

International Team Shows that Primes Can Be Found in Surprising Places

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Providence, RI – Prime numbers are the most basic objects in mathematics. They also are among the most mysterious, for after centuries of study, the structure of the set of prime numbers is still not well understood. Describing the distribution of primes is at the heart of much mathematics and has many important applications to such areas as cryptography. To study primes, researchers have developed what might be termed mathematical ‘lenses’ that allow them to bring into focus certain aspects of prime numbers.

Recently two mathematicians – John Friedlander of the University of Toronto and Henryk Iwaniec of Rutgers University in New Jersey – astounded the mathematical world by announcing that they had developed a new, more refined lens for the prime numbers. Their work was particularly surprising because it solved an important mathematics problem on which no progress had been made for the past one hundred years.

The significance of the work of Friedlander and Iwaniec can be seen in its historical context. It was Euclid, in ancient Greece, who first showed that there are an infinity of

primes among the integers. Much later, in 1837, Gustave Lejeune Dirichlet showed that there are infinitely many primes among the numbers $a, a+d, a+2d, a+3d, \dots$. Here a and d must have no common divisors greater than 1.

Following Dirichlet’s work two questions come to mind:

“In what other ‘natural’ sequences of numbers can one find infinitely many primes?”

“Can one determine how frequently primes occur in such sequences?”

Techniques invented in the 1890s allowed mathematicians to give a pretty good approximation of how often primes occur among the integers, as well as in the sequences that Dirichlet examined. The same techniques can be modified to show that there are infinitely many primes in all sorts of different sequences of numbers, such as, for example, among numbers of the form $a^2 + 21b^2$, with a and b integers and other similar sequences. However, all sequences considered by such techniques have one thing in common – they are not ‘sparse’. That is, they contain so many non-prime numbers that, as lenses for examining prime numbers, they do not provide a very sharp focus.

In the last hundred years no one has significantly improved upon this situation. No one was able to determine a sparse sequence of numbers which could be proven

to contain an infinite sequence of prime numbers.

This is why this new work is such a surprise. What Friedlander and Iwaniec did was to show that there are infinitely many primes among numbers of the form $a^2 + b^4$. This set of numbers is far sparser than the others that were known to contain infinitely many primes. For example, among the whole numbers up to 1 trillion (10^{12}), there are about 27 billion different numbers of the form $a^2 + b^2$, but less than a billion different numbers of the form $a^2 + b^4$. Moreover Friedlander and Iwaniec are able to accurately determine the frequency of primes in their sequence.

To get hold of such a comparatively thin sequence, Friedlander and Iwaniec have had to develop a whole new theory to attain a deeper level of understanding. Their achievement, recently announced in the *Proceedings of the National Academy of Sciences* (<http://www.pnas.org/content/vol94/issue4>) was greeted with astonishment from other experts, who had thought such a development was far from feasible. A full account of their results has been accepted for publication in the most prestigious mathematics journal, *Annals of Mathematics*.

The key to their work lies in a revolutionary method: Friedlander and Iwaniec's refinements of a tool called the 'Asymptotic Sieve', which was created by Enrico Bombieri, a

mathematician at the Institute for Advanced Study. This allowed them to break through the notorious 'parity problem', which in the past had been a barrier against the use of sieve theory in questions about the distribution of primes. There is a whole host of further brilliant technical innovations in their ground-breaking articles, as well as some significant conceptual advances. There is little doubt that when this powerful new lens is better understood and fully focused, our knowledge of the distribution of primes will be far richer and deeper.

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