HALÁSZ IN SHORT INTERVALS

ANDREW GRANVILLE, ADAM HARPER, KAISA MATOMÄKI, AND MAKSYM RADZIWIŁŁ

1. Introduction

Let $\mathbb{U} := \{z : |z| \leq 1\}$ be the complex unit-disk. For a multiplicative function $f : \mathbb{N} \to \mathbb{U}$ and $\mathrm{Re}(s) > 0$, write

$$F_x(s) = \sum_{\substack{n \\ p|n \implies p \leq x}} \frac{f(n)}{n^s} = \prod_{p \leq x} \left(1 + \frac{f(p)}{p^s} + \frac{f(p^2)}{p^{2s}} + \dots\right).$$

Given $\varepsilon \in [0, \frac{1}{1000})$ and a $\theta \in (\frac{1}{2} + \varepsilon, 1]$ we define,

$$\tau_\varepsilon(\theta) = \begin{cases} \theta & \text{if } \theta > \frac{7}{12} + \varepsilon, \\ \frac{2\theta}{2m+1} & \text{if } \frac{1}{2} + \frac{1}{8m+12} + \varepsilon \leq \theta \leq \frac{1}{2} + \frac{1}{8m+4} + \varepsilon, m \in \mathbb{N} \end{cases}$$

Let $\tau(\theta) = \tau_0(\theta)$. Of particular interest to us are the end points of θ close to 1 and θ close to $\frac{1}{2}$. Notice that for $\theta > \frac{7}{12}$ we have,

We shift a bit to avoid issues at the discontinuities

$$\tau(\theta) = \theta$$

while for $\theta \to \frac{1}{2}^+$

$$\tau(\theta) = 4(\theta - \tfrac{1}{2}) + O((\theta - \tfrac{1}{2})^2).$$

Below is a graphical representations of $\tau(\theta)$. The black lines represent $\tau(\theta)$. The dotted blue line represents the lines $y = x$, while the dotted black line represents the line $y = 4(x - \frac{1}{2})$ which corresponds to the asymptotic approximation to $\tau(\theta)$ as $\theta \to \frac{1}{2}$. `It is confusing that there are two sorts of dotted black lines!`

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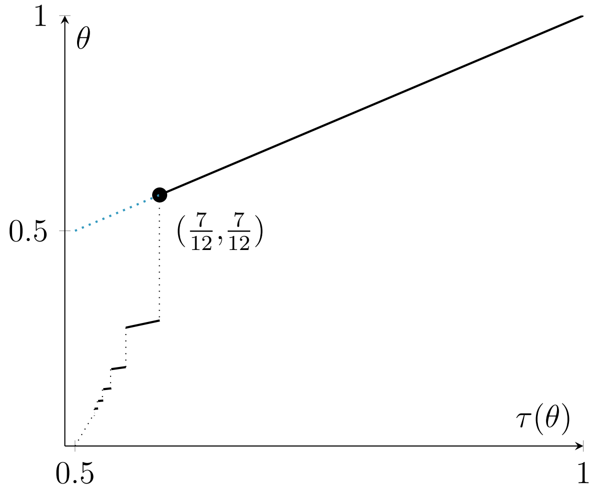

We will now state our main technical result, from which various corollaries quickly follow.

Theorem 1. *Let $\varepsilon \in (0, \frac{1}{1000})$ and $\eta > 0$ be given. Let $\theta \in (\frac{1}{2} + \varepsilon, 1]$. For a given large x, let*

$$y := x^\theta \leq Y := x/(\log x)^{3\eta/2}.$$

Let $f : \mathbb{N} \to \mathbb{U}$ be a multiplicative function such that $f(p) = 0$ for $p \geq x^{\tau_\varepsilon(\theta)}$. Let t_0 be a real number with $|t_0| \leq (x/y)\log^{20} x$. Then

I find the formulation without the maximal condition more easy/flexible

$$\sum_{x \leq n \leq x+y} f(n) = \int_x^{x+y} u^{it_0} du \cdot \frac{1}{Y} \sum_{x \leq n \leq x+Y} f(n) n^{-it_0}$$
$$+ O\left(\frac{y(\log\log x)^{10/\tau_\varepsilon(\theta)}}{(\log x)^{\eta/2}} \cdot \prod_{p \leq x} \left(1 + \frac{|f(p)| - 1}{p}\right) + \frac{y(\log\log x)^{10}}{\log x} \right)$$
$$+ O\left(\frac{y(\log\log x)^{10/\tau_\varepsilon(\theta)}}{\log x} \cdot \max_{\substack{|t-t_0| \geq (\log x)^\eta \\ |t| \leq 2(x/y)\log^{20} x}} |F_y(1+it)| \right).$$

Remark 1. An inspection of our proof shows that for $\theta < \frac{7}{12}$ the smoothness cut-off $\tau(\theta)$ can be increased. However this comes at the cost of a rather complicated $\tau(\theta)$, while the gains are small.

It is advantageous to take t_0 that maximizes $|F(1+it)|$. Otherwise the error term might dominate over the main term.

There is a trade-off in the choice of the parameter $\eta > 0$. A large η leads to (optimally) good error term, but leaves us with more difficulty in understanding the main term over the longer interval $[x, x+Y]$. A choice of a small $\eta > 0$ makes understanding the main term easy, but leads to weaker error terms. The various corollaries that we will state are concerned with finding the right balance.

If one is not interested in the term

$$\prod_{p \leq x} \Big(1 + \frac{|f(p)| - 1}{p}\Big)$$

and is willing to trivially bound it by $\ll 1$, then it is possible to replace

$$\max_{|t - t_0| \geq (\log x)^{\eta}} |F_x(1+it)| \text{ by } \max_{|t-t_0| \geq (\log x)^{\eta/2}} |F_x(1+it)|.$$

This then leads to an error term that is reminiscent of the error term one gets in Halasz's theorem. Precisely we then get that for $T_0 \geq 1$,

$$\frac{1}{y} \sum_{x \leq n \leq x+y} f(n) = \frac{1}{y} \int_x^{x+y} u^{it_0} du \cdot \frac{T_0^{3/2}}{x} \sum_{x \leq n \leq x + x/T_0^{3/2}} f(n) n^{-it_0}$$
$$+ O\Big(\frac{\log\log x}{T_0} + \frac{\log\log x}{\log x} \max_{\substack{|t-t_0| \geq T_0 \\ |t| \leq (x/y)(\log x)^{20}}} |F_x(1+it)|\Big)$$

We leave these modifications to the reader.

To do: some further remarks that need to be properly formatted:

Remark 2. We restrict our attention to y-smooth integers n since a prime $> y$ can hit at most one integer in the interval. There is a drop at $7/12$ which is unavoidable given the current technology, etc. Discussion of the best result one can hope for, etc.

Remark 3. I think we could extend these results to unbounded multiplicative functions. Any interest in doing that? (I would count the coefficients of Dedekind zeta functions or $|\lambda_{\pi}(n)|^2$ with π an cuspidal automorphic representation of $GL(n)$ as among the more interesting cases)

Remark 4. The power of $\log\log x$ in the error term could be reduced with more care. However we cannot avoid an error term of at least $O(y/\log x)$. To clarify why we note that one might alter the values of the $f(p)$ for $y/2 < p < y$ so as to maximize the error term.

We are now ready to discuss the various corollaries of our technical result.

For real-valued multiplicative functions our results take a particularly simple form.

Corollary 1. *Let $\varepsilon \in (0, \frac{1}{1000})$ and $\theta \in (\frac{1}{2} + \varepsilon, 1]$ be given. Let $f : \mathbb{N} \to [-1, 1]$ be multiplicative with $f(p) = 0$ for $p > x^{\tau_{\varepsilon}(\theta)}$. Then, for $y = x^{\theta}$,*

$$\frac{1}{y} \sum_{x \leq n \leq x+y} f(n) = \frac{1}{x} \sum_{x \leq n \leq 2x} f(n) + O\Big(\frac{1}{(\log x)^{\frac{1}{4}(1 - \frac{2}{\pi} + o(1))}}\Big).$$

Note that the constant is slightly worse than what we get from the repulsion estimates, this is due to the Lipschitz stuff – perhaps with more care we could reach a better constant, for instance by playing a bit with smoothings in the Perron formula section

Another immediate consequence of our result is an improvement of an old result of Hildebrand on mean-values of multiplicative functions.

Corollary 2. *Let $f : \mathbb{N} \to [-1, 1]$ be multiplicative. Then, uniformly in $1 \leq w \leq x$,*

$$\Big| \frac{w}{x} \sum_{x \leq n \leq x + x/w} f(n) - \frac{1}{x} \sum_{x \leq n \leq 2x} f(n) \Big| \ll \frac{\log w}{\log x} + (\log x)^{-\frac{1}{4}(1 - \frac{2}{\pi} + o(1))}$$

Both results admit extensions to complex multiplicative functions. To do: We should probably add this later

We now turn our attention to multiplicative functions that are frequently zero.

Corollary 3. *Let $\kappa, C, \delta > 0$, $\varepsilon \in (0, \frac{1}{1000})$ and $\theta \in (\frac{1}{2} + \varepsilon, 1]$ be given. Let $f : \mathbb{N} \to [-1, 1]$ be a multiplicative function such that for any $2 \leq w \leq z \leq x^{\delta}$, we have,*

$$\sum_{w \leq p \leq z} \frac{|f(p)|}{p} \geq \kappa \sum_{w \leq p \leq z} \frac{1}{p} - \frac{C}{\log w}.$$

and such that $f(p) = 0$ for $p > x^{\tau_{\varepsilon}(\theta)}$. Then, there exists an $\eta = \eta(\kappa) > 0$, such that for $y = x^{\theta}$,

$$\frac{1}{y} \sum_{x \leq n \leq x + y} f(n) = \frac{1}{x} \sum_{x \leq n \leq 2x} f(n) + O_{C,\delta,\kappa}\Big(\frac{1}{(\log x)^{\eta}} \prod_{p \leq x} \Big(1 + \frac{|f(p)| - 1}{p} \Big) \Big).$$

This immediately implies a similar generalization of Hildebrand's result to multiplicative functions that possibly vanish frequently.

Corollary 4. *Let $\kappa, C, \delta > 0$ be given. Let $f : \mathbb{N} \to [-1, 1]$ be a multiplicative function such that for any $2 \leq w \leq z \leq x^{\delta}$, we have,*

$$\sum_{w \leq p \leq z} \frac{|f(p)|}{p} \geq \kappa \sum_{w \leq p \leq z} \frac{1}{p} - \frac{C}{\log w}.$$

Then, there exists an $\eta = \eta(\kappa) > 0$ such that uniformly in $1 \leq w \leq x$,

$$\Big| \frac{w}{x} \sum_{x \leq n \leq x + x/w} f(n) - \frac{1}{x} \sum_{x \leq n \leq 2x} f(n) \Big| \ll_{C,\kappa,\delta} \Big(\frac{\log w}{\log x} \Big)^{\eta} \cdot \prod_{p \leq x} \Big(1 + \frac{|f(p)| - 1}{p} \Big).$$

Building on Corollary 3 we notice that if we do not insist on an asymptotic then we can obtain the order of magnitude of rather general functions.

Corollary 5. *Let $\kappa, C, \delta > 0$ be given. Let $\theta \in (\frac{1}{2}, 1]$ be given. Let $f : \mathbb{N} \to [0, 1]$ be a multiplicative function such that for any $2 \leq w \leq z \leq x^{\delta}$ we have,*

$$\sum_{w \leq p \leq z} \frac{|f(p)|}{p} \geq \kappa \sum_{w \leq p \leq z} \frac{1}{p} - \frac{C}{\log w}.$$

Then, for $y = x^{\theta}$,

$$\frac{1}{y} \sum_{x \le n \le x+y} f(n) \asymp_{C,\kappa,\delta} \frac{1}{x} \sum_{x \le n \le 2x} f(n).$$

Combining Corollary 3 and Corollary 5 gives consequences for norm-forms.

Corollary 6. *Let* K *be a number field over* $\mathbb{Q}$*. Let* $n_1 < n_2 < \dots$ *be the sequence of norm forms of* K*. Let* $\theta \in (\frac{1}{2}, 1]$ *be given. Then, for* $y = x^{\theta}$,

$$\frac{1}{y} \sum_{x \le n_i \le x+y} 1 \asymp_{K,\theta} \prod_{\substack{p \le x \\ p \ne N\mathfrak{a}}} \Big(1 - \frac{1}{p}\Big)$$

where the product is taken over all the primes that are not integral ideals of K*.*

Corollaries 2 and 4 admit quick direct proofs, that we sketch in the appendix To do: needs to be added later

To prove Theorem 1 we will first dispose of some sparse subsets of the integers that do not factor nicely, and then we factorise the remaining integers in a convenient way. Then we use a consequence of Perron's formula (Lemma 2 below) to relate the average of f on short interval to an average of Dirichlet polynomials. These averages can be understood through mean and large value results of Dirichlet polynomials thanks to the factorisation which allows us to study suitable products of Dirichlet polynomials. To do: Needs to be expanded in particular with the discussion of the exponent $\frac{7}{12}$

Lior Bary-Soroker tells me that one of his students recently proved an asymptotic in function fields (of course using the Riemann Hypothesis which is known in function fields).

2. Standard Lemmas

2.1. **Shiu's/Henriot's bounds.** We will freely use (without further reference) the following bound originally due to Shiu.

Lemma 1. *Let* $\varepsilon > 0$ *be given. Let* $f : \mathbb{N} \to [0, 1]$ *be a multiplicative function. Then, for* $H > x^{\varepsilon}$,

$$\sum_{x \le n \le x+H} f(n) \ll_{\varepsilon} H \prod_{p \le x} \Big(1 + \frac{f(p) - 1}{p}\Big).$$

Proof. See [?] or [?]. $\square$

2.2. **Perron's formula.**

Lemma 2. *Let $\eta > 0$ be given. Let $|a_n| \le 1$ and $A(s) := \sum_{n \asymp x} a_n n^{-s}$. If $1 \le y \le Y = x/(\log x)^{3\eta/2}$ then for any $|t_0| \le (x/y)(\log x)^{20}$,*

$$\Big|\sum_{x\le n\le x+y} a_n - \int_x^{x+y} u^{it_0}du \cdot \frac{1}{Y}\sum_{x\le n\le x+Y} a_n n^{-it_0}\Big| \ll \frac{y}{(\log x)^\eta}\cdot \prod_{p\le x}\Big(1+\frac{|f(p)|-1}{p}\Big)$$
$$+\log\log x \max_{x/y\le T\le (x/y)(\log x)^{20}} \frac{x}{T}\int_{\substack{|t|\le T\\ |t-t_0|>(\log x)^\eta}} |A(1+it)|\ dt + \frac{y}{(\log x)^{10}}.$$

Proof. We begin by using Perron to get that

$$\sum_{x\le n\le x+y} a_n = \int_{-(x/y)(\log x)^{20}}^{(x/y)(\log x)^{20}} A(1+it)\frac{(x+y)^{1+it}-x^{1+it}}{1+it}dt + O(y(\log x)^{-10}).$$

The "main term" comes from the integral

$$\int_{t_0-(\log x)^\eta}^{t_0+(\log x)^\eta} A(1+it)\frac{(x+y)^{1+it}-x^{1+it}}{1+it}dt.$$

Now

$$\frac{(x+y)^{1+it}-x^{1+it}}{1+it} = \int_x^{x+y} u^{it}du,$$

and writing $t = t_0+t'$ with $|t'| \le (\log x)^\eta$, we have

$$u^{it} = u^{it_0}u^{it'} = u^{it_0}x^{it'}(1+O((y/x)(\log x)^\eta)).$$

Therefore the main term equals

$$\int_x^{x+y} u^{it_0}du\int_{-\log x}^{\log x} A(1+it_0+it')x^{it'}dt + O\left(y(y/x)(\log x)^{2\eta}\cdot\prod_{p\le x}\Big(1+\frac{|f(p)|-1}{p}\Big)\right)$$

and this error term is

$$\ll \frac{y}{(\log x)^\eta}\prod_{p\le x}\Big(1+\frac{|f(p)|-1}{p}\Big)$$

provided $y \le x/(\log x)^{3\eta/2}$. Now

$$\sum_{x\le n\le x+Y} a_n n^{-it_0}$$

has main term

$$\int_{-(\log x)^\eta}^{(\log x)^\eta} A(1+it+it_0)\frac{(x+Y)^{1+it}-x^{1+it}}{1+it}dt$$
$$= Y\int_{-(\log x)^\eta}^{(\log x)^\eta} A(1+it+it_0)x^{it}dt + O\left((Y^2/x)\int_{-(\log x)^\eta}^{(\log x)^\eta}|A(1+it+it_0)||t|dt\right)$$

using the Taylor series expansion, and this last error term is $O((Y^2/x)(\log x)^{2\eta})$ which is

$$\ll \frac{Y}{(\log x)^\eta} \prod_{p \le x} \Big(1 + \frac{|f(p)|-1}{p} \Big)$$

provided $Y \le x/(\log x)^{3\eta/2}$. Therefore

$$\sum_{x \le n \le x+y} a_n - \int_x^{x+y} u^{it_0} du \cdot \frac{1}{Y} \sum_{x \le n \le x+Y} a_n n^{-it_0} \ll$$

$$\ll \frac{y}{(\log x)^\eta} \prod_{p \le x} \Big(1 + \frac{|f(p)|-1}{p} \Big) + \int_{\substack{|t| \le (x/y)(\log x)^{20} \\ |t-t_0| > (\log x)^\eta}} |A(1+it)| \min \Big\{ y, \frac{x}{1+|t|} \Big\} dt$$

since $|\frac{(x+y)^{1+it}-x^{1+it}}{1+it}| \ll \min\{y, x/(1+|t|)\}$, and in the Y-integral we have an analogous bound but the integral only goes up to $(x/Y)(\log x)^{20}$. This last term is

$$\ll \sum_{k=1}^{20 \log\log x} \frac{y}{e^k} \int_{\substack{|t| \le (x/y)e^k \\ |t-t_0| > (\log x)^\eta}} |A(1+it)| \; dt$$

$$\ll \log\log x \max_{x/y \le T \le (x/y)(\log x)^{20}} \frac{x}{T} \int_{\substack{|t| \le T \\ |t-t_0| > (\log x)^\eta}} |A(1+it)| \; dt.$$

□

3. Large values of Dirichlet polynomials

In this section and later we say that $\mathcal{U} \subset \mathbb{R}$ is well-spaced if $|u-v| \ge 1$ for all distinct $u, v \in \mathcal{U}$. Halász's large value result for Dirichlet polynomials states that for $A(s) = \sum_{N \le n \le 2N} a_n n^{-s}$ and $\mathcal{U} \subset [-T,T]$ a sequence of well-spaced points, one has

$$\sum_{t \in \mathcal{U}} |A(it)|^2 \ll \Big(N + |\mathcal{U}| \sqrt{T} \log 2T \Big) \sum_{N \le n \le 2N} |a_n|^2. \tag{1}$$

We need a variant of this which works well when the length of the polynomial N is quite small compared to T, the coefficients a_n are sparsely supported and we have a very good upper bound for $|\mathcal{U}|$. The following lemma does this.

Lemma 3. *Let $\mathcal{P}$ be a subset of the primes $\le N$ and let $A(s) = \sum_{N \le n \le 2N} a_n n^{-s}$ be a Dirichlet polynomial with a_n supported on integers n with no prime factors from $\mathcal{P}$. Assume that $|a_n| \le 1$ for all n. Let $\mathcal{U} \subset [-T,T]$ be a sequence of well-spaced points. Then, for any $\eta > 0$,*

$$\sum_{t \in \mathcal{U}} |A(1+it)|^2 \ll \prod_{p \in \mathcal{P}} \Big(1 - \frac{2}{p} \Big) + |\mathcal{U}| T^{9\eta^{3/2}/2} (\log T)^{2/3} N^{-\eta/2}.$$

Proof. This follows from [**?**, Proof of Lemma 8?] taking $\eta' = 1/4$ and changing the sieve weights slightly to pick up the condition $p \mid n \implies p \notin \mathcal{P}$ instead of the condition $n \in \mathbb{P}$. The proof follows the proof of orginal Halász-Montgomery (see [**?**, Proof of Theorem 9.6]) with a few changes: One restricts the sums over n to its support, and then after using duality, one inserts sieve weights, and furthermore one estimates the appearing $\sum_m m^{it_{r_1}-it_{r_2}}$ using a result of Ford [**?**, Theorem 1] on bounding ζ-function close to Re $s = 1$. □

When we apply Lemma 3, we need a good upper bound for a certain choice of $\mathcal{U}$.

Lemma 4. *Let*

$$P(s) = \sum_{P \le p \le 2P} \frac{a_p}{p^s} \quad \textit{with } |a_p| \le 1.$$

Let $\mathcal{U} \subset [-T, T]$ be a sequence of well-spaced points such that $|P(1+it)| \ge V^{-1}$ for every $t \in \mathcal{U}$. Then

$$|\mathcal{U}| \ll T^{2\frac{\log V}{\log P}} V^2 \exp\left(2\frac{\log T}{\log P} \log\log T\right).$$

Proof. This consequence of the discrete mean value theorem for Dirichlet polynomials is proven in [**?**, Lemma 8]. □

Actually we will want to have a version of the previous result for a polynomial over the integers rather than primes.

Lemma 5. *There exists an absolute constant $c > 0$ such that the following holds. Let $Q \ge P \ge 1$ and $H \ge 1$, and write $U = (\log(Q/P) + 3)H$. Let a_m, b_m and c_p be 1-bounded sequences such that*

$$\begin{cases} a_{mp} = b_m c_p & \textit{whenever } P < p \le Q \textit{ and } p \nmid m \\ a_m = 0 & \textit{if } m \textit{ has no prime factor in the interval } (P, Q]. \end{cases}$$

Write

$$A(s) = \sum_{N < n \le 2N} \frac{a_n}{n^s}$$

Let $1 \le V \le c \cdot \min\{H, P\}$, and let $\mathcal{U} \subset [-T, T]$ be a sequence of well-spaced points such that $|A(1+it)| \ge 1/V$ for every $t \in \mathcal{U}$. Then

$$|\mathcal{U}| \ll T^{2\frac{\log(2UV)}{\log P}} U^3 V^2 \exp\left(2\frac{\log T}{\log P} \log\log T\right).$$

Proof. Write

$$\omega_{(P,Q]}(n) = \sum_{\substack{P < p \le Q \\ p \mid n}} 1.$$

By a Buchstab/Ramaré type identity (see also [**?**, Lemma 12] for a related argument) we can write

$$\begin{aligned}A(s) &= \sum_{P<p\le Q}\sum_{N/p<m\le 2N/p}\frac{a_{mp}}{(\omega_{(P,Q]}(m)+\mathbf{1}_{(p,m)=1})(mp)^s}\\ &= \sum_{P<p\le Q}\sum_{N/p<m\le 2N/p}\frac{b_m c_p}{(\omega_{(P,Q]}(m)+1)(mp)^s}+O(1/P)\\ &= \sum_{\lfloor H\log P\rfloor\le j\le H\log Q}\sum_{\substack{p\in(e^{j/H},e^{(j+1)/H}]\\ P<p\le Q}}\frac{c_p}{p^s}\sum_{N/e^j<m\le 2N/e^j}\frac{b_m}{(\omega_{(P,Q]}(m)+1)m^s}+O\left(\frac{1}{P}+\frac{1}{H}\right).\end{aligned}$$

Writing

$$P_j(s)=\sum_{\substack{p\in(e^{j/H},e^{(j+1)/H}]\\ P<p\le Q}}\frac{c_p}{p^s},$$

we see that

$$|A(1+it)|\le U\max_{\lfloor H\log P\rfloor\le j\le H\log Q}|P_j(1+it)|+O\left(\frac{1}{P}+\frac{1}{H}\right).$$

Hence, for any $t\in\mathcal{U}$, there exists $j\in[\lfloor H\log P\rfloor, H\log Q]$ such that $|P_j(1+it)|\ge 1/(2UV)$. The claim follows then from Lemma 4. ∎

In particular if $H,V\asymp(\log T)^B$ and $T^{10}\gg Q>P\gg(\log T)^A$ in the situation of Lemma 5, then $|\mathcal{U}|\ll T^{\frac{4B+2+o(1)}{A}}$.

4. A variant of Halász's theorem

In this section we shall deduce a slight variant of Halász theorem from the orginal. For this we need the following lemma about truncated Euler series.

Lemma 6. *Let $|a_p|\le 1$ be complex numbers and let $x\ge u>\log x$. Then*

$$\Big|\prod_{p\le u}\Big(1+\frac{a_p}{p}\Big)\Big|\ll\max_{|t|\le 1}\Big|\prod_{p\le x}\Big(1+\frac{a_p}{p^{1+it}}\Big)\Big|.$$

Proof. Let h be an even, smooth analytic function, with supp $\widehat{h}\subset[-1,1]$. Then

$$\sum_{p\le u}\frac{a_p}{p}\cdot\widehat{h}\Big(\frac{\log p}{\log u}\Big)=\frac{\log u}{2\pi}\int_{\mathbb{R}}\sum_{p\le x}\frac{a_p}{p^{1+it}}\ h\Big(\frac{t\log u}{2\pi}\Big)dt$$

We pick $\widehat{h}(x) = 1 - |x|$ for $|x| \leq 1$ and $\widehat{h}(x) = 0$ otherwise. Notice that $h(x) \geq 0$, and $h(x) \ll (1 + |x|)^{-2}$ and

$$\sum_{p \leq u} \frac{a_p \log p}{p \log u} = O(1)$$

Therefore we get for $T \geq 1$,

$$\begin{aligned} \mathrm{Re} \sum_{p \leq u} \frac{a_p}{p} &\leq \frac{\log u}{2\pi} \int_{-T}^{T} \Big(\mathrm{Re} \sum_{p \leq x} \frac{a_p}{p^{1+it}} \Big) \cdot h\Big(\frac{t \log u}{2\pi} \Big) dt + O\Big(\frac{\log \log x}{T \log u} \Big) + O(1) \\ &\leq \widehat{h}(0) \max_{|t| \leq T} \ \mathrm{Re} \sum_{p \leq x} \frac{a_p}{p^{1+it}} + O\Big(\frac{\log \log x}{T \log u} + 1 \Big) \end{aligned}$$

Since $\widehat{h}(0) = 1$ and $u > \log x$, the claim follows by taking $T = 1$. $\square$

The previous lemma allows us to obtain the following variant of Halász's theorem.

Lemma 7. *Let $\eta > 0$. Let f be a multiplicative function taking values in the unit disc and let $x \geq w \geq 1$ and $z \leq \exp(\frac{\log x}{\log \log x})$. Then*

$$\Bigg| \sum_{\substack{n \sim x \\ p | n \implies p \leq w \\ \exists p \geq z \colon p | n}} \frac{f(n)}{n^{1+it_0}} \Bigg| \ll \frac{\log \log x}{(\log x)^{\eta/2}} \prod_{p \leq x} \Big(1 + \frac{|f(p)| - 1}{p} \Big) + \frac{\log \log x}{\log x} \cdot \max_{|t - t_0| \leq \frac{1}{10} (\log x)^{\eta}} |F_x(1 + it)|$$

Proof. First note that

$$\Bigg| \sum_{\substack{n \sim x \\ p | n \implies p \leq z}} \frac{1}{n} \Bigg| \ll (\log x)^{-100},$$

so it suffices to show the claimed upper bound for

$$\Bigg| \sum_{\substack{n \sim x \\ p | n \implies p \leq w}} \frac{f(n)}{n^{1+it_0}} \Bigg| .$$

Now we need a slightly modified version of Halasz's theorem. Let

$$H(\beta)^2 = \sum_{k \in \mathbb{Z}} \frac{1}{k^2 + 1} \cdot \max_{|\tau - k| \leq \frac{1}{2}} |F_w(1 + \beta + it + i\tau)|^2$$

According to Montgomery,

$$\frac{1}{x}\Big|\sum_{\substack{x<n\le 2x\\ p|n \implies p\le w}} f(n)n^{-it}\Big| \ll \frac{1}{\log x}\int_{1/\log x}^{1}\frac{H(\beta)}{\beta}d\beta. \tag{2}$$

We also have,

$$|F_w(1+\beta+it+i\tau)| \ll \exp\Big(\Re\sum_{p\le\max(\exp(1/\beta),w)}\frac{f(p)}{p^{1+it+i\tau}}\Big)$$

On the part of the sum defining $H(\beta)$ corresponding to $|k| > \frac{1}{10}(\log x)^{\eta}-1$ we simply apply the trivial bound and get that the total contribution of this part to the right hand side of (2) is bounded by,

$$\frac{\log\log x}{(\log x)^{\eta/2}}\prod_{p\le x}\Big(1+\frac{|f(p)|-1}{p}\Big).$$

On terms with $|k| \le \frac{1}{10}(\log x)^{\eta}$ and $|\beta| < 1/\log\log x$ we use Lemma 6. This shows that the total contribution of this part to the right-hand side of (2) is

$$\ll \frac{\log\log x}{\log x}\max_{|t-t_0|\le\frac{1}{10}(\log x)^{\eta}}|F_x(1+it)|.$$

Finally we bound trivially the terms with $|k| \le \frac{1}{10}(\log x)^{\eta}$ and $1 \ge |\beta| \ge (\log\log x)^{-1}$, getting that the total contribution of such terms to the right-hand side of (2) is

$$\ll \frac{\log\log x}{\log x}.$$

Combining the above bounds we conclude that,

$$\frac{1}{x}\Big|\sum_{\substack{n\sim x\\ p|n\implies p\le w}}\frac{f(n)}{n^{1+it_0}}\Big| \ll \frac{\log\log x}{(\log x)^{\eta/2}}\prod_{p\le x}\Big(1+\frac{|f(p)|-1}{p}\Big)+\frac{\log\log x}{\log x}\max_{|t-t_0|\le\frac{1}{10}(\log x)^{\eta}}|F_x(1+it)|$$

as claimed (one can show that the maximum of the Euler product is always $\gg 1$, allowing us to omit the term $\log\log x/\log x$) $\square$

To see this simply note that the integral of the real-part of sum over the primes is close $o(1)$, therefore there exists a value in the neighborhood of t_0 that is $\ge -o(1)$

5. Analyzing means of Dirichlet polynomials

Throughout this section we shall assume that each $|a_\ell|, |b_m|, |c_n| \le 1$.

Proposition 1. *Let $M, N, T \ge 2$. Suppose that, for some $\eta \in (0, 1/2)$, $\mathcal{U} \subset [-T, T]$ be well-spaced and such that*

$$|\mathcal{U}| \ll \frac{\min\{\frac{M^{\eta/2}}{(\log M)^2}, \frac{N^{\eta/2}}{(\log N)^2}\}}{T^{9\eta^{3/2}/2}(\log T)^{2/3}}.$$

Suppose that b_m and c_n are only supported on integers m, n whose prime factors do not belong to subsets of primes $\mathcal{P}_B \subset [1, 2M]$ and $\mathcal{P}_C \subset [1, 2N]$ respectively. Then

$$\sum_{t \in \mathcal{U}} \left| \sum_{\substack{\ell \sim L \\ m \sim M \\ n \sim N}} \frac{a_\ell b_m c_n}{(\ell m n)^{1+it}} \right| dt \ll \left(\max_{t \in \mathcal{U}} \left| \sum_{\ell \sim L} \frac{a_\ell}{\ell^{1+it}} \right| \right) \prod_{p \in \mathcal{P}_B} \left(1 - \frac{1}{p}\right) \cdot \prod_{p \in \mathcal{P}_C} \left(1 - \frac{1}{p}\right).$$

Proof. We first estimate the sum over ℓ by the maximal t, then Cauchy and apply Lemma 3, which gives

$$\left(\sum_{t \in \mathcal{U}} \left| \sum_{m \sim M} \frac{b_m}{m^{1+it}} \right|^2 \right)^{1/2} \ll \prod_{p \in \mathcal{P}_B} \left(1 - \frac{1}{p}\right),$$

and similarly for the Dirichlet polynomial with coefficients c_n. □

The following proposition follows from the proof of [**?**, Lemma 7.3].

Proposition 2. *Let $C > 0$ be given. Let $\eta \in (0, \frac{1}{2}]$. Let $\mathcal{T} \subset [-T, T]$, and let $L, M, N \geq 1$ be such that $LMN \asymp x$ and*

$$\max\{M/N, N/M\} \leq C \cdot \frac{x^{2\eta}}{(\log x)^{1000}} \text{ and } L \leq C \cdot x^\gamma$$

where

$$\gamma = \begin{cases} \frac{4\eta+1}{3} & \text{if } \frac{1}{8} \leq \eta \leq \frac{1}{2} \\ 4\eta & \text{if } \frac{1}{4 \cdot (2g+1)} \leq \eta \leq \frac{1}{8g}\ ,\ g \in \mathbb{N} \\ 4\eta - \frac{1 - 4\eta(2g+1)}{4g-1} & \text{if } \frac{1}{8g+6-\frac{4}{4g+1}} \leq \eta \leq \frac{1}{4 \cdot (2g+1)}\ ,\ g \in \mathbb{N} \\ 4\eta - \frac{4\eta(2g+2)-1}{4g+3} & \text{if } \frac{1}{4 \cdot (2g+2)} \leq \eta \leq \frac{1}{8g+6-\frac{4}{4g+1}}\ ,\ g \in \mathbb{N} \end{cases}$$

Then,

$$\int_{\mathcal{T}} \Big| \sum_{\substack{\ell \sim L \\ m \sim M \\ n \sim N}} \frac{a_\ell b_m c_n}{(\ell m n)^{1+it}} \Big| dt \ll_C (\log x)^{20} \cdot \Big((\log x)^{-100} + \max_{t \in \mathcal{T}} \Big| \sum_{\ell \sim L} \frac{a_\ell}{\ell^{1+it}} \Big| \Big).$$

Remark 5. In all cases $4\eta \geq \gamma \geq 4\eta - 4\eta^2 + O(\eta^3)$. In particular if $\frac{1}{4 \cdot (2g+3)} \leq \eta \leq \frac{1}{4 \cdot (2g+1)}$ with $g \in \mathbb{N}$, then $\frac{1}{2g+3} \leq \gamma \leq 4\eta \leq \frac{1}{2g+1}$.

6. Combinatorial decompositions

In order to apply Proposition 2 we need to be able to split an integer into a product of three factors of convenient sizes. For this we will use the following two combinatorial lemma. The first lemma below will be used when addressing intervals of length $y > x^{7/12}$.

Lemma 8. *Suppose that $\eta \geq \frac{1}{12}$. If $\frac{1}{2} + \eta \geq a_1 \geq \ldots \geq a_r > 0$ with $a_1 + \ldots + a_r = 1$ then one can partition $\{1, 2, \ldots, r\}$ into union of three disjoint subsets I, J and K such that*

$$\left|\sum_{i\in I} a_i - \sum_{i \in J} a_i\right| \leq 2\eta \quad \textit{and} \quad \left|\sum_{i\in K} a_i\right| \leq \frac{1}{3}.$$

This bound on the a_i's is best possible, since otherwise let $a_1 > \frac{1}{2} + \eta$ and then the sum of the other α_i's is $< \frac{1}{2} - \eta$ and their difference is $> 2\eta$.

Proof of Lemma 8. We may assume that $a_{r-1} + a_r > \frac{1}{2} + \eta$ else we may replace a_{r-1} and a_r by their sum, and deduce the partition needed here from the partition there. Therefore we may assume $r \leq 3$ else $1 \geq a_1 + a_2 + a_3 + a_4 \geq 2(\frac{1}{2} + \eta)$, a contradiction.

Now $a_1 < \frac{1}{2} - \eta$ else we can take $I = \{1\}$, $J = \{1, \ldots, r\} \setminus I$ and $K = \emptyset$. But then $r \geq 3$ as $ra_1 \geq a_1 + \ldots + a_r = 1$, and so $r = 3$.

Now $a_2 > \frac{1}{2}(\frac{1}{2} + \eta)$ (as $a_2 + a_3 > \frac{1}{2} + \eta$), which is $> \frac{1}{2} - 3\eta$, and so $a_1 - a_2 < 2\eta$. We also have $a_3 \leq \frac{1}{3}$ as $3a_3 \leq a_1 + a_2 + a_3 = 1$. Hence we can take $I = \{1\}, J = \{2\}$ and $K = \{3\}$. $\square$

In order to tackle intervals that are shorter than $x^{7/12}$ we need the following refinement.

Lemma 9. *Fix integer $m \geq 1$ and suppose that $\frac{1}{2m+3} \leq \tau \leq 4\eta \leq \frac{1}{2m+1}$. Let $y = y(\eta, \tau) := \max\{\tau, \frac{1+2\eta}{2m+2}\}$. For any $0 < a_r \leq \ldots \leq a_2 \leq a_1 \leq y$, such that $a_1 + \cdots + a_r = 1$, there exists a partition of $\{1, \ldots, r\} = I \cup J \cup K$ such that*

$$\left|\sum_{i\in I} a_i - \sum_{j \in J} a_j\right| \leq 2\eta \quad \textit{and} \quad \sum_{k\in K} a_k \leq \tau.$$

This is best possible for if $a_1 = \ldots = a_{2m+1} = y(\eta, \tau) + \epsilon$ and $a_{2m+2} = 1 - (2m+1)a_1$ for some arbitrarily small $\epsilon > 0$ then $a_1 > a_{2m+2} + 2\eta$.

Proof. We may assume that $a_r + a_{r-1} > y$ for if not we may replace a_r and a_{r-1} by their sum, use the rsult, and then deduce the result for our original a_i's. Therefore $a_{r-1} > y/2 \geq a_1/2$.

If one has a set of integers $c_1, ..., c_s$ with each $c_i \leq 2\eta$ then we can assign signs so that $| \pm c_1 \pm c_2 \pm \cdots \pm c_s| \leq 2\eta$ simply by being greedy.

Suppose that $a_1 \le \tau$. Since $a_{r-1} \ge y/2 \ge \tau/2$ the differences $a_i - a_{i+1} \in [0, \tau/2] \in [0, 2\eta]$. So the c_j's in the last paragraph can be taken to be $a_{2\ell-1} - a_{2\ell}$ with $1 < 2\ell \le r-1$. If r is odd let $K = \{r\}$ and we are done. If r is even we either have $a_{r-1} - a_r \le 2\eta$ or $a_r < a_{r-1} - 2\eta \le \tau - 2\eta \le 2\eta$. In the first case we let $a_{r-1} - a_r$ be a c_j and we are done, in the second we let a_r be a c_j, and let $K = \{r-1\}$.

We may now assume that $a_1 = y > \tau$.

If every $a_{2\ell-1} - a_{2\ell} \le 2\eta$ then the inequalities are satisfied, unless r is odd and $a_r > \tau$ (else we let $K = \{a_r\}$). But then $\frac{r}{2m+3} \le r\tau < 1 \le ry \le \frac{r}{2m+1}$ and so $r = 2m+1$ and each $a_i = \frac{1}{2m+1}$. However this is impossible since $a_1 > y$ (for if $\frac{1+2\eta}{2m+2} \ge \frac{1}{2m+1}$ then $2\eta \ge \frac{1}{2m+1}$, contradicting the hypothesis).

Now assume $a_{2\ell+1} - a_{2\ell+2} > 2\eta$, so that $a_{2\ell+2} < y - 2\eta$.

If $r > 2\ell+2$ then $a_{2\ell+2} + a_r - a_1 \in (0, y - 4\eta)$. We now take this and $a_2 - a_3, a_4 - a_5, \ldots, a_{2\ell} - a_{2\ell+1}$ as c_i's as these are all $\le 2\eta$ (for, if not, then $y - 4\eta > a_{2\ell+2} > y/2$ which is false). For the remaining a_i's no two can differ by more than 2η else they would sum to less than y. Therefore we can pair them up and any one left over becomes the set K.

Hence $r = 2\ell+2$. If $a_r \le 2\eta$ then $a_{2\ell+1} > \tau$, else we can place it in K. If $a_r > 2\eta$ then $a_{2\ell+1} > a_r + 2\eta > 4\eta \ge \tau$. Therefore

$$\frac{2\ell+1}{2m+3} \le (2\ell+1)\tau < \sum_{i=1}^{2\ell+2} a_i = 1 \le (2\ell+2)y - 2\eta < \frac{2\ell+2}{2m+1}$$

and so $\ell = m$.

Now $a_{2m+2} = 1 - \sum_{i=1}^{2m+1} a_i \le 1 - (2m+1)y = y - 2\eta$. We can therefore use the differences $a_{2\ell-1} - a_{2\ell}$ for $\ell = 1, \ldots, m+1$ as our c_i. $\square$

7. Proof of the theorem

We need to split into two cases according to whether n has a large prime factor or not.

7.1. Case $P^+(n) < x^{4/7}$. Write $z = \exp(\lfloor \frac{\log x}{\log\log x} \rfloor)$. We will want to exclude some subset of the following small subsets of the integers $n \in (x, x+y]$:

- The integers that have two prime factors $p, q \in [z, 4y^{1/2}]$ such that $p, q \in (e^i, e^{i+1}]$ for some i. The number of such integers in an interval of length y is
$$\ll \sum_{\lfloor \log z \rfloor \le i \le \log(4y^{1/2})} \sum_{p,q \in (e^i, e^{i+1}]} \frac{y}{pq} \ll y \sum_{\lfloor \log z \rfloor \le i \le \log(4y^{1/2})} \frac{1}{i^2} \ll y \frac{\log\log x}{\log x}.$$
using the prime number theorem.
- Integers that have no prime factors in $[(\log x)^A, z]$ for some fixed $A \ge 2$. The number of these is $\ll_A y(\log\log x)/\log z \ll y(\log\log x)^2/\log x$.

- Integers whose z-smooth part is $\geq x^{\epsilon/100}$. Any such n must have a z-smooth divisor $r \in R := [x^{\epsilon/100}, x^{\epsilon/100}z)$, and so the number of such n is $\ll y \sum_r 1/r$ over the z-smooth integers $r \in R$. Now the density of such integers r is $\ll 1/u^u$ where $u := \frac{\log x^{\varepsilon/100}}{\log z} \gg \log\log x$; therefore number of such n is $O(y/(\log x)^{10})$.

Let

$$\varrho = \varrho(\theta) = \min\left\{\frac{4}{7}, \tau_\varepsilon(\theta)\right\}$$

Let us write

$$\mathcal{N}_0 := \{x < n \leq x+y \colon P^+(n) < x^\varrho \text{ and the second largest prime factor of } n \text{ is at most } 4y^{1/2}\}$$

and

$$\mathcal{M} = \{m \in \mathbb{N} \colon \text{There is no } i \text{ and } p_1, p_2 \in (e^i, e^{i+1}] \text{ such that } p_1p_2 \mid m\}.$$

Write $\mathcal{N}$ for the set of integers $n \in \mathcal{N}_0$ such that $n = mv$, where

- v is z-smooth and has a prime factor $\geq (\log x)^A$
- $v \leq x^{\epsilon/100}$
- $m \in \mathcal{M}$ is z-rough.

By the above

$$\sum_{\substack{x<n\leq x+y \\ n\in\mathcal{N}_0\setminus\mathcal{N}}} 1 \ll y\,\frac{(\log\log x)^2}{\log x}.$$

We shall factor m in such a way that, after an application of Lemma 2, we can split the integral coming from that lemma in such a way that for the typical t we can use Proposition 2 together with Lemma 8 and Lemma 9 and for some exceptional t we can use Proposition 1. To faciliate this, we just need to factor m in an unique way to a product of a bounded number of factors shorter than x^ϱ that lie on e-adic intervals and do not have any cross-conditions between them. Essentially we write $m = r_1 \cdots r_{k+1} q_1 \cdots q_k$ where to r_1 we collect largest prime factors of m until it is $\geq x^{\varrho/2}$, take q_1 the next largest prime factor of m, then again collect remaining largest prime factors to r_2 until it is $\geq x^{\varrho/2}$ etc. The following technical construction does this rigorously.

For each $n \in \mathcal{N}$, there exists unique $k \leq \lceil \frac{2}{\varrho} \rceil + 1$ and $\lceil \log x^\varrho \rceil = i_0 > i_1 > \ldots > i_k > i_{k+1} = \log z$ such that $n = vr_1 \cdots r_{k+1} q_1 \ldots q_k$, where

- v is z-smooth and has a prime factor $\geq (\log x)^A$
- $v \leq x^{\epsilon/100}$
- $q_j \in (e^{i_j}, e^{i_j+1}]$ for all $j = 1, \ldots, k$.
- $p \mid r_j \implies p \in (e^{i_j+1}, e^{i_{j-1}}]$
- $r_j \in [x^{\varrho/2}, x^\varrho) \cap \mathcal{M}$ for all $j = 1, \ldots, k$

- $r_j/P^-(r_j) < x^{\varrho/2}$ for all $j = 1, \dots, k+1$.
- $r_{k+1} \in (\{1\} \cup (z, x^{\varrho}]) \cap \mathcal{M}$.

We only treat the case $r_{k+1} \neq 1$, the opposite case being treated completely similarly with one less factor.

Furthermore we split r_j into e-adic intervals $r_j \in (e^{h_j}, e^{h_j+1}]$. Then $v \asymp xe^{-(i_1+\dots+i_k+h_1+\dots+h_{k+1})}$ and

$$\log x^{1-\epsilon/100} - O(1) \le i_1 + \dots + i_k + h_1 + \dots + h_{k+1} \le \log \frac{x}{(\log x)^A}.$$

By Lemma 2 we then have

$$\sum_{\substack{x \le n \le x+y \\ n \in \mathcal{N} \\ r_{k+1} \neq 1}} f(n) - \int_x^{x+y} u^{it_0} du \cdot \frac{1}{Y} \sum_{\substack{x \le n \le x+Y \\ n \in \mathcal{N} \\ r_{k+1} \neq 1}} f(n) n^{-it_0} \ll \frac{y}{(\log x)^{\eta}} \prod_{p \le x} \Big(1 + \frac{|f(p)| - 1}{p}\Big) + \frac{y}{(\log x)^{10}}$$

plus $\log\log x$ times the maximum, over $x/y \le T \le (x/y)(\log x)^{20}$ of,

(3)

$$\frac{x}{T} \sum_{k=1}^{\lceil 2/\varrho \rceil + 1} \sum_{\substack{\log x^{\varrho} > i_1 > \dots > i_k > \log z \\ \log x^{\varrho/2} \le h_1, \dots, h_k \le \log x^{\varrho} \\ \log z < h_{k+1} \le \log x^{\varrho} \\ \log x^{1-\epsilon/100} - O(1) \le i_1 + \dots + i_k + h_1 + \dots + h_{k+1} \le \log \frac{x}{(\log x)^A}}} \int_{\substack{|t| \le T \\ |t-t_0| > (\log x)^{\eta}}} |Q_{i_1}(1+it) \dots Q_{i_k}(1+it)$$
$$\cdot R_{i_1, i_0, h_1}(1+it) \cdots R_{i_{k+1}, i_k, h_{k+1}}(1+it) V_{\log x - (i_1 + \dots + i_k + h_1 + \dots + h_{k+1})}(1+it)| dt,$$

where

$$Q_i(s) = \sum_{p \in (e^i, e^{i+1}]} \frac{f(p)}{p^s}, \quad R_{i,j,h}(s) = \sum_{\substack{r \in (e^h, e^{h+1}] \cap \mathcal{M} \\ p | r \implies p \in (e^{i+1}, e^j] \\ r/P^-(r) \le x^{\varrho/2}}} \frac{f(r)}{r^s} \quad \text{and} \quad V_i(s) = \sum_{\substack{n \asymp e^i \\ n \text{ is } z\text{-smooth} \\ \exists p \ge (\log x)^A :\, p | n}} \frac{f(n)}{n^s}.$$

We write

$$\mathcal{T} = \{|t| \le T \colon |V_{\log x - i}| \le (\log x)^{-1000} \text{ for each } i \in \{1, 2, \dots, \log x - A \log\log x\}\}$$

and $\mathcal{U} = [-T, T] \setminus \mathcal{T}$.

Consider first the integral in (3) over $t \in \mathcal{U}$. By Lemma 5 we know that, for any $\delta > 0$, once A is large enough in terms of δ, we have $|\mathcal{U}| \ll T^{\delta}$, and by Lemma 7 we know that, for all such t' and i

$$|V_{\log x - i}(1+it')| \ll \frac{\log\log x}{(\log x - i)^{\eta/2}} \prod_{p \le x} \Big(1 + \frac{|f(p)| - 1}{p}\Big) + \frac{\log\log x}{\log x - i} \max_{|t-t'| \le \frac{1}{10}(\log x)^{\eta}} |F_x(1+it)|$$
$$\ll \frac{\log\log x}{(\log x - i)^{\eta/2}} \prod_{p \le x} \Big(1 + \frac{|f(p)| - 1}{p}\Big) + \frac{\log\log x}{\log x - i} \max_{|t-t_0| > \frac{1}{2}(\log x)^{\eta}} |F_x(1+it)|,$$

since $|t-t'| \le \frac{1}{10}(\log x)^\eta$ and $|t'-t_0| > (\log x)^\eta$ and so $|t-t_0| > \frac{1}{2}(\log x)^\eta$.

Furthermore $e^{h_1} \ge x^{\varrho/2}$ and either $e^{h_2} \ge x^{\varrho/4}$ or $e^{i_1} \ge x^{\varrho/4}$. For simplicity let us assume we are in the first case — the second case is treated similarly. Bounding trivially

$$|Q_i(1+it)| \ll \frac{1}{i} \quad \text{and} \quad |R_{i_j,i_{j-1},h_j}(1+it)| \ll \frac{1}{\log z}$$

for $i = 1,\dots,k$ and $j = 3,\dots,k+1$ in the integral over $\mathcal{U}$ in (3), we see that this part contributes to the integral at most

$$\frac{x}{T}\sum_{k=1}^{\lceil 2/\varrho\rceil+1} \sum_{\substack{\log x^\varrho > i_1 > \dots > i_k > \log z \\ \log x^{\varrho/2} \le h_1,\dots \le \log x^\varrho \\ \log z < h_{k+1} \le \log x^\varrho \\ \log x^{1-\epsilon/100} - O(1) \le i_1+\dots+i_k+h_1+\dots+h_{k+1} \le \log \frac{x}{(\log x)^A}}} \frac{1}{i_1\cdots i_k}\cdot\frac{1}{(\log z)^{k-1}}$$
$$\cdot \int_{\substack{t\in\mathcal{U} \\ |t-t_0|>(\log x)^\eta}} |R_{i_1,i_0,h_1}(1+it)R_{i_2,i_1,h_2}(1+it)V_{\log x-(i_1+\dots+i_k+h_1+\dots+h_{k+1})}|dt.$$

By the above bound for $V(1+it)$ and Proposition 1 applied to a well-spaced subset $\mathcal{U}'\subset\mathcal{U}$, this is

$$\frac{x}{T}\sum_{k=1}^{\lceil 2/\varrho\rceil+1} \sum_{\substack{\log x^\varrho > i_1 > \dots > i_k > \log z \\ \log x^{\varrho/2} \le h_1,\dots,h_{k+1} \le \log x^\varrho \\ (1-\epsilon/100)\log x - O(1) \le i_1+\dots+i_k+h_1+\dots+h_{k+1} \le \log x - A\log\log x}} \frac{1}{i_1\cdots i_k}\cdot\frac{(\log\log x)^{k+1}}{(\log x)^{k+1}}$$
$$\cdot\Big(\frac{\log\log x}{(\log x - i_1-\dots-i_k-h_1-\dots h_{k+1})^{\eta/2}}\prod_{p\le x}\Big(1+\frac{|f(p)|-1}{p}\Big)$$
$$+\frac{\log\log x}{\log x}\max_{|t-t_0|\ge\frac{1}{2}(\log x)^\eta}|F_x(1+it)|\Big)$$
$$\ll \frac{(\log\log x)^{2k+3}}{(\log x)^{\eta/2}}\prod_{p\le x}\Big(1+\frac{|f(p)|-1}{p}\Big)+\frac{(\log\log x)^{2k+3}}{\log x}\cdot\max_{|t-t_0|\ge\frac{1}{2}(\log x)^\eta}|F_x(1+it)|\Big)$$

Now we can concentrate to the integral over $\mathcal{T}$ in (3). If $\theta > \frac{7}{12}+\varepsilon$ then we apply Lemma 8 with k replaced by $2k+1$ to the set

$$\{\alpha_1,\dots,\alpha_{2k+1}\} = \left\{\frac{i_1}{\log x},\dots,\frac{i_k}{\log x},\frac{h_1}{\log x},\dots,\frac{h_{k+1}}{\log x}\right\}.$$

We see that, for any appearing combination of i_j, h_j, we can write

$$Q_{i_1}(1+it)\dots Q_{i_k}(1+it)R_{i_1,i_0,h_1}(1+it)\cdots R_{i_{k+1},i_k,h_{k+1}}(1+it) = M(1+it)N(1+it)A(1+it)$$

such that the lengths M, N, A of the polynomials satisfy

$$\max\{M/N, N/M\} \ll x^{2\theta-1} \quad \text{and } A \ll x^{\frac{1}{3}}$$

Then we can apply Proposition 2 with $\eta > \frac{1}{12} + \frac{\varepsilon}{4}$ to the Dirichlet polynomials $M(s)$, $N(s)$ and $A(s)V(s)$ (note that $V(s)$ add only $o(1)$ to γ which is acceptable since $\eta > \frac{1}{12} + \frac{\varepsilon}{4}$). This leads to a satisfactory bound for the integral over $\mathcal{T}$ in (3).

Now consider the case $\frac{1}{2} < \theta < \frac{7}{12} + \varepsilon$. Let $\gamma = \gamma(\theta)$ be the exponent appearing in Proposition 2 with $\eta = \theta - \frac{1}{2}$. Since $\tau_\varepsilon(\theta) < \max\{\gamma(\theta - \varepsilon), \tau_0(\theta - \varepsilon)\}$, since the length of each polynomial Q, R does not exceed $O(x^{\tau_\varepsilon(\theta)})$ and since there exists an m such that $\frac{1}{2m+3} \leq \gamma(\theta - \varepsilon) \leq 4(\theta - \varepsilon - \frac{1}{2}) \leq \frac{1}{2m+1}$ it follows from Lemma 9 that we can group these polynomials into a product

$$M(s)N(s)A(s)$$

where the lengths M, N, A of the polynomials $M(s), N(s), A(s)$ satisfy

$$\max\{M/N, N/M\} = O(x^{2\theta-1}) \quad \text{and } A = O(x^{\gamma(\theta-\varepsilon)}) = O(x^{\gamma(\theta)-\varepsilon^{10}})$$

Therefore we can apply Proposition 2 to the polynomials $M(s)$, $N(s)$ and $A(s)V(s)$ (with as usual V adding at most $o(1)$ to γ). This leads to a satisfactory bound for the integral over $\mathcal{T}$ in (3).

Next we consider

$$\mathcal{N}_1 := \{x < n \leq x+y \colon P^+(n) \leq x^{4/7} \text{ and the second largest prime factor of } n \text{ is at least } y^{1/2}\}.$$

This works completely similarly except we first write

$$\sum_{n \in \mathcal{N}_1} f(n) = \frac{1}{2!} \sum_{\substack{x < p_1 p_2 n' \leq x+y \\ 4y^{1/2} < p_1, p_2 \leq y \\ p | n' \implies p \leq 4y^{1/2}}} f(p_1 p_2 n') + \frac{1}{3!} \sum_{\substack{x < p_1 p_2 n' \leq x+y \\ 4y^{1/2} < p_1, p_2, p_3 \leq y \\ p | n' \implies p \leq 4y^{1/2}}} f(p_1 p_2 p_3 n') + O(x^{1/2})$$

and then write $n' = vm$ where v is as before and m is decomposed as before. Now the Dirichlet polynomials over p_i play similar roles in the arguments as those over q_i and r_i. The reason we needed to separate this case is that we were not able to show that the numbers with two factors from an interval $(e^i, e^{i+1}] \subset [4y^{1/2}, 2x^{1/2}]$ are rare To do: Did not really think about whether we could actually do this .

7.2. **Case $P^+(n) \geq x^{4/7}$.** Write $y' = y/\exp((\log\log x)^2)$. The number of integers in an interval of length y that have a prime factor in $[y', y]$ is $O(y(\log\log x)^2/(\log x))$ which is negligible.

We consider

$$\mathcal{N}_2 := \{x < n \leq x + y \colon x^{4/7} \leq P^+(n) < y'\} = \bigcup_{\frac{4}{7}\log x \leq j \leq \log y'} \mathcal{N}_{2,j},$$

where

$$\mathcal{N}_{2,j} = \{x < n \leq x + y \colon P^+(n) \in (e^j, e^{j+1}]\}$$

are disjoint. Let $\mathcal{N}$ be the set of integers such that there exists $j \in [\frac{4}{7}\log x, \log y']$ such that $n \in \mathcal{N}_{2,j}$ and $n = pmv$, where

- $p \in (e^j, e^{j+1}]$
- v is $z_j := \exp(\frac{\log y - j}{\log\log x})$-smooth and has a prime factor $\geq (\log x)^A$
- $v \leq (y/e^j)^{1/4}$
- m is z_j-rough.

Here we are ignoring two sorts of numbers

- Those that do not have a prime factor from $[(\log x)^A, z_j]$, the number of these is

$$\sum_{(\frac{4}{7}-2\varepsilon)\log x \leq j \leq \log y'} \sum_{e^j < p \leq e^{j+1}} \sum_{\substack{x/p < m < (x+y)/p \\ q \mid m \implies q \notin [(\log x)^A, z_j]}}$$
$$\ll \sum_{\frac{4}{7}\log x \leq j \leq \log y'} \sum_{e^j < p \leq e^{j+1}} \frac{y \log\log x}{p \log z_j}$$
$$\ll y(\log\log x)^2 \cdot \sum_{\frac{4}{7}\log x \leq j \leq \log y'} \frac{1}{j \cdot (\log y - j)} \ll y\frac{(\log\log x)^3}{\log x}.$$

- Those whose z_j-smooth part is at least $(y/e^j)^{1/4}$. The number of these is

$$\sum_{\frac{4}{7}\log x \leq j \leq \log y'} \sum_{\substack{(y/e^j)^{1/4} \leq v \leq 2x/e^j \\ v\ z_j\text{-smooth}}} \sum_{\substack{m \asymp 2x/(ve^j) \\ m\ z_j\text{-rough}}} \sum_{x/(mv) < p \leq (x+y)/mv} 1$$
$$\ll \sum_{\frac{4}{7}\log x \leq j \leq \log y'} \sum_{\substack{(y/e^j)^{1/4} \leq v \leq 2x/e^j \\ v\ z_j\text{-smooth}}} \frac{1}{v},$$

where we estimated sums over p and m trivially. Since $\frac{\log(y/e^j)^{1/4}}{\log z_j} \gg \log\log x$, the last sum here is $O((\log x)^{-100})$ by an estimate for the number of smooth numbers (see e.g. [**?**, formula (1.12)]).

By above

$$\sum_{\substack{x < n \leq x+y \\ n \in \mathcal{N}_2 \setminus \mathcal{N}}} 1 \ll y\frac{(\log\log x)^2}{\log x}.$$

Splitting also m to intervals $(e^h, e^{h+1}]$ and applying Lemma 2, we are led to studying

$$\frac{x}{T} \sum_{\substack{\frac{4}{7}\log x \le j \le \log y' \\ h \le \log 2x/e^j \\ \log \frac{x}{(y/e^j)^{1/4}} \le h+j \le \log \frac{x}{(\log x)^A}}} \int_{\substack{|t|\le T \\ |t-t_0|>(\log x)^\eta}} |Q_j(1+it) M_{j,h}(1+it) V_{j,\log x - j - h}(1+it)| \, dt,$$

where

$$Q_j(s) = \sum_{p \in (e^j, e^{j+1}]} \frac{f(p)}{p^s}, \quad M_{j,h}(s) = \sum_{\substack{m \in (e^h, e^{h+1}] \\ m \ z_j\text{-rough}}} \frac{f(m)}{m^s} \quad \text{and} \quad V_{j,h}(s) = \sum_{\substack{v \asymp e^h \\ v \ z_j\text{-smooth} \\ \exists p \ge (\log x)^a : \, p | v}} \frac{f(v)}{v^s}.$$

Like previously, we write

$$\mathcal{T} = \{|t| \le T \colon |V_{\log x - i}| \le (\log x)^{-1000} \text{ for each } i \in [1, \log x - A \log\log x]\}$$

and $\mathcal{U} = [-T, T] \setminus \mathcal{T}$.

Consider first the integral in (3) over $\mathcal{U}$. By Lemma 5 we know that, for any $\delta > 0$, once A is large enough in terms of δ, we have $|\mathcal{U}| \ll T^\delta$, and by Lemma 7 we know that, for all t and j, h

$$|V_{\log x - j - h}(1+it)| \ll \frac{\log\log x}{(\log x - h - j)^{\eta/2}} \prod_{p \le x} \Big(1 + \frac{|f(p)| - 1}{p}\Big) + \frac{\log\log x}{\log x} \cdot \max_{|t-t_0| \le \frac{1}{10}(\log x)^\eta} |F_x(1+it)|.$$

Applying this and Proposition 1 we see that the integral over $\mathcal{U}$ contributes

$$\sum_{\substack{\frac{4}{7}\log x \le j \le \log y' \\ h \le \log 2x/e^j \\ \log \frac{x}{(y/e^j)^{1/4}} \le h+j \le \log \frac{x}{(\log x)^A}}} \Big(\frac{\log\log x}{(\log x - h - j)^{\eta/2}} \prod_{p \le x} \Big(1 + \frac{|f(p)| - 1}{p}\Big)$$
$$+ \frac{\log\log x}{\log x} \cdot \max_{|t-t_0| > \frac{1}{10}(\log x)^\eta} |F_x(1+it)| \Big) \cdot \frac{1}{j} \frac{1}{\log z_j}$$

which is acceptable.

For the integral over $\mathcal{T}$ we can apply Proposition 2 with m-polynomial $M_{j,h}(s)$, n-polynomial $Q_j(s)$ and ℓ-polynomial $V_{j,h}(s)$ since

$$\frac{e^j}{e^h} \asymp \frac{e^{2j} x/(e^{j+h})}{x} \le \frac{e^{2j}}{x} \cdot \Big(\frac{y}{e^j}\Big)^{1/4} \le \frac{y'^2}{x} \Big(\frac{y}{y'}\Big)^{1/4} \le \frac{y^2}{x (\log x)^{1000}}.$$

since $e^j \le y'$.

8. Proof of the Corollaries

8.1. **Proof of Corollary 1 and Corollary 2.** Corollary 1 follows from the lemma below and the Vinogradov repulsion estimate proven in the appendix.

Lemma 10. *Let $f : \mathbb{N} \to [-1,1]$ be a multiplicative function. Let $\eta > 0$ be given. Then,*

$$\Big|\frac{(\log x)^{\eta}}{x}\sum_{x\le n\le x+x/(\log x)^{\eta}} f(n) - \frac{1}{x}\sum_{x\le n\le 2x} f(n)\Big| \ll \Big(\frac{1}{\log x}\Big)^{1-2/\pi}\cdot(\log x)^{\eta+o(1)}$$

Proof. Let $w = 1 + (\log x)^{-\eta}$. Then,

$$\frac{(\log x)^{\eta}}{x}\sum_{x\le n\le x+x/(\log x)^{\eta}} f(n) = \frac{1}{x(w-1)}\sum_{n\le xw} f(n) - \frac{1}{x(w-1)}\sum_{n\le x} f(n)$$

By a result of Granville-Soundararajan,

$$\frac{1}{x}\sum_{n\le xw} f(n) = \frac{w}{x}\sum_{n\le x} f(n) + O\Big(\Big(\frac{\log 2w}{\log x}\Big)^{1-\frac{2}{\pi}+o(1)}\Big)$$

And so it follows that,

$$\frac{(\log x)^{\eta}}{x}\sum_{x\le n\le x+x/(\log x)^{\eta}} f(n) = \frac{1}{x}\sum_{n\le x} f(n) + O\Big((\log x)^{\eta}\cdot\Big(\frac{1}{\log x}\Big)^{1-\frac{2}{\pi}+o(1)}\Big).$$

By another, similar application of Granville-Soundararajan, we also find that,

$$\frac{1}{x}\sum_{n\le x} f(n) = \frac{1}{x}\sum_{x\le n\le 2x} f(n) + O\Big((\log x)^{-(1-\frac{2}{\pi}+o(1))}\Big).$$

and the result follows. □

Proof of Corollary 1. We choose $t_0 = 0$ in Theorem 1. Subsequently we apply Lemma 10 and the repulsion estimate from the appendix. This shows that for any $\eta > 0$,

$$\frac{1}{y}\sum_{x\le n\le x+y} f(n) = \frac{1}{x}\sum_{x\le n\le 2x} f(n) + \\ + O\Big((\log x)^{3\eta/2-(1-\frac{2}{\pi}+o(1))} + (\log x)^{-\eta/2+o(1)} + (\log x)^{-\frac{1}{3}\cdot(1-\frac{2}{\pi})+o(1)}\Big)$$

The optimal choice is $\eta = \frac{1}{2}\cdot(1-\frac{2}{\pi})$ and the claim follows. □

Proof of Corollary 2. Without loss of generality we can assume that $w \leq x^{1/100}$. Corollary 2 follows from Corollary 1 upon noticing that the number of integers that have a prime factor $> x/w$ is less than the number of integers that don't have a prime factor in $[w, x^{1-1/100}]$ and the number of such integers in a short interval $[x, x+h]$ (with $h > x^{1/2}$) is by a sieve bound

$$\ll \prod_{w \leq p \leq x^{1/10}} \Big(1 - \frac{1}{p}\Big) \asymp \frac{\log w}{\log x}.$$

□

8.2. **Proof of Corollary 3 and Corollary 4.** We will use the following Lipschitz estimate.

Lemma 11. *Let $\kappa, C, \delta > 0$ be given. Suppose that $f : \mathbb{N} \to [-1,1]$ is a multiplicative function such that for any $2 \leq w \leq z \leq x^{\delta}$,*

$$\sum_{w \leq p \leq z} \frac{|f(p)|}{p} \geq \kappa \sum_{w \leq p \leq z} \frac{1}{p} - \frac{C}{\log w}.$$

Then, there exists an $\eta = \eta(\kappa) > 0$ such that for all $0 < \gamma < \eta$,

$$\Big| \frac{(\log x)^{\gamma}}{x} \sum_{x \leq n \leq x + x(\log x)^{-\gamma}} f(n) - \frac{1}{x} \sum_{x \leq n \leq 2x} f(n) \Big| \ll (\log x)^{\gamma - \eta} \prod_{p \leq x} \Big(1 + \frac{|f(p)| - 1}{p}\Big).$$

Proof. Similarly to the proof of Lemma 10 the result will follow if we are able to show that for $\frac{1}{10} \leq w \leq 10$,

$$\Big| \frac{w}{x} \sum_{n \leq x/w} f(n) - \frac{1}{x} \sum_{n \leq x} f(n) \Big| \ll (\log x)^{-\eta} \prod_{p \leq x} \Big(1 + \frac{|f(p)| - 1}{p}\Big).$$

Combining Proposition 3.3 and Lemma 2.2 in Granville-Soundararajan, we see that the left-hand side is bounded by

$$\frac{\log \log x}{\log x} \cdot \Big(\max_{|t| \leq \log x} |(1 - w^{-it}) \cdot F_x(1+it)| + 1 \Big). \tag{4}$$

Note that $|1 - w^{-it}| \ll \min(|t|, 1)$. Using the repulsion estimate from Lemma 13 (see the Appendix) we find that the above is

$$\ll (\log \log x) \cdot \max_{|t| \leq \log x} \Big(\frac{\min(|t|, 1)}{(1 + |t| \log x)^{\eta}} + \frac{1}{(\log x)^{\eta}} \Big) \cdot \prod_{p \leq x} \Big(1 + \frac{|f(p)| - 1}{p}\Big) + \frac{\log \log x}{\log x}.$$

In turn this is,

$$\ll \frac{\log \log x}{(\log x)^{\eta/2}} \prod_{p \leq x} \Big(1 + \frac{|f(p)| - 1}{p}\Big) + \frac{\log \log x}{\log x}.$$

We can then eliminate the second term, by fiddling with the value of η, since it is allowed to depend on κ. $\square$

Proof of Corollary 3. Let $\eta = \eta(\kappa)$ be the smallest of the constants appearing in Lemma 11 and Lemma 13 . Then by Theorem 1 and the repulsion estimate from Lemma 13 (see the Appendix) we find,

in reality they are the same constants

$$\frac{1}{y}\sum_{x\le n\le x+y} f(n) = \frac{(\log x)^{\eta/2}}{x}\sum_{x\le n\le x+x(\log x)^{-\eta/2}} f(n)$$
$$+ O\Big((\log x)^{-\eta/3+o(1)}\prod_{p\le x}\Big(1+\frac{|f(p)|-1}{p}\Big) + \frac{1}{(\log x)^{\eta+o(1)}}\prod_{p\le x}\Big(1+\frac{|f(p)|-1}{p}\Big)\Big)$$

Then using Lemma 11 allows us to conclude. $\square$

Proof of Corollary 4. Without loss of generality we assume that $w \le x^{1/100}$. Corollary 4 follows from Corollary 3 on noticing that for $H > x^{1/2}$,

$$\Big|\sum_{\substack{x\le n\le x+H\\ n=ap\\ p>x/w}} f(n)\Big| \le \sum_{\substack{x\le n\le x+H\\ p|n \implies p\notin[w,x^{1/4}]}} |f(n)| \ll H\frac{\log w}{\log x}\prod_{p\le w}\Big(1+\frac{|f(p)|-1}{p}\Big)$$

In turn by our assumption this is

$$\ll \prod_{p\le x}\Big(1+\frac{|f(p)|-1}{p}\Big)\cdot\prod_{w\le p\le x^{\delta}}\Big(1-\frac{|f(p)|}{p}\Big) \ll \prod_{p\le x}\Big(1+\frac{|f(p)|-1}{p}\Big)\cdot\Big(\frac{\log w}{\log x}\Big)^{\kappa}.$$

$\square$

8.3. **Proof of Corollary 5 and Corollary 6.**

Proof of Corollary 5. The upper bound follows immediately from Shiu's bound. To obtain the lower bound we restrict to number that are x^{ε} smooth and apply Corollary 3

$\square$

Proof of Corollary 6. This is in fact immediate from the previous Corollary for number fields K with class number one. For number fields with class number > 1 we appeal to a Lemma from a pre-print of Matomaki-Radziwill to see that the indicator function of norm-forms is a linear combination of complex valued multiplicative functions. To do: Add it later?

$\square$

9. Acknowledgements

Appendix A. Vinogradov repulsion

We state below two repulsion estimates.

Lemma 12. *Let f be a multiplicative function with each $|f(n)| \leq 1$. Select $t_1 \in \mathbb{R}$ with $|t_1 - t_0| \geq \log x$ and $|t_1| \leq 2(x/y)\log^{20} x$ which maximize s $|F_y(1+it_1)|$. Then*

$$\frac{|F_y(1+it_1)|}{\log x} \ll \left(\frac{(\log\log x)^4}{\log x}\right)^{\frac{1}{3}(1-\frac{2}{\pi})}$$

If $f(n) \geq 0$ for all n then we obtain the analogous result with $\frac{1}{\pi}$ in place of $\frac{2}{\pi}$.

Proof. Now

$$\begin{aligned}\log|F_y(1+it_1)| &\leq \frac{1}{2}\left(\log|F_y(1+it_0)| + \log|F_y(1+it_1)|\right)\\ &= \sum_{p\leq y} \mathrm{Re}\left(\frac{f(p)}{p^{1+iu}} \frac{(p^{i\tau}+p^{-i\tau})}{2}\right) + O(1)\\ &\leq \sum_{p\leq y} \frac{1}{p}|\cos(\tau\log p)| + O(1),\end{aligned}$$

where $\tau = |t_0 - t_1|/2$ and $u = (t_0+t_1)/2$.

Now if $T \geq 1$ and $\exp(C(\log T)^{2/3}(\log\log 10T)^{4/3}) \leq N \leq y$, we have

$$\sum_{N\leq p\leq y} \frac{1}{p}|\cos(T\log p)| = \frac{2}{\pi}\log\left(\frac{\log y}{\log N}\right) + O(1)$$

by approximating $|\cos(T\log n)|$ by a polynomial in $\cos(T\log n)$, and then applying the explicit formula for the number of primes up to x with the Vinogradov–Korobov zero-free region, $\sigma \geq 1 - c/((\log|t|)^{2/3}(\log\log|t|)^{1/3})$, and noting that $\int_0^1 |\cos(2\pi t)|dt = \frac{2}{\pi}$.

We apply this with $N = \exp(C(\log x)^{2/3}(\log\log x)^{4/3})$, and the trivial bound for smaller p, to obtain

$$\log\frac{|F_y(1+it_1)|}{\log y} \leq -\left(1-\frac{2}{\pi}\right)\log\left(\frac{\log y}{\log N}\right) + O(1)$$

If $f(n) \geq 0$ for all n, then $t_0 = 0$ and $F_y(1-it_1) = \overline{F_y(1+it_1)}$, so

$$\log|F_y(1+it_1)| = \frac{1}{2}\sum_{p\leq y}\frac{1}{p}\mathrm{Re}\left(f(p)(p^{-it_1}+p^{it_1})\right) \leq \sum_{p\leq y}\frac{\max\{0,\cos(t\log p)\}}{p}$$

and we get the same results, with $\int_0^1 \max\{0,\cos(2\pi t)\}dt = \frac{1}{\pi}$ replacing $\frac{2}{\pi}$. ∎

Lemma 13. *Let $f : \mathbb{N} \to [-1, 1]$ be a multiplicative function. Suppose that, there exists a constant $\kappa, \delta, C > 0$ such that,*

$$\sum_{w \le p \le z} \frac{|f(p)|}{p} \ge \kappa \sum_{w \le p \le z} \frac{1}{p} - \frac{C}{\log w}$$

for all $3 \le w \le z \le y^{\delta}$. Then, there exists a $\eta = \eta(\kappa) > 0$ such that for all $|t| \le y$,

$$\frac{|F_y(1 + it)|}{\log y} \ll \Big(\frac{1}{(1 + |t| \log y)^{\eta}} + \frac{1}{(\log y)^{\eta}} \Big) \cdot \prod_{p \le y} \Big(1 + \frac{f(p) - 1}{p} \Big).$$

Proof. Let $\varepsilon \in (0, \frac{1}{1000})$ be a small fixed quantity. Notice that,

$$\begin{aligned}
\log |F_y(1 + it)| &= \Re \sum_{p \le y} \frac{f(p) \cos(t \log p)}{p} + O(1) \\
&\le \sum_{p \le y} \frac{f(p)}{p} - \sum_{p \le y} \frac{f(p)(1 - |\cos(t \log p)|)}{p} \\
&\le \sum_{p \le y} \frac{f(p)}{p} - \kappa\varepsilon \sum_{\substack{|\cos(t \log p)| \le 1 - \varepsilon \\ N \le p \le y}} \frac{1}{p}
\end{aligned}$$

where $N = \max(e^{1/|t|}, (\log y)^{2/3+\varepsilon})$. Let $\Phi(x) = \sum_{|\ell| \le \Delta} a(\ell) e(x\ell)$ be a trigonometric polynomial of degree Δ such that,

$$\mathbf{1}_{|\cos(t \log p)| \le 1 - \varepsilon} \ge \Phi(t \log p)$$

with Δ choosen large enough so that, $a(0) \ge \frac{1}{2}$. It follows from this and Vinogradov-Korobov that,

$$\sum_{\substack{N \le p \le y \\ |\cos(t \log p)| \le 1 - \varepsilon}} \frac{1}{p} \ge \sum_{N \le p \le y} \frac{\Phi(t \log p)}{p} = a(0) \log \Big(\frac{\log y}{\log N} \Big) + O(1).$$

Thus

$$\log |F_y(1 + it)| \le \sum_{p \le y} \frac{f(p)}{p} - \frac{\kappa\varepsilon}{2} \log \Big(\frac{\log y}{\log N} \Big) + O(1)$$

and the claim follows. □