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On Complementary Decompositions of the Complete Graph

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Abstract. We consider decompositions $K_n \to H$, where H is either P_3 (the path with 3 edges) or the complete bipartite graph $K_{1,3}$, with the property that upon taking the complement of each graph in the decomposition one obtains a new decomposition $K_n \to H^c$.

1. Introduction

Let G, H be graphs. An *H-decomposition* of G, denoted $G \to H$ is a collection H_1, H_2, …, H_t of subgraphs of G, each H_i isomorphic to H, such that each edge in G appears in exactly one of the H_i. Let H^c denote the complement of H and suppose that the collection H_1^c, H_2^c, …, H_t^c forms an H^c-decomposition of G. We will say that the two decompositions are *complementary* and write $G \to (H, H^c)$. Note that under these circumstances the set $\{H_i \cup H_i^c\}$ is a collection of complete subgraphs of G with the property that each edge in G appears in exactly two of its members; in particular a complementary decomposition $K_n \to (H, H^c)$ gives rise to a balanced incomplete block design $(n, k, 2)$-BIBD where k is the number of vertices in H.

Perhaps the most extensively studied examples of complementary decompositions are the nested Steiner Triple Systems. A *Steiner Triple System* STS(v) is a $(v, 3, 1)$-BIBD, i.e. a decomposition of K_v into triangles. If one can add a point to each block in an STS(v) to obtain a $(v, 4, 2)$-BIBD, the STS is said to be *nested*. A moment's reflection will convince the reader that a nested STS(v) is equivalent to a complementary decomposition $K_v \to (K_{1,3}, K_{1,3}^c)$. D. Stinson [6] has proven that there exist nested STS(v) if and only if $v \equiv 1$ modulo 6. A *Steiner Pentagon System* SPS(n) is a decomposition $K_n \to C_5$ (where C_k denote the cycle of length k) such that any pair of vertices are joined by a path of length two in exactly one of the cycles in the decomposition. These systems have an algebraic motivation, coexisting with a certain class of quasigroups. We mention them here because an SPS(n) is just a complementary decomposition $K_n \to (C_5, C_5)$. It is shown in [4] that SPS(n) exist if and only if $n \equiv 1$ or 5 modulo 10, except when $n = 15$.

It is easily seen that if H is a graph on k vertices and there exists a decomposition

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$G \to (H, H^c)$ then $k \equiv 0$ or 1 modulo 4 since each of H and H^c contains $\frac{1}{2}\binom{k}{2}$ edges. In Section 2 we will be concerned with decompositions $K_n \to (H, H^c)$ where H has 4 vertices (and therefore 3 edges). There are two possibilities for the pairs (H, H^c):

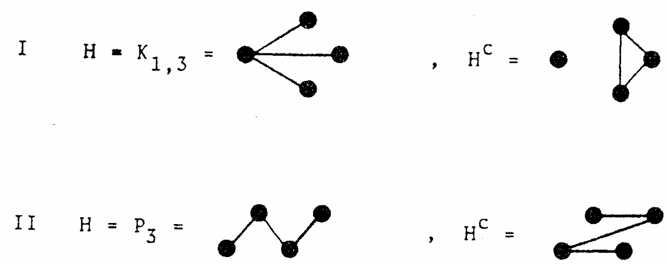

Fig. 1

We will prove that there is a complementary decomposition $K_n \to (P_3, P_3)$ if and only if $n \equiv 1$ modulo 3. We have already indicated that a complementary decomposition $K_n \to (K_{1,3}, K_{1,3}^c)$ exists if and only if $n \equiv 1$ modulo 6. We will take this one step further and show that for each integer $n \equiv 1$ modulo 6 there exists a pair of complementary decompositions $K_n \to (K_{1,3}, K_{1,3}^c)$ and $K_n \to (P_3, P_3)$ which give rise to the *same* $(n, 4, 2)$-BIBD; that is, the graphs $H_1, \ldots, H_t, H_1^c, \ldots, H_t^c$ and $J_1, \ldots, J_t, J_1^c, \ldots, J_t^c$ corresponding to these decompositions can be numbered so that $H_i \cup H_i^c = J_i \cup J_i^c$ for $i = 1, \ldots, t$. The $(n, 4, 2)$-BIBD's arising in this manner will be called *pandecomposable* (since they can simultaneously generate all possible complementary decompositions of K_n into 4-vertex graphs).

2. Complementary Decompositions

We will use the methods of [5] and [6], relying on the following well known results (see [3], [1]).

Lemma 2.1 [Hanani]. *If $n \equiv 1$ or 4 modulo 12 then there is a K_4-decomposition of K_n (i.e. an $(n, 4, 1)$-BIBD).*

Lemma 2.2 [Brouwer]. *If $n \equiv 7$ or 10 modulo 12 and $n \neq 10$ or 19 then there is a decomposition of K_n into K_4's together with one K_7.*

Lemma 2.3. *There exist pandecomposable $(7, 4, 2)$-, $(13, 4, 2)$- $(19, 4, 2)$- and $(37, 4, 2)$-BIBD's.*

Proof. In each case we give a set of base blocks, each block a, b, c, d giving rise to the graphs

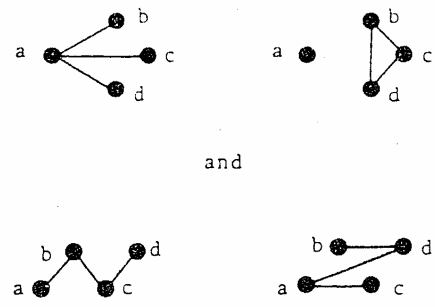

Fig. 2

$n = 7$	0, 4, 2, 1	mod 7		
$n = 13$	0, 9, 1, 3;	0, 6, 5, 2	mod 13	
$n = 19$	0, 11, 1, 7;	0, 4, 6, 9;	0, 5, 17, 16	mod 19
$n = 37$	0, 15, 2, 20;	0, 16, 9, 12;	0, 26, 1, 10;	
	0, 29, 31, 14;	0, 32, 18, 24;	0, 33, 34, 7	mod 37

Lemma 2.4. *Let $G = K_{2,2,2,2}$ or $K_{2,2,2,2,2,2,2}$. There exist complementary deco positions $G \to (K_{1,3}, K^c_{1,3})$ and $G \to (P_3, P_3)$ giving rise to the same double ed covering of G by K_4's.*

Proof. In each case we give the K_4's, which are to be interpreted as in lemma 2 with the exception of the four starred (*) blocks, whose $K_{1,3}$, $K^c_{1,3}$ subgraphs a to be interpreted by interchanging a and b in the top two figures of lemma 2.3.

$G = K_{2,2,2,2}$:

0, 2, 7, 4;	1, 3, 6, 5;	2, 1, 5, 7;	3, 0, 4, 6;
4, 2, 6, 1;	5, 3, 7, 0;	6, 0, 5, 2;	7, 1, 4, 3.

$G = K_{2,2,2,2,2,2,2}$: We write the vertices as $1, 1', 2, 2', \ldots, 7, 7'$.

1, 5, 6, 2′	4, 1, 2, 3	7, 1, 4′, 5	3′, 5, 7′, 2	6′, 1′, 7′, 5′
1, 6′, 5′, 2	4, 5′, 6, 3′	7, 2′, 4, 6′	3′, 7, 6, 1	6′, 3′, 4, 5
2, 1′, 3, 6	5, 2, 4, 1′	1′, 7, 4, 3	4′, 2′, 1, 3′	7′, 4′, 2, 6
2, 6′, 7, 4′	5, 2′, 3, 7′	1′, 4′, 3′, 7′	4′, 6′, 5, 3	*4, 7′, 1, 5′
3, 7, 5′, 2′	6, 1′, 5, 7	2′, 7′, 6, 4	5′, 3′, 2, 7	
*1, 3, 6′, 7′	6, 4′, 3, 5′	*6′, 2′, 3′, 1′	*4′, 5′, 1′, 2′	

In what follows we will denote by $K_{g_1 \times t_1, \ldots, g_r \times t_r}$ the complete multipartite gra with t_i holes of size g_i, $i = 1, \ldots, r$.

Theorem 2.5. *There exists a pandecomposable* $(n, 4, 2)$-BIBD *if and only if* $n \equiv$ *modulo* 6.

Proof. $n \equiv 1$ or 7 modulo 24. Let $m = \dfrac{n-1}{2}$. Remove a point from an $(m + 1, 4,$ BIBD (Lemma 2.1) to yield a partition of $K_{3 \times m/3}$ into K_4's. Replace each vert by two new ones and each K_4 by a $K_{2,2,2,2}$ to yield a partition of $K_{6 \times m/3}$ in

$K_{2,2,2,2}$'s. Add a new vertex to this design; each $K_{2,2,2,2}$ is replaced by the design of Lemma 2.4 while each of the holes are to be filled with the seven point designs of Lemma 2.3.

$n \equiv 13$ or 19 modulo 24. Let $m = \frac{n-1}{2}$. From Lemma 2.3 we may assume that $m \neq 6$, 9 or 18 so that by Lemma 2.2 we can construct a decomposition of K_{m+1} into K_4's together with one K_7. Remove a vertex not in the K_7 to yield a partition of $K_{3\times m/3}$ into K_4's and a K_7. Now proceed exactly as above, noting that both designs of Lemma 2.4 will be used in the final step. □

Remark. By essentially the same procedure as the foregoing, Lemmas 2.1, 2.2 and 2.4 can be seen to imply the following: given any integer $n \equiv 1$ modulo 3, $n \neq 10$, 19, the cocktail party graph T_{2n} (i.e. K_{2n} minus the edges of a perfect matching) admits a pair of complementary decompositions $T_{2n} \to (K_{1,3}, K_{1,3}^c)$ and $T_{2n} \to (P_3, P_3)$ giving rise to the same double edge covering of T_{2n} by K_4's.

Lemma 2.6. *If $n = 4, 7, 10$ or 19 there is a complementary decomposition $K_n \to (P_3, P_3)$.*

Proof. The cases $n = 7$ or 19 are a consequence of Theorem 2.5. In each of the remaining cases we list the blocks of the corresponding $(n, 4, 2)$-BIBD, each block a, b, c, d giving rise to the graphs

Fig. 3

$n = 4$	0, 1, 2, 3;	2, 0, 3, 1.		
$n = 10$	0, 1, 2, 7;	8, 0, 4, 3;	5, 0, 3, 1;	2, 0, 6, 4;
	0, 9, 6, 5;	0, 7, 8, 9;	9, 1, 6, 3;	9, 2, 8, 1;
	2, 4, 9, 5;	4, 7, 9, 3;	6, 2, 3, 8;	2, 5, 3, 7;
	1, 5, 4, 8;	4, 1, 7, 6;	6, 8, 5, 7.	

□

The (10,4, 2)-BIBD given above was constructed by deleting a block from the symmetric (16, 6, 2)-BIBD given in the appendix of Hall [2]. We remark that this design has another interesting property: for each $i = 0, \ldots, 9$ the set $\{B - i: i \in B\}$ can be arranged into two parallel classes of triples on the set $\{0, 1, \ldots, 9\} - \{i\}$.

Theorem 2.7. *There exists a complementary decomposition $K_n \to (P_3, P_3)$ if and only if $n \equiv 1$ modulo 3.*

Proof. The condition $n \equiv 1$ modulo 3 is a necessary one on the existence of an $(n, 4, 2)$-BIBD.

$n \equiv 1$ or 4 modulo 12. Take a decomposition of K_n into K_4's (Lemma 2.1) and replace each K_4 by the four-point design of Lemma 2.6.

$n \equiv 7$ or 10 modulo 12. From Lemma 2.6 we may assume that $n \neq 10$ or 19. Use

Lemma 2.2 to construct a partition of K_n into K_4's and a K_7. Build the four- and seven-point designs of Lemma 2.6 on the graphs in the partition.

4. Conclusion

The designs given here can be fit into a more general framework as follows. Let G be a simple graph. Let m, n and λ be positive integers with $m\lambda = \binom{n}{2}$ and let $G_1, \ldots, G_\lambda$ be a collection of m-edge subgraphs of K_n. The problem is to construct a K_n-decomposition D of λG which satisfies the following properties:

(i) each complete graph $K_n(i)$ in D admits a decomposition into m-edge subgraphs $G_1(i), \ldots, G_\lambda(i)$ (each $G_j(i)$ isomorphic to G_j), and

(ii) for each $j = 1, \ldots, \lambda$, the collection $\{G_j(i)\}$ is a G_j-decomposition of G.

A particularly interesting sub-problem of the above occurs when we let $G = K_v$ and $m = n\left(\text{so that } n \text{ is odd and } \lambda = \frac{n-1}{2}\right)$; then let G_1 be an n-cycle, and for each $j = 2, \ldots, \lambda$ let G_j be the set of cycles on the vertices of G_1 formed by ruling that two vertices be adjacent if and only if they are at distance j from each other along G_1. D.R. Stinson has suggested the name *Steiner n-gon system of order v* for these designs since they are a natural generalization of Steiner Triple Systems (i.e. where $n = 3$) and Steiner Pentagon Systems (i.e. where $n = 5$). These designs are of interest because it is not difficult to show that a Steiner n-gon system of order v is equivalent to a Cyclic Perpendicular Array CPA(v, n) (see e.g. [7] for the definition and some applications of perpendicular arrays). In a forthcoming paper ('Nested Steiner n-gon Systems and Perpendicular Arrays') we will discuss this problem in some detail and obtain many new examples of these designs.

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