{a+d+1=N\\ a\geq 2,\ d\geq 1}}(\zeta(a,d,1)+\zeta(a,1,d))\right)\\
&\qquad\qquad+3\sum_{\substack{c+d=N\\ d\geq 4,\ c\geq 2}}(d-3)\zeta(c)\zeta(d)-2\binom{N-4}{2}\zeta(N)\\
&\quad=6\zeta(N-2,1,1)-(N-1)(N-8)\zeta(N)-3\zeta(N-2)\zeta(2)-12\zeta(N-1,1)\\
&\qquad\qquad-\frac{3}{2}\sum_{\substack{c+d=N\\ d\geq 2,\ c\geq 2}}(N-6)\zeta(c)\zeta(d)
\end{aligned}$$

using (9) and (10), and combining the (c,d) and (d,c) terms in the final sum. Using (7) to remove the $\zeta(N-1,1)$ terms, we obtain (8).

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